MAC2233 Chapter 6 Review

1. Find the indefinite integral $\int 7x^3\ dx$.

2. Find the indefinite integral $\int 6\sqrt{x} - 9x^{-2} + 5e^x\ dx$.

3. Find the indefinite integral $\int \frac{2x^2-7}{x^3}\ dx$.

4. Find the indefinite integral $\int (x+2)^3\ dx$.

5. Given $f'(x) = e^x - 2x$ and $f(0) = 2$, find $f(x)$.

6. Find the indefinite integral $\int \frac{3x^2+2}{(x^3+2x)^2}\ dx$.

7. Find the indefinite integral $\int e^{-x}(1+e^{-x})\ dx$.

8. Find the indefinite integral $\int \frac{5}{x(\ln x)^4}\ dx$.

9. The population of a certain city is projected to grow at the rate of

$$r(t) = 400\left(1 + \frac{2t}{24+t^2}\right) \quad (0 \le t \le 5)$$

people/year, t years from now. The current population is 60,000. What will be the population 5 years from now?

10. Approximate the area under the curve $f(x) = 4 - x^2,\ x \in [0,2]$, using four rectangles of equal width and the midpoint of each resulting interval as the sample point. Use a definite integral to find the exact area.

11. Approximate the area under the curve $f(x) = 2x,\ x \in [0,3]$ using six rectangles and the left endpoint of each interval as the sample point. Use a definite integral to find the exact area.

12. Find the area of the region under the graph of $f(x) = 2x + x^6$ on the interval $[0,2]$.

13. Evaluate the definite integral $\int_1^9 \frac{x - 1}{\sqrt{x}}\, dx$.

14. The management of Ditton Industries has determined that the daily marginal revenue function associated with selling x units of their deluxe toaster ovens is given by

$$R'(x) = -0.1x + 40$$

where $R'(x)$ is measured in dollars/unit. Find the daily total revenue realized from the sale of 200 units of the toaster oven. Find the additional revenue realized when the production (and sales) level is increase from 200 to 300 units.

15. Evaluate the definite integral $\int_0^2 \frac{x}{\sqrt{x^2+5}}\, dx$.

16. Evaluate the definite integral $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

17. Find the average value of $f(x) = 4 - x^2$ on $[-2, 3]$.

18. Find the area of the region enclosed by $f(x) = 9 - x^2$, $g(x) = -x + 3$, $x = -1$, and $x = 5$.

19. Find the area between the curve $y = (x - 1)(x + 2)$ and the x-axis for $x \in [0, 3]$.

20. Because of the increasingly important role played by coal as a viable alternative energy source, the production of coal has been growing at the rate of

$$3.5e^{0.05t}$$

billion metric tons/year, t years from 1980 (which corresponds to $t = 0$). Had it not been for the energy crisis, the rate of production since 1980 might have been only

$$3.5e^{0.01t}$$

billion metric tons/year, t years from 1980. Determine how much additional coal was produced between 1980 and the end of the century as an alternative energy source.

Solutions:

1.
$$\int 7x^3 \, dx = 7\Big(\frac{1}{1+3}\Big)x^{1+3} + C = \frac{7}{4}x^4 + C.$$

2.
$$\int 6\sqrt{x} - 9x^{-2} + 5e^x \, dx = \int 6x^{1/2} - 9x^{-2} + 5e^x \, dx$$
$$= 6\Big(\frac{1}{(1/2)+1}\Big)x^{(1/2)+1} - 9\Big(\frac{1}{-2+1}\Big)x^{-2+1} + 5e^x + C = 4x^{3/2} + 9x^{-1} + 5e^x + C.$$

3.
$$\int \frac{2x^2-7}{x^3} \, dx = \int \frac{2x^2}{x^3} - \frac{7}{x^3} \, dx = \int 2x^{-1} - 7x^{-3} \, dx$$
$$= 2(\ln|x|) - 7\Big(\frac{1}{-3+1}\Big)x^{-3+1} + C = 2\ln|x| + \frac{7}{2}x^{-2} + C = 2\ln|x| + \frac{7}{2x^2} + C$$

