Lab 11: Practice Final

1. Evaluate $\int_0^1 \ y(y^2+1)^5 \ dy$.

2. Find the absolute max and min of $f(x) = (x^2+2x)^3$ on the interval $[-3, 3]$.

3. Evaluate the limit: $\lim_{x \to 0^+} \ x^2 \ln x$.

4. Find the shortest distance between the line $y = 3x + 2$ and the point (1,1).

5. Find the derivative of $y = 8e^{\sin\theta}$.

6. Find the equation of the tangent line to the function $f(x) = \frac{x^2-1}{x^2+1}$ at the point (0,-1).

7. Determine y' given $xy^4 + x^2y = x + 3y$.

8. Find the two integers x and y which will maximize the product xy given that $x + 4y = 1000$.

9. Evaluate $\int_0^\pi \ 8x^2 - \tan x \sec x \ dx$.

10. The volume of a cylinder is given by $V = \pi r^2 h$ and is increasing at a constant rate of 10 cm^3/sec. Find $\frac{dh}{dt}$ if $\frac{dr}{dt} = 1 \ cm/sec$ and $r = 3 \ cm$, $h = 2 \ cm$.

11. Sketch the graph of the function $f(x) = \frac{x^2}{x+8}$.

Solutions:

1. To find an antiderivative we will use u substitution. Letting $u = y^2 + 1$ we have $du = 2ydy$ or $(1/2)du = ydy$. Thus

$$\int y(y^2+1)^5\, dy = \int (y^2+1)^5\, ydy = \int u^5\,(1/2)du = (1/2)\int u^5\, du = (1/2)(u^6/6) = u^6/12 = (y^2+1)^6/12$$

and we evaluate the definite integral as

$$\int_0^1 y(y^2+1)^5\, dy = (y^2+1)^6/12|_0^1 = ((1)^2+1)^6/12-((0)^2+1)^6/12 = 2^6/12-1/12 = (64-1)/12 = 63/12 = 21/4.$$

2. Taking the derivative we find

$$f'(x) = 3(x^2+2x)^2(2x+2) = 6(x+1)(x^2+2x)^2 = 6x^2(x+1)(x+2)^2$$

so that the critical points occur at $x = 0, -1, -2$. Evaluating the function at the critical points as well as the endpoints of the interval gives

$$f(-3) = 27, \quad f(-2) = 0, \quad f(-1) = -1, \quad f(0) = 0, \quad f(3) = 3375.$$

Thus the absolute max is 3375 at $x = 3$ and the absolute min is -1 at $x = -1$.

3. The limit $\lim_{x\to 0^+} x^2 \ln x$ is of the indeterminate form $0 \cdot \infty$. After rearranging we obtain

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2}.$$

Applying L' Hospital's rule gives

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} -x^2/2 = 0.$$

4. The distance between the point and the line is given by

$$D = \sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-1)^2 + ((3x+2)-1)^2} = \sqrt{(x-1)^2 + (3x+1)^2} = \sqrt{10x^2+4x+2}.$$

If we minimize the quantity D^2, we also minimize D. Hence

$$D^2(x) = 10x^2 + 4x + 2 \qquad \frac{d}{dx}D^2(x) = 20x + 4.$$

Thus the only critical point for D^2 is $x = -1/5$ and the domain of the function is $x \in (-\infty, +\infty)$. We note that $D^2(-1/5) = 1.6$ and that $\lim_{x\to\pm\infty} D^2(x) = +\infty$. Hence the function D^2 and thus the function D has a minimal value when $x = -1/5$; finally we calculate $D(-1/5) = \sqrt{1.6} \approx 1.265$.

5. Using the chain rule,

$$y' = 8e^{\sin\theta}(\sin\theta)' = 8e^{\sin\theta}\cos\theta = 8\cos\theta e^{\sin\theta}.$$

6. We begin by finding the derivative of $f(x)$:

$$f'(x) = \frac{(x^2-1)'(x^2+1) - (x^2-1)(x^2+1)'}{(x^2+1)^2} = \frac{(2x)(x^2+1) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Thus $f'(0) = 0$ and the equation of the tangent line at (0,-1) has the form

$$y = f'(0)x + b = (0)x + b = b;$$

we note, therefore, that $b = -1$ and the tangent line is horizontal and is given by $y = -1$.

7. Taking the derivative of both sides of the equation gives

$$\frac{d}{dx}[xy^4 + x^2y] = \frac{d}{dx}[x + 3y]$$
$$(x)'(y^4) + (x)(y^4)' + (x^2)'(y) + (x^2)(y)' = 1 + 3y'$$
$$y^4 + 4xy^3y' + 2xy + x^2y' = 1 + 3y'$$
$$4xy^3y' + x^2y' - 3y' = 1 - y^4 - 2xy$$
$$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy$$
$$y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}.$$

8. Since $x+4y = 1000$ we solve for x and obtain $x = 1000-4y$. Thus we may write the product as $xy = y(1000-4y) = 1000y - 4y^2 = g(y)$. Taking the derivative we find $g'(y) = 1000-8y$ so that a critical point for the function is $y = 125$. Since $g''(y) = -8$, the function is concave downward for all values of y and hence a local max occurs at $y = 125$. For $y = 125$ we find $x = 1000 - 4(125) = 500$. Thus the max value of the product xy is given by $500 \cdot 125 = 62,500$.

9. $\int_0^\pi \ 8x^2 - \tan x \sec x \ dx = (8x^3/3 - \sec x)|_0^\pi = (8(\pi)^3/3 - \sec\pi) - (8(0)^3/3 - \sec 0)$

$= (8(\pi)^3/3 - (-1)) - (0-1) = 8(\pi)^3/3 + 2.$

10. For the equation $V = \pi r^2 h$, we assume that V, r, and h are functions of t and we take derivatives with respect to t to obtain

$$V' = 2\pi r r' h + \pi r^2 h'.$$

Substituting known values into this equation gives (we have suppressed units for clarity)

$$10 = 2\pi(3)(1)(2) + \pi(3)^2 h'$$

$$10 = 12\pi + 9\pi h'$$

so that $h' = (10 - 12\pi)/(9\pi) \ cm/sec.$

11. We begin by taking derivatives and find

$$f'(x) = \frac{x(x+16)}{(x+8)^2}, \qquad f''(x) = \frac{128}{(x+8)^3}.$$

From the form of the original equation we deduce that $x = -8$ is a vertical asymptote and that $\lim_{x\to -8^-} f(x) = -\infty$ and $\lim_{x\to -8^+} f(x) = +\infty$. We also deduce that there are no horizontal asymptotes since $\lim_{x\to\pm\infty} f(x) = \pm\infty$.

Based on the second derivative, there exists no inflection points for the graph , $f''(x) < 0$ (the curve is concave downward) for $x \in (-\infty, -8)$, and $f''(x) > 0$ (the curve is concave upward) for $x \in (-8, \infty)$.

Based on the first derivative we see there exists two critical points, $x = 0, -16$. Since the curve is concave downward at $x = -16$ this point corresponds to a local max of value -32; since the curve is concave upward at $x = 0$ this point corresponds to a local min of value 0. The function is increasing ($f'(x) > 0$) on $(-\infty, -16) \cup (0, +\infty)$ and decreasing ($f'(x) < 0$) on $(-16, 0)$.