Section 6.1

1. $F(x)$ is an anti derivative of $f(x)$ if $F'(x) = f(x)$. Taking the derivative we find

$$F'(x) = x^2 + 4x - 1 = f(x)$$

so that $F(x)$ is an anti derivative of $f(x)$.

3. $F(x)$ is an anti derivative of $f(x)$ if $F'(x) = f(x)$. Taking the derivative we find

$$F'(x) = (1/2)(2x^2 - 1)^{-0.5}(4x) = \frac{2x}{\sqrt{2x^2 - 1}} = f(x)$$

so that $F(x)$ is an anti derivative of $f(x)$.

5. $G(x)$ is an anti derivative of $f(x)$ if $F'(x) = f(x)$. Taking the derivative we find

$$G'(x) = 2 = f(x)$$

so that $G(x)$ is an anti derivative of $f(x)$. The set of all anti derivatives of $f(x)$ is found by adding an arbitrary constant to $G(x)$. Thus any anti derivative of $f(x)$ is of the form $2x + C$.

7. $G(x)$ is an anti derivative of $f(x)$ if $F'(x) = f(x)$. Taking the derivative we find

$$G'(x) = x^2 = f(x)$$

so that $G(x)$ is an anti derivative of $f(x)$. The set of all anti derivatives of $f(x)$ is found by adding an arbitrary constant to $G(x)$. Thus any anti derivative of $f(x)$ is of the form $x^2 + C$.

9. $\int 6\ dx = 6x + C$. (Rule 1)

11. $\int x^3\ dx = \left(\frac{1}{1+3}\right)x^{1+3} + C = \frac{1}{4}x^4 + C$. (Rule 2)

13. $\int x^{-4}\ dx = \left(\frac{1}{1+(-4)}\right)x^{1+(-4)} + C = \frac{-1}{3}x^{-3} + C$. (Rule 2)

15. $\int x^{2/3}\ dx = \left(\frac{1}{1+(2/3)}\right)x^{1+(2/3)} + C = \frac{3}{5}x^{5/3} + C$. (Rule 2)

17. $\int x^{-5/4}\ dx = \left(\frac{1}{1+(-5/4)}\right)x^{1+(-5/4)} + C = -4x^{-1/4} + C$. (Rule 2)

19. $\int \frac{2}{x^2}\ dx = \int 2x^{-2}\ dx = 2\int x^{-2}\ dx = (2)\left(\frac{1}{1+(-2)}\right)x^{1+(-2)} + C = (2)(-1)x^{-1} + C = -2x^{-1} + C$. (Rules 2 and 3)

21. $\int \pi\sqrt{t}\ dt = \pi \int t^{1/2}\ dt = (\pi)\left(\frac{1}{1+(1/2)}\right)t^{1+(1/2)} + C = (\pi)\left(\frac{2}{3}\right)t^{3/2} + C = \left(\frac{2\pi}{3}\right)t^{3/2} + C$. (Rules 2 and 3)

23. $\int (3-2x)\ dx = \int 3\ dx - 2\int x\ dx = 3x - (2)\left(\frac{1}{1+1}\right)x^{1+1} = 3x - x^2 + C$. (Rules 1, 2, 3, and 4)

25. $\int (x^2 + x + x^{-3})\ dx = \int x^2\ dx + \int x\ dx + \int x^{-3}\ dx$

$= \left(\frac{1}{1+2}\right)x^{1+2} + \left(\frac{1}{1+1}\right)x^{1+1} + \left(\frac{1}{1+(-3)}\right)x^{1+(-3)} + C = \left(\frac{1}{3}\right)x^3 + \left(\frac{1}{2}\right)x^2 + \left(\frac{-1}{2}\right)x^{-2} + C$. (Rules 2 and 4)

27. $\int 4e^x\ dx = 4\int e^x\ dx = 4e^x + C$. (Rules 3 and 5)

29. $\int 1 + x + e^x\ dx = \int 1\ dx + \int x\ dx + \int e^x\ dx = x + \left(\frac{1}{1+1}\right)x^{1+1} + e^x + C$

$= x + \left(\frac{1}{2}\right)x^2 + e^x + C$. (Rules 1, 2, 4, and 5)

31. $\int 4x^3 - \frac{2}{x^2} - 1\ dx = 4\int x^3\ dx - 2\int x^{-2}\ dx - \int 1\ dx$

$= (4)\left(\frac{1}{1+3}\right)x^{1+3} - (2)\left(\frac{1}{1+(-2)}\right)x^{1+(-2)} - x + C = x^4 + 2x^{-1} - x + C$. (Rules 1, 2, 3, and 4)

33. $\int x^{5/2} + 2x^{3/2} - x\ dx = \int x^{5/2}\ dx + 2\int x^{3/2}\ dx - \int x\ dx$

$= \left(\frac{1}{1+(5/2)}\right)x^{1+(5/2)} + (2)\left(\frac{1}{1+(3/2)}\right)x^{1+(3/2)} - \left(\frac{1}{1+1}\right)x^{1+1} + C = (2/7)x^{7/2} + (4/5)x^{5/2} - (1/2)x^2 + C$.