4.
$$\int (x+2)^3 \, dx = \int x^3 + 6x^2 + 12x + 8 \, dx$$
$$= \Big(\frac{1}{3+1}\Big)x^{3+1} + 6\Big(\frac{1}{2+1}\Big)x^{2+1} + 12\Big(\frac{1}{1+1}\Big)x^{1+1} + 8(x) + C$$
$$= \frac{1}{4}x^4 + 2x^3 + 6x^2 + 8x + C.$$

5. We begin by finding the general antiderivative:

$$f(x) = e^x - 2\Big(\frac{1}{1+1}\Big)x^{1+1} + C = e^x - x^2 + C$$

and using the condition $f(0) = 2$ obtain

$$2 = e^0 - (0)^2 + C$$
$$2 = 1 + C$$
$$1 = C$$

so that we write $f(x) = e^x - x^2 + 1$.

6. We begin with the following u substitution:

$$u = x^3 + 2x$$
$$\frac{du}{dx} = 3x^2 + 2$$
$$du = (3x^2 + 2)\ dx$$

and rewrite the integral as

$$\int \frac{3x^2+2}{(x^3+2x)^2}\ dx = \int \frac{1}{(x^3+2x)^2} \cdot (3x^2+2)\ dx = \int \frac{1}{u^2}\ du.$$

Evaluating this integral we find

$$\int \frac{1}{u^2}\ du = \int u^{-2}\ du = \frac{-1}{u} + C.$$

Substituting $u = x^3 + 2x$ into the antiderivative we find

$$\int \frac{3x^2+2}{(x^3+2x)^2}\ dx = \frac{-1}{x^3+2x} + C.$$

7. We begin with the following u substitution:

$$u = 1 + e^{-x}$$
$$\frac{du}{dx} = -e^{-x}$$
$$du = -e^{-x} dx$$
$$-du = e^{-x} dx$$

and rewrite the integral as

$$\int e^{-x}(1+e^{-x})\ dx = \int (1+e^{-x})e^{-x}\ dx = \int u\ (-du) = \int -u\ du.$$

Evaluating this integral we find

$$\int -u\ du = \frac{-u^2}{2} + C.$$

Substituting $u = 1 + e^{-x}$ into the antiderivative we find

$$\int e^{-x}(1+e^{-x})\ dx = \frac{-(1+e^{-x})^2}{2} + C.$$

8. We begin with the following u substitution:

$$u = \ln x$$
$$\frac{du}{dx} = \frac{1}{x}$$
$$du = \frac{1}{x}\ dx$$

and rewrite the integral as

$$\int \frac{5}{x(\ln x)^4}\ dx = \int \frac{5}{(\ln x)^4} \cdot \frac{1}{x}\ dx = \int \frac{5}{u^4}\ du.$$

Evaluating this integral we find

$$\int \frac{5}{u^4}\ du = \int 5u^{-4}\ du = \frac{-5}{3}u^{-3} + C = \frac{-5}{3u^3} + C.$$

Substituting $u = \ln x$ into the antiderivative we find

$$\int \frac{5}{x(\ln x)^4}\ dx = \frac{-5}{3(\ln x)^3} + C.$$

9. Since the antiderivative of $r(t)$ is a function which represents the population at a given time t, we need to find the antiderivative of $r(t)$ which satisfies $r(0) = 60,000$ (the population at $t = 0$).

