MAS 4105 Test 4

(12 pts) 1.

a. If u and v are vectors in an inner product space $\mathbf{V}$, express the Cauchy-Schwartz Inequality in terms of u and v.

b. If u and v are vectors in an inner product space $\mathbf{V}$, express the Triangle Inequality in terms of u and v.

c. If $\mathbf{R^3}$ is considered as an inner product space with the standard inner product, give an example of an orthogonal set in $\mathbf{R^3}$ which is not linearly independent.

d. Let $\mathbf{T}$ be a linear operator on a vector space $\mathbf{V}$; give two conditions which will insure that the transformation is diagonalizable.

e. Let $\mathbf{T}$ be a linear operator on a vector space $\mathbf{V}$ and let λ_1 and λ_2 be two distinct eigenvalues for $\mathbf{T}$; what is $\mathbf{E}_{\lambda_1} \cap \mathbf{E}_{\lambda_2}$?

f. Let $\mathbf{T}$ be a linear operator on a vector space $\mathbf{V}$ and let λ be an eigenvalues for $\mathbf{T}$; if the multiplicity of λ is four, what are the possible dimensions of $\mathbf{E}_{\lambda_1}$?

(8 pts) 2. Given the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 9 & 0 & 0 \end{pmatrix}$$

Find the eigenvalues and eigenvectors for the matrix. If the matrix $\mathbf{A}$ is diagonalizable, give the corresponding matrices $\mathbf{Q}$ and $\mathbf{D}$.

(6 pts) 3. In the proof of the theorem below explain the underlined statement.

Theorem 5.8. Let $\mathbf{T}$ be a linear operator on a vector space $\mathbf{V}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of $\mathbf{T}$. For each $i = 1, 2, \ldots, k$, let S_i be a finite linearly independent subset of the eigenspace $\mathbf{E}_{\lambda_\mathrm{i}}$. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of $\mathbf{V}$.

Proof. Suppose that for each i

$$S_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}.$$

Then $S = \{v_{ij} : 1 \leq j \leq n_i, \ \text{ and } \ 1 \leq \mathrm{i} \leq \mathrm{k}\}$. Consider any scalars $\{a_{ij}\}$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = \mathbf{0}.$$

For each i, let

$$w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}.$$

<u>Then $w_i \in \mathbf{E}_{\lambda_\mathrm{i}}$ for each i</u>, and $w_1 + w_2 + \cdots + w_k = \mathbf{0}$. Therefore by the lemma, $w_i = \mathbf{0}$ for all i. But each S_i is linearly independent, and hence $a_{ij} = 0$ for all j. We conclude that S is linearly independent.

(8 pts) 4. Let $\mathbf{T}$ be a linear operator on a vector space $\mathbf{V}$, and let λ be an eigenvalue of $\mathbf{T}$. Prove that a vector $v \in \mathbf{V}$ is an eigenvector of $\mathbf{T}$ corresponding to λ if and only if $v \neq \mathbf{0}$ and $v \in \mathbf{N}(\mathbf{T}-\lambda\mathrm{I})$.

(8 pts) 5. Let $\mathbf{R^3}$ be considered as an inner product space with the standard inner product. Convert the following basis of $\mathbf{R^3}$ into an orthonormal basis:

$$\beta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \right\}.$$

(8 pts) 6. Let **V** be an inner product space and $\mathbf{S} = \{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\mathrm{k}\}$ be an orthonormal subset of **V**. First prove that if $y \in$span(**S**), then

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle \, v_i.$$

Use this result to prove that the vectors in **S** are linearly independent.