(Rules 2, 3, and 4)

35. $\int \sqrt{x} + \frac{3}{\sqrt{x}}\ dx = \int x^{1/2}\ dx + 3\int x^{-1/2}\ dx = \left(\frac{1}{1+(1/2)}\right)x^{1+(1/2)} + (3)\left(\frac{1}{1+(-1/2)}\right)x^{1+(-1/2)} + C$

$= (2/3)x^{3/2} + 6x^{1/2} + C$. (Rules 2, 3, and 4)

37. $\int \frac{u^3+2u^2-u}{3u}\ du = \int \frac{u^3}{3u} + \frac{2u^2}{3u} - \frac{u}{3u}\ du = \int \frac{u^2}{3} + \frac{2u}{3} - \frac{1}{3}\ du = (1/3)\int u^2\ du + (2/3)\int u\ du - \int \frac{1}{3}\ du$

$= (1/3)\left(\frac{1}{1+2}\right)u^{1+2} + (2/3)\left(\frac{1}{1+1}\right)u^{1+1} - (1/3)u + C = (1/9)u^3 + (1/3)u^2 - (1/3)u + C$.

(Rules 1, 2, 3, and 4)

39. $\int\ (2t+1)(t-2)\ dt = \int\ (2t)(t-2)+(1)(t-2)\ dt = \int\ 2t^2-3t-2\ dt$

$= 2\int\ t^2\ dt - 3\int\ t\ dt - \int\ 2\ dt = (2)\left(\frac{1}{1+2}\right)t^{1+2} - (3)\left(\frac{1}{1+1}\right)t^{1+1} - 2t + C = (2/3)t^3 - (3/2)t^2 - 2t + C.$

(Rules 1, 2, 3, and 4)

41. $\int\ \frac{1}{x^2}(x^4-2x^2+1)\ dx = \int\ \frac{1}{x^2}(x^4) + \frac{1}{x^2}(-2x^2) + \frac{1}{x^2}(1)\ dx = \int\ x^2 - 2 + \frac{1}{x^2}\ dx$

$= \int\ x^2\ dx - \int\ 2\ dx + \int\ x^{-2}\ dx = \left(\frac{1}{1+2}\right)x^{1+2} - 2x + \left(\frac{1}{1+(-2)}\right)x^{1+(-2)} + C = (1/3)x^3 - 2x - x^{-1} + C.$

(Rules 1, 2, 3, and 4)

43. $\int\ \frac{1}{(s+1)^{-2}}\ ds = \int\ (s+1)^2\ ds = \int\ s^2+2s+1\ ds = \int\ s^2\ ds + 2\int\ s\ ds + \int\ 1\ ds$

$= \left(\frac{1}{1+2}\right)s^{1+2} + (2)\left(\frac{1}{1+1}\right)s^{1+1} + s + C = (1/3)s^3 + s^2 + s + C.$ (Rules 1, 2, 3, and 4)

45. $\int\ (e^t + t^e)\ dt = \int\ e^t\ dt + \int\ t^e\ dt = e^t + \left(\frac{1}{1+e}\right)t^{1+e} + C$ (Rules 2, 3, and 4)

47. $\int\ \frac{x^3+x^2-x+1}{x^2}\ dx = \int\ \frac{x^3}{x^2} + \frac{x^2}{x^2} + \frac{-x}{x^2} + \frac{1}{x^2}\ dx = \int\ x + 1 + \frac{-1}{x} + \frac{1}{x^2}\ dx$

$= \int\ x\ dx + \int\ 1\ dx - \int\ \frac{1}{x}\ dx + \int\ x^{-2}\ dx = \left(\frac{1}{1+1}\right)x^{1+1} + x - \ln|x| + \left(\frac{1}{1+(-2)}\right)x^{1+(-2)} + C$

$= (1/2)x^2 + x - \ln|x| - x^{-1} + C.$ (Rules 1, 2, 3, 4, 5, and 6)