We begin by rewriting the integral as

$$\int 400\left(1 + \frac{2t}{24 + t^2}\right)\ dt = \int 400 + 400\left(\frac{2t}{24 + t^2}\right)\ dt$$

$$= \int 400\ dt + \int 400\left(\frac{2t}{24 + t^2}\right)\ dt = 400t + \int 400\left(\frac{2t}{24 + t^2}\right)\ dt.$$

To evaluate the second integral we use the following u substitution:

$$u = 24 + t^2$$
$$\frac{du}{dt} = 2t$$
$$du = 2t\ dt$$

and rewrite the second integral as

$$\int 400\Big(\frac{1}{24+t^2}\Big)\cdot 2t\ dt = \int 400\Big(\frac{1}{u}\Big)\ du = \int \frac{400}{u}\ du = 400\ln|u| + C.$$

Substituting $u = 24 + t^2$ into the antiderivative we find

$$\int 400\Big(\frac{2t}{24+t^2}\Big)\ dt = 400\ln|24+t^2| + C$$

so we can write

$$R(t) = \int r(t)\ dt = 400t + 400\ln|24+t^2| + C = 400(t + \ln|24+t^2|) + C.$$

To evaluate C we write

$$R(0) = 60,000 = 400((0) + \ln(24+(0)^2) + C = 400\ln 24 + C$$

so that $C = 60,000 - 400\ln 24$ and the function representing population can be written as

$$R(t) = 400(t + \ln|24+t^2|) + 60,000 - 400\ln 24.$$

To find the population 5 years from now, we evaluate

$$R(5) = 400((5)+\ln(24+(5)^2))+60,000-400\ln 24 = 2000+400\ln 49+60,000-400\ln 24 \approx 62,286.$$

10. The interval $[0,2]$ is divided into the four subintervals $[0,0.5]$, $[0.5,1.0]$, $[1.0,1.5]$, $[1.5,2.0]$, each of length $\Delta x = \frac{2-0}{4} = 0.5$ and with midpoints given by 0.25, 0.75, 1.25, and 1.75. Hence an approximation of the area under the curve is

$$A \approx f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x$$

$$A \approx 3.9375(0.5) + 3.4375(0.5) + 2.4375(0.5) + 0.9375(0.5) = 5.375.$$

The exact area is found from the integral

$$\int_0^2 4-x^2\ dx = (4x-x^3/3)|_0^2 = (4(2)-(2)^3/3)-(4(0)-(0)^3/3) = 16/3 \approx 5.333.$$

11. The interval $[0,3]$ is divided into the six subintervals $[0,0.5]$, $[0.5,1]$ $[1,1.5]$, $[1.5,2]$, $[2,2.5]$, $[2.5,3]$ each of length $\Delta x = \frac{3-0}{6} = 0.5$ and with left endpoints given by 0, 0.5, 1, 1.5, 2, and 2.5. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x$$

$$A \approx 0(0.5) + 1(0.5) + 2(0.5) + 3(0.5) + 4(0.5) + 5(0.5) = 7.5.$$

The exact area is given by the integral

$$A = \int_0^3 2x\ dx = x^2|_0^3 = (3)^2 - (0)^2 = 9.$$

12. The area is given by the following definite integral:

$$\int_0^2 2x + x^6\ dx = (x^2 + x^7/7)|_0^2 = ((2)^2 + (2)^7/7) - ((0)^2 + (0)^7/7) = 156/7$$

13.

$$\int_1^9 \frac{x\ -\ 1}{\sqrt{x}}\ dx = \int_1^9 \frac{x\ -\ 1}{x^{1/2}}\ dx = \int_1^9 \frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\ dx = \int_1^9 x^{1/2} - x^{-1/2}\ dx = ((2/3)x^{3/2} - 2x^{1/2})|_1^9$$

$$= ((2/3)(9)^{3/2} - 2(9)^{1/2}) - ((2/3)(1)^{3/2} - 2(1)^{1/2}) = ((2/3)(27) - 2(3)) - ((2/3)(1) - 2(1))$$

$$= (18 - 6) - ((2/3) - 2) = 40/3.$$

14. Since the revenue for selling zero units will be zero dollars (R(0) = 0), we find

$$R(200) = R(200) - R(0) = \int_0^{200} R'(x)\ dx = \int_0^{200} -0.1x + 40\ dx = (-0.05x^2 + 40x)|_0^{200}$$