Solutions:

1. a. $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$ b. $\|u+v\| \leq \|u\| + \|v\|$

c. any orthogonal set containing the zero vector

d. the characteristic polynomial splits and the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue

e. $\{\mathbf{0_V}\}$ f. $\{1, 2, 3, 4\}$

2.

$$\det(\mathbf{A} - \lambda \mathbf{I_3}) = \det \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 9 & 0 & -\lambda \end{pmatrix} = (-1)^2(-\lambda)\det \begin{pmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{pmatrix} + (1)^4(1)\det \begin{pmatrix} 0 & 2-\lambda \\ 9 & 0 \end{pmatrix}$$

$$= -\lambda(2-\lambda)(-\lambda) + (-1)(9)(2-\lambda) = (2-\lambda)(\lambda^2 - 9) = (2-\lambda)(\lambda-3)(\lambda+3).$$

Thus eigenvalues of $\mathbf{A}$ are 2, 3, -3.

$$\lambda = 2: \quad (\mathbf{A} - 2\mathbf{I_3}) = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 9 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \text{eigenvector} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda = 3: \quad (\mathbf{A} - 3\mathbf{I_3}) = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -1 & 0 \\ 9 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \text{eigenvector} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\lambda = -3: \quad (\mathbf{A} - (-3)\mathbf{I_3}) = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 5 & 0 \\ 9 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \text{eigenvector} \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.$$

Thus $\mathbf{A}$ is diagonalizable and

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 3 & 3 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

3. By definition, the vectors of the form v_{ij} are elements of $\mathbf{E}_{\lambda_\mathbf{i}}$. Since $\mathbf{E}_{\lambda_\mathbf{i}}$ is a subspace of $\mathbf{V}$, it is closed under vector addition and scalar multiplication. Since w_i is a finite sum of scalar multiples of vectors in $\mathbf{E}_{\lambda_\mathbf{i}}$, the vector therefore lies in $\mathbf{E}_{\lambda_\mathbf{i}}$.

4. Suppose first that $v \in \mathbf{V}$ is an eigenvector of $\mathbf{T}$ corresponding to λ. By the definition of an eigenvector, $v \neq \mathbf{0}$ and $\mathbf{T}(\mathrm{v}) = \lambda\mathrm{v}$. The last equation can be rewritten as

$$\mathbf{0} = \mathbf{T}(\mathrm{v}) - \lambda\mathrm{v} = \mathbf{T}(\mathrm{v}) - \lambda\mathbf{I}(\mathrm{v}) = (\mathbf{T} - \lambda\mathbf{I})(\mathrm{v})$$

so that $v \in \mathbf{N}(\mathbf{T} - \lambda\mathrm{I})$ by the definition of the null space.

Next suppose that $v \in \mathbf{V}$, $v \neq \mathbf{0}$, and $v \in \mathbf{N}(\mathbf{T} - \lambda\mathrm{I})$. By the definition of the null space, v satisfies $(\mathbf{T} - \lambda\mathbf{I})(\mathrm{v}) = \mathbf{0}$ which upon rearrangement can be written as $\mathbf{T}(\mathrm{v}) = \lambda\mathrm{v}$. Thus by the definition of an eigenvector, v is an eigenvector of $\mathbf{T}$ corresponding to the eigenvalue λ.

5.

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{6}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{-6}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus an orthonormal basis is given by

$$\left\{ \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \tfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

6. If $y \in \text{span}(\mathbf{S})$, then by definition there exists scalars $a_1, a_2, \ldots, a_k$ such that $y = \sum_{i=1}^{k} a_i v_i$. Then for $1 \leq j \leq k$, since the vectors in $\mathbf{S}$ are orthonormal, we have

$$\langle y, v_j \rangle = \left\langle \sum_{i=1}^{k} a_i v_i, v_j \right\rangle = \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle = a_j \langle v_j, v_j \rangle = a_j \|v_j\|^2 = a_j \cdot 1 = a_j.$$

So $a_j = \langle y, v_j \rangle$, and the result follows.

For the second part of the problem, we set the vector y equal to the zero vector in $\mathbf{V}$. By Theorem 6.1, $0 = \langle \mathbf{0_V}, \mathrm{c} \rangle$ for any vector $c \in \mathbf{V}$, so that

$$\mathbf{0_V} = \sum_{\mathrm{i=1}}^{\mathrm{k}} \langle \mathbf{0_V}, \mathrm{v_i} \rangle \, \mathrm{v_i} = \sum_{\mathrm{i=1}}^{\mathrm{k}} 0 \cdot \mathrm{v_i}.$$

Since the coefficients were uniquely determined by the inner product, the vectors in $\mathbf{S}$ are linearly independent by definition.