49. $\int\ \frac{(\sqrt{x}-1)^2}{x^2}\ dx = \int\ \frac{x-2\sqrt{x}+1}{x^2}\ dx = \int\ \frac{x}{x^2} + \frac{-2\sqrt{x}}{x^2} + \frac{1}{x^2}\ dx = \int\ \frac{1}{x} + \frac{-2}{x^{3/2}} + \frac{1}{x^2}\ dx$

$= \int\ x^{-1}\ dx - 2\int\ x^{-3/2}\ dx + \int\ x^{-2}\ dx = \ln|x| - (2)\left(\frac{1}{1+(-3/2)}\right)x^{1+(-3/2)} + \left(\frac{1}{1+(-2)}\right)x^{1+(-2)} + C$

$= \ln|x| + 4x^{-1/2} - x^{-1} + C.$ (Rules 2, 3, 4, and 6)

51. $f(x) = \int\ 2x+1\ dx = (2)\left(\frac{1}{1+1}\right)x^{1+1} + x + C = x^2 + x + C.$

$f(1) = (1)^2 + (1) + C = 3 \quad \longrightarrow \quad 2 + C = 3 \quad \longrightarrow \quad C = 1.$

$f(x) = x^2 + x + 1.$

53. $f(x) = \int\ 3x^2 + 4x - 1\ dx = (3)\left(\frac{1}{1+2}\right)x^{1+2} + (4)\left(\frac{1}{1+1}\right)x^{1+1} - x + C = x^3 + 2x^2 - x + C.$

$f(2) = (2)^3 + 2(2)^2 - (2) + C = 9 \quad \longrightarrow \quad 14 + C = 9 \quad \longrightarrow \quad C = 5.$

$f(x) = x^3 + 2x^2 - x + 5.$

55. $f(x) = \int\ 1 + \frac{1}{x^2}\ dx = \int\ 1 + x^{-2}\ dx = x + \left(\frac{1}{1+(-2)}\right)x^{1+(-2)} + C = x - x^{-1} + C.$

$f(1) = (1) - (1)^{-1} + C = 2 \quad \longrightarrow \quad 0 + C = 2 \quad \longrightarrow \quad C = 2.$

$f(x) = x - x^{-1} + 2.$

57. $f(x) = \int\ \frac{x+1}{x}\ dx = \int\ 1 + x^{-1}\ dx = x + \ln|x| + C.$

$f(1) = (1)\ln(1) + C = 1 \quad \longrightarrow \quad 1 + C = 1 \quad \longrightarrow \quad C = 0.$

$f(x) = x + \ln|x|.$

59. $f(x) = \int\ \frac{1}{2}x^{-1/2}\ dx = (1/2)\int\ x^{-1/2}\ dx = (1/2)\left(\frac{1}{1+(-1/2)}\right)x^{1+(-1/2)} + C = x^{1/2} + C.$

$f(2) = (2)^{1/2} + C = \sqrt{2} \quad \longrightarrow \quad \sqrt{2} + C = \sqrt{2} \quad \longrightarrow \quad C = 0.$

$f(x) = x^{1/2}.$

61. $f(x) = \int\ e^x + x\ dx = e^x + \left(\frac{1}{1+1}\right)x^{1+1} + C = e^x + (1/2)x^2 + C.$

$f(0) = e^{(0)} + (1/2)(0)^2 + C = 3 \quad \longrightarrow \quad 1 + C = 3 \quad \longrightarrow \quad C = 2.$

$f(x) = e^x + (1/2)x^2 + 2.$

67. Since $C'(x) = 0.000009x^2 - 0.0009x + 8$, the cost function is given by

$$C(x) = \int\ 0.000009x^2 - 0.009x + 8\ dx = 0.000003x^3 - 0.0045x^2 + 8x + C.$$

Since the fixed cost is given by $C(0)$, we find

$$C(0) = 0.000003(0)^3 - 0.0045(0)^2 + 8(0) + C = C = \$120$$

and therefore the cost function is $C(x) = 0.000003x^3 - 0.0045x^2 + 8x + 120$ dollars. By direct calculation, we find $C(500) = \$3370$.

69. Since $P'(x) = -0.004x + 20$, the monthly profit function is given by

$$P(x) = \int -0.004x + 20 \, dx = -0.002x^2 + 20x + C.$$

Since the fixed cost is given by $-P(0)$, we find

$$P(0) = -0.002(0)^2 + 20(0) + C = C = \$ - 16,000$$

and therefore the monthly profit function is $P(x) = -0.002x^2 + 20x - 16,000$ dollars. The only critical point for the monthly profit function is found from the equation

$$P'(x) = -0.004x + 20 = 0$$

or $x = 5000$; since $P''(x) = -0.004 < 0$, the profit function is concave downward so that the critical point corresponds to an absolute maximum. Therefore the maximum monthly profit is $P(5000) = \$34,000$.