$$= (-0.05(200)^2 + 40(200)) - (-0.05(0)^2 + 40(0)) = 6,000\ dollars.$$

The additional revenue realized when the production (and sales) level is increase from 200 to 300 units is given by

$$R(300) - R(200) = \int_{200}^{300} R'(x)\ dx = \int_{200}^{300} -0.1x + 40\ dx = (-0.05x^2 + 40x)|_{200}^{300}$$

$$= (-0.05(300)^2 + 40(300)) - (-0.05(200)^2 + 40(200)) = 1,500 \ dollars.$$

15. We begin with the following u substitution:

$$\begin{aligned} u &= x^2 + 5 \\ \frac{du}{dx} &= 2x \\ du &= 2x \ dx \\ \frac{du}{2} &= x \ dx \\ x = 0 \quad \rightarrow \quad u &= (0)^2 + 5 = 5 \\ x = 2 \quad \rightarrow \quad u &= (2)^2 + 5 = 9 \end{aligned}$$

and rewrite the integral as

$$\int_0^2 \frac{x}{\sqrt{x^2+5}} \ dx = \int_0^2 \frac{1}{\sqrt{x^2+5}} \cdot x \ dx = \int_5^9 \frac{1}{\sqrt{u}} \cdot \frac{1}{2} \ du = \int_5^9 \frac{u^{-0.5}}{2} \ du.$$

Evaluating this integral we find

$$\int_5^9 \frac{u^{-0.5}}{2} \ du = (u^{0.5})|_5^9 = ((9)^{0.5}) - ((5)^{0.5}) = 3 - \sqrt{5}.$$

16. We begin with the following u substitution:

$$\begin{aligned} u &= \sqrt{x} = x^{0.5} \\ \frac{du}{dx} &= \frac{1}{2\sqrt{x}} \\ du &= \frac{1}{2\sqrt{x}} \ dx \\ 2 \ du &= \frac{1}{\sqrt{x}} \ dx \\ x = 1 \quad \rightarrow \quad u &= \sqrt{1} = 1 \\ x = 4 \quad \rightarrow \quad u &= \sqrt{4} = 2 \end{aligned}$$

and rewrite the integral as

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^4 e^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\,dx = \int_1^2 e^u \cdot 2\,du = \int_1^2 2e^u\,du.$$

Evaluating this integral we find

$$\int_1^2 2e^u\,du = (2e^u)|_1^2 = (2e^2) - (2e^1) = 2e^2 - 2e = 2e(e-1).$$

17. The average value is given by

$$\frac{1}{3-(-2)}\int_{-2}^3 4-x^2\,dx = \frac{1}{5}\Big(4x-\frac{1}{3}x^3\Big)\Big|_{-2}^{3} = \frac{1}{5}\Big(4(3)-\frac{1}{3}(3)^3\Big)-\frac{1}{5}\Big(4(-2)-\frac{1}{3}(-2)^3\Big) = \frac{5}{3}.$$

18. We begin by finding the points of intersection of the graphs of the two functions on $[-1,5]$; setting the functions equal to each other gives

$$9-x^2 = -x+3$$

$$0 = x^2 + -x - 6$$

$$0 = (x-3)(x+2)$$

so that the graphs intersect at $x = 3$ and $x = -2$. Hence we subdivide the interval $[-1,5]$ into the subintervals $[-1,3]$ and $[3,5]$ and evaluate the functions at a test point in the interior of each of the subintervals. We find $f(0) = 9$, $g(0) = 3$, $f(4) = -7$, and $g(4) = -1$ so that on $[-1,3]$ the graph of $f(x)$ is above that of $g(x)$ and on $[3,5]$ the graph of $g(x)$ is above that of $f(x)$. Hence the area is given by

$$\int_{-1}^3 (9-x^2)-(-x+3)\,dx+\int_3^5 (-x+3)-(9-x^2)\,dx = \int_{-1}^3 -x^2+x+6\,dx+\int_3^5 x^2-x-6\,dx$$

