MAC2311 Lab 2: The Definition of the Derivative

Find the first derivatives of the given functions.

1. Use the definition of the derivative to calculate $f'(x)$ given $f(x) = 2x^2 + 3x$; find the equation of the tangent line to the graph of the above function at $x = 3$.

2. Use the definition of the derivative to calculate $f'(x)$ given $f(x) = 3x^2 - 9x + 2$; at which values of x is the tangent line horizontal?

3. Use the definition of the derivative to calculate $f'(x)$ given $f(x) = 1/(x + 1)$; determine the domain of $f(x)$ and $f'(x)$.

4. Use the definition of the derivative to calculate $f'(x)$ given $f(x) = 1/\sqrt{x}$; determine the domain of $f(x)$ and $f'(x)$.

5. Use the definition of the derivative to calculate $f'(x)$ given $f(x) = x^2 - 1$; for which values of x is $f(x)$ increasing; for which values of x is $f(x)$ decreasing.

6. Is it possible to have two functions $f(x)$ and $g(x)$ such that $f(x) \neq g(x)$ and $f'(x) = g'(x)$ for all x? (Hint, consider $f(x) = 2x + 3,\ g(x) = 2x + 5$.)

7. Let $f(x) = |x|$; is $f(x)$ differentiable? If so calculate $f'(x)$ for all points at which the derivative exists. (Hint, write $f(x)$ as a piecewise function.)

8. Use the definition of the derivative to calculate $g'(t)$ given $g(t) = 3t/(t + 1)$; determine the domain of $g(t)$ and $g'(t)$; find the equation of the tangent line to the graph of the above function at $t = 0$; find values of t at which the tangent line is horizontal; find the values of t where is $g(t)$ increasing; find the values of t where is $g(t)$ decreasing.

1. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[2(x+h)^2+3(x+h)]-[2x^2+3x]}{h} = \lim_{h\to 0} \frac{[2x^2+4xh+2h^2+3x+3h]-[2x^2+3x]}{h}$

$= \lim_{h\to 0} \frac{4xh+2h^2+3h}{h} = \lim_{h\to 0} \frac{h(4x+3+2h)}{h} = \lim_{h\to 0} (4x+3+2h) = 4x+3.$

The equation of the tangent line in slope-intercept form at $x = 3$ is given by

$y = f'(3)x + b = 15x + b$. Since a point on the line is given by $(3, f(3)) = (3, 27)$, we find

$27 = 15(3) + b$ or $b = -18$; hence the tangent line is given by $y = 15x - 18$.

2. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)^2-9(x+h)+2]-[3x^2-9x+2]}{h} = \lim_{h\to 0} \frac{[3x^2+6xh+3h^2-9x-9h+2]-[3x^2-9x+2]}{h}$

$= \lim_{h\to 0} \frac{6xh+3h^2-9h}{h} = \lim_{h\to 0} \frac{h(6x-9+3h)}{h} = \lim_{h\to 0} (6x-9+3h) = 6x-9.$

The tangent line is horizontal when $f'(x) = 0$. Solving the equation $0 = 6x - 9$ gives $x = 3/2$.

3. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[1/((x+h)+1)]-[1/(x+1)]}{h} = \lim_{h\to 0} \frac{[(x+1)-(x+1+h)]/[(x+1)(x+1+h)]}{h}$

$= \lim_{h\to 0} \frac{[-h]/[(x+1)(x+1+h)]}{h} = \lim_{h\to 0} \frac{[-h]/[(x+1)(x+1+h)]}{h/1} = \lim_{h\to 0} \frac{-h}{h(x+1)(x+1+h)} = \lim_{h\to 0} \frac{-1}{(x+1)(x+1+h)}$

$= -1/(x+1)^2.$

The functions $f(x)$ and $f'(x)$ are defined for all x not equal to negative one so that we write

$D = \{x \mid x \neq -1\}$ for both.

4. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(1/\sqrt{x+h})-(1/\sqrt{x})}{h} = \lim_{h\to 0} \frac{[\ \sqrt{x}-\sqrt{x+h}\ ]/[\ \sqrt{x}\sqrt{x+h}\ ]}{h}$

$= \lim_{h\to 0} \frac{[\ \sqrt{x}-\sqrt{x+h}\ ]/[\ \sqrt{x}\sqrt{x+h}\ ]}{h/1} = \lim_{h\to 0} \frac{\sqrt{x}\ -\ \sqrt{x+h}}{h(\sqrt{x}\sqrt{x+h})} = \lim_{h\to 0} \frac{(\sqrt{x}\ -\ \sqrt{x+h})}{h(\sqrt{x}\sqrt{x+h})} \frac{(\sqrt{x}\ +\ \sqrt{x+h})}{(\sqrt{x}\ +\ \sqrt{x+h})}$

$= \lim_{h\to 0} \frac{x-(x+h)}{h(\sqrt{x}\sqrt{x+h})(\sqrt{x}\ +\ \sqrt{x+h})} = \lim_{h\to 0} \frac{-h}{h(\sqrt{x}\sqrt{x+h})(\sqrt{x}\ +\ \sqrt{x+h})} = \lim_{h\to 0} \frac{-1}{(\sqrt{x}\sqrt{x+h})(\sqrt{x}\ +\ \sqrt{x+h})}$

$= \frac{-1}{2(\sqrt{x})^3} = (-1/2)x^{-3/2}.$

The domains of $f(x)$ and $f'(x)$ are the same and are given by $D = \{x \mid x > 0\}$.

5. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2-1]-[x^2-1]}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-1-x^2+1}{h}$

$= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x.$

The function is increasing when $f'(x) > 0$ or $2x > 0$ which yields $x > 0$; the function is decreasing when $f'(x) < 0$ or $2x < 0$ which yields $x < 0$.

6. Consider the two functions given in the hint and suppose there exists a value of x at which $f(x) = g(x)$; this would imply that for this x, $2x+3 = 2x+5$ or after simplification, $3 = 5$, which is clearly not possible; hence $f(x) \neq g(x)$ for any x. (Alternatively one could note that the graphs of the two functions are parallel lines and can never intersect which also implies $f(x) \neq g(x)$ for any x.)

Using the definition of the derivative one can directly verify that $f'(x) = 2 = g'(x)$ so that the derivatives of the two functions are the same for all x. (One can also determine $f'(x) = 2 = g'(x)$ since the graph of both functions are straight lines of slope two; hence the slope of the tangent lines and therefore the derivative must also equal two.)

7. The function $f(x) = |x|$ can be written piecewise as $f(x) = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}$
From this representation, it is clear that $f(x)$ can be written as a differentiable function for both $x < 0$ and $x > 0$. Direct calculation yields that $f'(x) = -1,\ x < 0$, and $f'(x) = 1,\ x > 0$; the function is not differentiable at the point $x = 0$ since the function is not "smooth" at this point.

8. $g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \frac{[3(t+h)/((t+h)+1)]-[3t/(t+1)]}{h} = \lim_{h\to 0} [\frac{3(t+h)}{((t+h)+1)} - \frac{3t}{(t+1)}]\ \frac{1}{h}$

$= \lim_{h\to 0} [\frac{(t+1)3(t+h)}{(t+1)((t+h)+1)} - \frac{((t+h)+1)3t}{((t+h)+1)(t+1)}]\ \frac{1}{h} = \lim_{h\to 0} [\frac{(t+1)3(t+h)-((t+h)+1)3t}{((t+h)+1)(t+1)}]\ \frac{1}{h}$

$= \lim_{h\to 0} [\frac{3t^2+3th+3t+3h-3t^2-3ht-3t}{((t+h)+1)(t+1)}]\ \frac{1}{h} = \lim_{h\to 0} [\frac{3h}{((t+h)+1)(t+1)}]\ \frac{1}{h} = \lim_{h\to 0} \frac{3}{((t+h)+1)(t+1)} = \frac{3}{(t+1)^2}.$

Both the function $g(t)$ and $g'(t)$ are defined at all points where $t \neq -1$ so the domains are given by $D = \{t \mid t \neq -1\}$.

The equation of the tangent line in slope-intercept form at $t = 0$ is given by

$y = g'(0)x + b = 3x + b$. Since a point on the line is given by $(0, g(0)) = (0, 0)$, we find

$0 = 3(0) + b$ or $b = 0$; hence the tangent line is given by $y = 3x$.

The tangent line is horizontal when $g'(t) = \frac{3}{(t+1)^2} = 0$; since this equation can not be satisfied for any real values of t, there does not exist a horizontal tangent line.

The function is increasing when $g'(t) > 0$; since the numerator in the derivative is always positive and the denominator is positive for $t \neq 0$, the function is increasing as long as $t \neq 0$; since $g'(t)$ is never negative, the function is never decreasing.