MAC2311 Lab 8: Curve Sketching and Optimization

For each of the functions below find: intervals upon which the function is increasing and decreasing; intervals upon which the function is concave upward and downward; local max and mins; inflection points; and vertical and horizontal asymptotes.

1. $f(x) = x^4 - 1$.

2. $f(x) = \frac{x^2}{x^2-9}$.

3. $f(x) = \cos^2 x + 2\sin x,\ x \in [0, 2\pi]$.

4. $f(x) = 8\ln x - x^2$.

Solve the following optimization problems.

5. A rectangular rain gutter is to be constructed from a metal sheet of width 30 cm by bending the metal at a distance of x cm from each side. How should x be chosen so that the gutter will carry the maximum amount of water?

6. If a resistor of R ohms is connected across a battery of E volts with an internal resistance of r ohms, then the power in Watts in the external resistor is

$$P = \frac{E^2 R}{(R+r)^2}.$$

If E and r are fixed but R varies, what is the maximum value of the power.

7. A cylindrical can without a top is constructed to hold a volume of V cm^3; find the dimensions of the can which will minimize the surface area of the can.

Solutions:

1. We begin by taking derivatives:

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2.$$

The function $f(x)$ is continuous on the interval $(-\infty, +\infty)$ and so has no vertical asymptotes and since $\lim_{x\to\pm\infty} f(x) = +\infty$, the function has no horizontal asymptotes.

From the second derivative we deduce that the only possible inflection point is at $x = 0$; by evaluating $f''(x)$ at 'test points' we find that $f''(x) > 0$ (and hence the curve is concave upward) on $(-\infty, 0) \cup (0, \infty)$. Thus the curve is concave upward for all x and $x = 0$ is not an inflection point.

From the first derivative we conclude that the only possible critical point is $x = 0$; since the curve is concave upward for all x, this point is a local min with value -1. By evaluating $f'(x)$ at 'test points' we find that $f'(x) > 0$ (and hence the function is increasing) on $(0, +\infty)$ and $f'(x) < 0$ (and hence the function is decreasing) on $(-\infty, 0)$.

2. We begin by taking derivatives:

$$f'(x) = \frac{-18x}{(x^2-9)^2}$$

$$f''(x) = \frac{54(x^2+3)}{(x^2-9)^3}.$$

It is clear that the function has vertical asymptotes at $x = \pm 3$, and since $\lim_{x\to\pm\infty} f(x) = 1$, the function has a horizontal asymptote at $y = 1$.

From the second derivative we deduce that there are no possible inflection points, however, we check the sign of the second derivative on either side of the vertical asymptotes. By evaluating $f''(x)$ at 'test points' we find that $f''(x) > 0$ (and hence the curve is concave upward) on $(-\infty, -3) \cup (3, +\infty)$ and $f''(x) < 0$ (and hence the curve is concave downward) on $(-3, 3)$.

From the first derivative we conclude that critical points are $x = 0, \pm 3$; since the function is continuous and concave downward at $x = 0$, this point is a local max of value 0 and as previously stated, $x = \pm 3$ are vertical asymptotes. By evaluating $f'(x)$ at 'test points' we find that $f'(x) > 0$ (and hence the function is increasing) on $(-\infty, -3) \cup (-3, 0)$ and $f'(x) < 0$ (and hence the function is decreasing) on $(0, 3) \cup (3, +\infty)$.

3. We begin by taking derivatives:

$$f'(x) = -2\cos x \sin x + 2\cos x = 2\cos x(1 - \sin x)$$

$$f''(x) = -2\sin x + 2\sin^2 x - 2\cos^2 x = 4\sin^2 x - 2\sin x - 2 = 2(2\sin x + 1)(\sin x - 1).$$

The function $f(x)$ is continuous on the interval $[0, 2\pi]$ and so has no vertical asymptotes; since the function is defined on a finite interval, there are no horizontal asymptotes.

From the second derivative we deduce that on the interval $[0, 2\pi]$, $\sin x = 1$ when $x = \pi/2$ and $\sin x = -1/2$ when $x = 7\pi/6$ and $11\pi/6$; hence the possible inflection points are $x = \pi/2,\ 7\pi/6,\ 11\pi/6$. By evaluating $f''(x)$ at 'test points' we find that $f''(x) > 0$ (and hence the curve is concave upward) on $(7\pi/6, 11\pi/6)$ and $f''(x) < 0$ (and hence the curve is concave downward) on $(0, \pi/2) \cup (\pi/2, 7\pi/6) \cup (11\pi/6, 2\pi)$. Thus the inflection points are $x = \ 7\pi/6,\ 11\pi/6$.