$$= \Big(-\frac{x^3}{3}+\frac{x^2}{2}+6x\Big)\Big|_{-1}^{3} + \Big(\frac{x^3}{3}-\frac{x^2}{2}-6x\Big)\Big|_{3}^{5}$$

$$= \Big(-\frac{3^3}{3}+\frac{3^2}{2}+6(3)\Big)-\Big(-\frac{(-1)^3}{3}+\frac{(-1)^2}{2}+6(-1)\Big)+\Big(\frac{5^3}{3}-\frac{5^2}{2}-6(5)\Big)-\Big(\frac{3^3}{3}-\frac{3^2}{2}-6(3)\Big) = \frac{94}{3}.$$

19. The function $y = (x-1)(x+2)$ is negative for $x \in [0,1)$ and is positive for $x \in (1,3]$. Hence the area between the graph of the function and the x-axis is given by

$$\int_0^1 -(x-1)(x+2)\,dx+\int_1^3 (x-1)(x+2)\,dx = \int_0^1 -x^2-x+2\,dx+\int_1^3 x^2+x-2\,dx$$

$$= \left(\frac{-x^3}{3} - \frac{x^2}{2} + 2x\right)\Big|_0^1 + \left(\frac{x^3}{3} + \frac{x^2}{2} - 2x\right)\Big|_1^3$$

$$\left(\frac{-(1)^3}{3} - \frac{(1)^2}{2} + 2(1)\right) - \left(\frac{-(0)^3}{3} - \frac{(0)^2}{2} + 2(0)\right) + \left(\frac{3^3}{3} + \frac{3^2}{2} - 2(3)\right) - \left(\frac{(1)^3}{3} + \frac{(1)^2}{2} - 2(1)\right) = \frac{59}{6}.$$

20. The additional coal produced between 1980 and 2000 will be the difference between the amount of coal produced at the greater rate, $3.5e^{0.05t}$, and the amount of coal produced at the lesser rate, $3.5e^{0.01t}$. Hence the additional coal is given by the integral

$$\int_0^{20} 3.5e^{0.05t} - 3.5e^{0.01t}\ dt.$$

To evaluate this integral we write as the difference of two integrals

$$\int_0^{20} 3.5e^{0.05t} - 3.5e^{0.01t}\ dt = 3.5\int_0^{20} e^{0.05t}\ dt - 3.5\int_0^{20} e^{0.01t}\ dt$$

and use u substitution on each of the resulting integrals.

For the first integral we have

$$u = 0.05t$$
$$\frac{du}{dt} = 0.05$$
$$du = 0.05\ dt$$
$$20\ du = dt$$
$$t = 0 \quad \rightarrow \quad u = 0.05(0) = 0$$
$$t = 20 \quad \rightarrow \quad u = 0.05(20) = 1$$

and rewrite this integral as

$$3.5\int_0^1 e^u\ 20\ du = 3.5\int_0^1 20e^u\ du = 3.5(20e^u|_0^1) = 3.5(20e^1 - 20e^0) = 70e - 70.$$

For the second integral we have

$$u = 0.01t$$
$$\frac{du}{dt} = 0.01$$
$$du = 0.01\ dt$$

$$100\ du = dt$$

$$t = 0 \quad \rightarrow \quad u = 0.01(0) = 0$$

$$t = 20 \quad \rightarrow \quad u = 0.01(20) = 0.2$$

and rewrite this integral as

$$3.5\int_0^{0.2} e^u\ 100\ du = 3.5\int_0^{0.2} 100e^u\ du = 3.5(100e^u|_0^{0.2}) = 3.5(100e^{0.2} - 100e^0) = 350e^{0.2} - 350.$$

Thus, the increase in coal production is given by

$(70e - 70) - (350e^{0.2} - 350) = 70e - 350e^{0.2} + 280 \approx 42.79$ *billion metric tons*.