MAP2302 Final

(2 pts) 1. Give the solution to $y' + (1/x)y = 1,\ y(2) = 1$.

(2 pts) 2. If the equation $y'' + y' - 6y = \tan t$ is to be solved using the Method of Variation of Parameters, write the resulting system of equations used in the technique. (Do not solve the system!)

(2 pts) 3. Find a particular solution to $y'' + 2y' + y = te^{2t}$.

(2 pts) 4. Find the solution to the following IVP, expressing the answer in terms of a convolution:

$y'' + 2y' + 2y = g(t),\ y(0) = 0,\ y'(0) = 0.$

(2 pts) 5. Solve: $(2x + y)\ dx + (x + y - 2)\ dy = 0$.

(2 pts) 6. Classify (separable, etc.) the following equation:

$$y' = \frac{2x + 3y}{5x - 2y}.$$

(2 pts) 7. If y_1 and y_2 are solutions to the equation $ay'' + by' + cy = 0$, then what is the name of the the function $y_1 y_2' - y_1' y_2$?

(2 pts) 8. Determine the inverse transform of $F(s) = \frac{e^{-2s}(s^2 - 5s + 13)}{s^3 - 6s^2 + 13s}$.

(2 pts) 9. Determine the Laplace Transform of $f(t) = te^t u(t-2)$.

(2 pts) 10. Solve: $x^2y'' - xy' + 4y = 0, \; x > 0$.

(2 pts) 11. Given

$$y'' + e^x y' + y = 0, \; y(0) = -2, \; y'(0) = 2, \; e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

find the value of the coefficient a_3 in a series expansion about $x_0 = 0$ for y.

(2 pts) 12. Given the equation $(x^2 - 9)^2 y'' + (x-3)y' + 5y = 0$ and using the nomenclature from Section 8.6, categorize the point $x_0 = 3$.

(2 pts) 13. Give the general solution to $y'' + 3y' - y = 0$.

(2 pts) 14. Given $3x^2y'' + xy = 0$, use the Method of Frobenius to find a recurrence relation for a series solution about $x_0 = 0$ for y.

(2 pts) 15. If a particular solution to the equation $y'' - 4y' + 13y = t^2e^{2t}\cos 3t + (t-4)e^{2t}\cos 3t$ is to be found using the Method of Undetermined Coefficients, give the correct form of y_p. (Do not solve for the value of the coefficients.)

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Solutions:

1. $y' + (1/x)y = 1,\ y(2) = 1.$

The equation is first order linear in standard form. The integrating factor is $\mu(x) = e^{\int (1/x)dx} = x$; thus

$$\begin{aligned}(yx)' &= x \\ yx &= (1/2)x^2 + C \\ y &= (1/2)x + C/x \\ 1 &= (1/2)(2) + C \\ C &= 0\end{aligned}$$

an the solution is therefore $y = (1/2)x$.

2. For the ODE, the auxiliary equation is $r^2 + r - 6 = (r+3)(r-2) = 0$ so that the roots are 2 and -3; thus we choose $y_1 = e^{2t}$ and $y_2 = e^{-3t}$. Thus the system of equations is given by:

$$v_1'e^{2t} + v_2'e^{-3t} = 0$$

$$2v_1'e^{2t} - 3v_2'e^{-3t} = \tan t.$$

3. For the ODE, the auxiliary equation is $r^2 + 2r + 1 = (r+1)^2 = 0$ so that the roots are -1 and -1; from the right hand side of the equation we determine that $m = 1$ and $r = 2$ so that $s = 0$. Therefore a particular solution has the form :

$$y_p = (At + B)e^{2t};\ y_p' = (2At + 2B + A)e^{2t};\ y_p'' = (4At + 4B + 4A)e^{2t}.$$

inserting these into the ODE gives

$$(4At + 4B + 4A)e^{2t} + 2(2At + 2B + A)e^{2t} + (At + B)e^{2t} = te^{2t}$$

$$(4At + 4B + 4A) + 2(2At + 2B + A) + (At + B) = t$$

$$(4At + 4B + 4A) + (4At + 4B + 2A) + (At + B) = t$$

$$9At + 9B + 6A = t;$$

this yields the systems of equations

$$9A = 1 \qquad 6A + 9B = 0$$

so that $A = 1/9$ and $B = -2/27$ and a particular solution is given by $y_p = (1/9)(t - 2/3)e^{2t}$.

4. Taking the Laplace Transform of both sides of the ODE gives

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2Y(s) = G(s)$$

$$(s^2 + 2s + 2)Y(s) = G(s)$$

$$Y(s) = G(s)\frac{1}{s^2 + 2s + 2}$$

$$Y(s) = G(s)\frac{1}{(s+1)^2 + 1}$$

$$y(t) = g(t) * e^{-t}\sin t.$$

5. Since

$$\frac{\partial}{\partial y}(2x + y) = 1 = \frac{\partial}{\partial x}(x + y - 2),$$

the equation is exact so that

$$F(x, y) = \int (2x + y)\ dx + g(y) = x^2 + xy + g(y)$$

$$\frac{\partial}{\partial y}F(x, y) = x + g'(y) = x + y - 2$$

$$g'(y) = y - 2 \quad \rightarrow \quad g(y) = 0.5y^2 - 2y$$

$$F(x, y) = x^2 + xy + 0.5y^2 - 2y$$

and the solution to the ODE is given implicitly by

$$x^2 + xy + 0.5y^2 - 2y = C.$$

6. Since the equation may be rewritten in the form

$$y' = \frac{2 + 3(y/x)}{5 - 2(y/x)}$$

it is clearly homogeneous. A quick calculation reveals the equation is not exact; in addition, based on the form of the equation it is not separable, first order linear, Bernoulli, or of the form $y' = g(ax+by)$.