From the first derivative we conclude that on the interval $[0, 2\pi]$, $\sin x = 1$ when $x = \pi/2$ and $\cos x = 0$ when $x = \pi/2$ and $3\pi/2$; hence critical points are $x = \pi/2,\ 3\pi/2$; since the function is continuous and concave upward at $x = 3\pi/2$, this point is a local min of value -2; since the function is continuous and concave downward on either side of $x = \pi/2$, this point is a local max of value 2. By evaluating $f'(x)$ at 'test points' we find that $f'(x) > 0$ (and hence the function is increasing) on $(0, \pi/2) \cup (3\pi/2, 2\pi)$ and $f'(x) < 0$ (and hence the function is decreasing) on $(\pi/2, 3\pi/2)$.

4. We begin noting that the domain of the function is given by $D = \{x \mid x > 0\}$ and taking derivatives find:

$$f'(x) = 8/x - 2x = 2(4 - x^2)/x$$

$$f''(x) = -8/x^2 - 2 = -2(x^2 + 4)/x^2.$$

The function $f(x)$ is continuous on the interval $(0, +\infty)$ and $\lim_{x \to 0^+} f(x) = -\infty$ so that a vertical asymptote exists at $x = 0$; since $\lim_{x \to +\infty} f(x) = -\infty$, the function has no horizontal asymptotes.

From the second derivative we deduce that there are no possible inflection points. By evaluating $f''(x)$ at 'test points' we find that $f''(x) < 0$ (and hence the curve is concave downward) on $(0, +\infty)$. Thus the curve is concave downward for all x in the domain of the function.

From the first derivative we conclude that the only critical point is $x = 2$; since the curve is concave downward for all x, this point is a local max of value $(8\ln 2 - 4)$. By evaluating $f'(x)$ at 'test points' we find that $f'(x) > 0$ (and hence the function is increasing) on $(0, 2)$ and $f'(x) < 0$ (and hence the function is decreasing) on $(2, +\infty)$.

5. The sides of the rectangular gutter will each be x cm and the base will be $(30 - 2x)$ cm; hence the area formed by a cross-section will be $x(30 - 2x)$ cm^2. The volume that the gutter holds will be maximized when the area of cross-section is maximized. Thus we consider the function

$$A(x) = x(30 - 2x) = 30x - 2x^2, \ 0 \leq x \leq 15.$$

We find $A'(x) = 30 - 4x$; this function is equal to zero when $x = 7.5$; evaluating $A(x)$ at this critical point as well as the endpoints of the above interval gives

$$A(0) = 0 \qquad A(7.5) = 112.5 \qquad A(15) = 0.$$

Hence the area of the cross-section and thus the volume of the gutter is maximized when $x = 7.5$ cm.

6. We begin by noting that the possible values of R lie in the interval $[0, +\infty)$. Taking the derivative of P with respect to R we find $\frac{dP}{dR} = \frac{E(r-R)}{(R+r)^3}$. There exists only one critical point for P' in the given domain of R, namely the point $R = r$. We note that for $R = 0$, $P = 0$; for $R = r$, $P = E^2/4R$; and $\lim_{R \to +\infty} P(R) = 0$. Hence the maximum power is $(E^2/4R)$ Watts.

7. The surface area of the can is given by $SA = \pi r^2 + 2\pi rh$ and the volume of the can is given by $V = \pi r^2 h$. Solving the second equation for h yields $h = V/(\pi r^2)$. Thus we can express surface area solely as a function of r: $SA = \pi r^2 + 2V/r$. We note that the possible values of r lie in the interval $(0, +\infty)$ and taking the derivative of surface area with respect to r yields $(SA)' = 2\pi r - 2V/r^2 = 2(\pi r^3 - V)/r^2$. The only possible critical point in the domain of r is $r = (V/\pi)^{1/3}$. We compute $\lim_{r \to 0^+} (\pi r^2 + 2V/r) = +\infty$, the surface area when $r = (V/\pi)^{1/3}$ is $(\pi(V/\pi)^{2/3} + 2V(\pi/V)^{1/3}) = 3(V^2\pi)^{1/3}$, and $\lim_{r \to +\infty} (\pi r^2 + 2V/r) = +\infty$. Hence the surface area is minimized when $r = (V/\pi)^{1/3}$ cm and $h = V/(\pi r^2) = (V/\pi)^{1/3}$ cm.