7. The function $W[y_1, y_2] = y_1 y_2' - y_1' y_2$ is known as the Wronskian.

8. Using partial fraction expansion we write

$$F(s) = \frac{e^{-2s}(s^2 - 5s + 13)}{s^3 - 6s^2 + 13s} = \frac{e^{-2s}(s^2 - 5s + 13)}{s(s^2 - 6s + 13)} = e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2 - 6s + 13}\right) = e^{-2s}\left(\frac{1}{s} - \frac{1}{(s-3)^2 + 2^2}\right)$$

so that

$$f(t) = u(t-2)[1 - (0.5)e^{3(t-2)} \sin 2(t-2)].$$

9.

$$L\{te^t u(t-2)\} = e^{-2s} L\{(t+2)e^{t+2}\} = e^{-2s} L\{(t+2)e^2 e^t\} = e^{-2s} e^2 L\{(t+2)e^t\}$$

$$= e^{-2s} e^2 L\{te^t + 2e^t\} = e^{2(1-s)}\left(\frac{1}{(s-1)^2} + \frac{2}{s-1}\right) = e^{2(1-s)}\left(\frac{2s-1}{(s-1)^2}\right).$$

10. The equation $x^2y'' - xy' + 4y = 0,\ x > 0$, is a Cauchy-Euler equation with an indicial equation

$$r^2 - 2r + 4 = 0$$

and thus roots $r = 1 \pm \sqrt{3}i$. Therefore the solution is

$$y = c_1 x \cos(\sqrt{3}\ln x) + c_2 x \sin(\sqrt{3}\ln x).$$

11. From the given information we write:

$$y = -2 + 2x + a_2x^2 + a_3x^3 + \cdots$$

$$y' = 2 + 2a_2x + 3a_3x^2 + \cdots$$

$$y'' = 2a_2 + 6a_3x + \cdots$$

$$e^x = 1 + x + x^2/2 + x^3/6 + \cdots$$

$$e^x y' = 2 + (2a_2 + 2)x + (3a_3 + 2a_2 + 1)x^2 + \cdots;$$

Substituting into the differential equation gives:

$$(2a_2 + 6a_3x + \cdots) + (2 + (2a_2 + 2)x + (3a_3 + 2a_2 + 1)x^2 + \cdots) + (-2 + 2x + a_2x^2 + a_3x^3 + \cdots) = 0.$$

Equating the coefficients of like powers of x yields:

$$x^0 : \quad 2a_2 + 2 - 2 = 2a_2 = 0 \quad \rightarrow \quad a_2 = 0$$

$$x^1 : \quad 6a_3 + (2a_2 + 2) + 2 = 6a_3 + 4 = 0 \quad \rightarrow \quad a_3 = -2/3.$$

Thus $a_3 = -2/3$.

12. Rewriting the equation in standard form gives

$$y'' + \left(\frac{1}{(x-3)(x+3)^2}\right)y' + \left(\frac{5}{(x-3)^2(x+3)^2}\right)y = 0$$

so that

$$\lim_{x\to 3}(x-3)\left[\frac{1}{(x-3)(x+3)^2}\right] = \frac{1}{36} \qquad \lim_{x\to 3}(x-3)^2\left[\frac{5}{(x-3)^2(x+3)^2}\right] = \frac{5}{36}$$

so the point $x = 3$ is regular singular.

13. The characteristic equation for the ODE is

$$r^2 + 3r - 1 = 0$$

with roots $r = (1/2)(-3 \pm \sqrt{13})$ so that the solution is

$$y = c_1 e^{(1/2)(-3+\sqrt{13})t} + c_2 e^{(1/2)(-3-\sqrt{13})t}.$$

14. The equation written in standard form is

$$y'' + 0 \cdot y' + \left(\frac{1}{3x}\right)y = 0$$

so that

$$\lim_{x\to 0} x[0] = 0 = p_0 \qquad \lim_{x\to 0} x^2\left[\frac{1}{3x}\right] = 0 = q_0$$

and the indicial equation is given by

$$r(r-1) + 0 \cdot r + 0 = r(r-1) = 0$$

with roots $r = 0,\ 1$. Choosing the larger of the two roots we write

$$y = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y' = \sum_{n=0}^{\infty} a_n(n+1)x^n$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+1)(n)x^{n-1}$$

and substituting into the ODE we find

$$3x^2\sum_{n=0}^{\infty} a_n(n+1)(n)x^{n-1} + x\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} 3a_n(n+1)(n)x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=1}^{\infty} 3a_n(n+1)(n)x^{n+1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+1} = 0$$

$$\sum_{n=1}^{\infty}[3a_n(n+1)(n) + a_{n-1}]x^{n+1} = 0.$$

Thus

$$3a_n(n+1)(n) + a_{n-1} = 0$$

so that the recurrence relation is

$$a_n = \frac{-a_{n-1}}{3n(n+1)},\ \ n \geq 1.$$

15. The characteristic equation for the ODE is

$$r^2 - 4r + 13 = 0$$

with roots $r = 2 \pm 3i$ so that $s = 1$; therefore a particular solution has the form

$$y_p = t(At^2 + Bt + C)e^{2t} \cos 3t + t(Dt^2 + Et + F)e^{2t} \sin 3t.$$