85. Since $R(t) = 3t^3 - 17.9445t^2 + 28.7222t + 26.632$ gives the rate of change of sales of organic milk, the actual sales is given by

$$S(t) = \int 3t^3 - 17.9445t^2 + 28.7222t + 26.632 \, dt = 0.75t^4 - 5.9815t^3 + 14.3611t^2 + 26.632t + C.$$

In 1999 (t = 0), the sales of milk was \$108 million. Therefore

$$S(0) = 0.75(0)^4 - 5.9815(0)^3 + 14.3611(0)^2 + 26.632(0) + C = C = 108$$

so that $S(t) = 0.75t^4 - 5.9815t^3 + 14.3611t^2 + 26.632t + 108$ million dollars. For the year 2004, $t = 5$, so $S(5) = 321.25$ million dollars.

Section 6.2

1.

$$\begin{aligned} u &= 4x + 3 \\ \frac{du}{dx} &= 4 \\ du &= 4\ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int 4(4x+3)^4\ dx = \int (4x+3)^4\ 4\ dx = \int (u)^4\ du.$$

Evaluating the new integral we find

$$\int u^4\ du = (1/5)u^5 + C.$$

Substituting $u = 4x + 3$ into the antiderivative yields

$$\int 4(4x+3)^4\ dx = (1/5)(4x+3)^5 + C.$$

3.

$$\begin{aligned} u &= x^3 - 2x \\ \frac{du}{dx} &= 3x^2 - 2 \\ du &= (3x^2 - 2)\ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int (x^3 - 2x)^2(3x^2 - 2)\ dx = \int u^2\ du.$$

Evaluating the new integral we find

$$\int u^2\ du = (1/3)u^3 + C.$$

Substituting $u = x^3 - 2x$ into the antiderivative yields

$$\int (x^3 - 2x)^2(3x^2 - 2)\ dx = (1/3)(x^3 - 2x)^3 + C.$$

5.

$$\begin{aligned} u &= 2x^2 + 3 \\ \frac{du}{dx} &= 4x \\ du &= 4x\ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int \frac{4x}{(2x^2+3)^3}\ dx = \int \frac{1}{(2x^2+3)^3}\ 4x\ dx = \int \frac{1}{u^3}\ du = \int u^{-3}\ du.$$

Evaluating the new integral we find

$$\int u^{-3}\ du = (-1/2)u^{-2} + C.$$

Substituting $u = 2x^2 + 3$ into the antiderivative yields

$$\int \frac{4x}{(2x^2+3)^3}\ dx = (-1/2)(2x^2+3)^{-2} + C.$$

7.

$$\begin{aligned} u &= t^3 + 2 \\ \frac{du}{dt} &= 3t^2 \\ du &= 3t^2\ dt. \end{aligned}$$

Substituting into the original integral gives

$$\int 3t^2\sqrt{t^3+2}\ dt = \int \sqrt{t^3+2}\ 3t^2\ dt = \int \sqrt{u}\ du = \int u^{1/2}\ du.$$

Evaluating the new integral we find

$$\int u^{1/2}\ du = (2/3)u^{3/2} + C.$$

Substituting $u = t^3 + 2$ into the antiderivative yields

$$\int\ 3t^2\sqrt{t^3+2}\ dt = (2/3)(t^3+2)^{3/2} + C.$$

9.

$$\begin{aligned} u &= x^2 - 1 \\ \frac{du}{dx} &= 2x \\ du &= 2x\ dx \\ (1/2)\ du &= x\ dx \end{aligned}$$

Substituting into the original integral gives

$$\int\ (x^2-1)^9 x\ dx = \int\ (x^2-1)^9\ x\ dx = \int\ u^9\ (1/2)\ du = (1/2)\int\ u^9\ du.$$

Evaluating the new integral we find

$$(1/2)\int u^9\ du = (1/2)[(1/10)u^{10} + C] = (1/20)u^{10} + C.$$

Substituting $u = x^2 - 1$ into the antiderivative yields

$$\int\ (x^2-1)^9 x\ dx = (1/20)(x^2-1)^{10} + C.$$

11.

$$\begin{aligned} u &= 1 - x^5 \\ \frac{du}{dx} &= -5x^4 \\ du &= -5x^4\ dx \\ (-1/5)du &= x^4\ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int \frac{x^4}{1-x^5}\,dx = \int \frac{1}{1-x^5}\,x^4\,dx = \int \frac{1}{u}\,(-1/5)\,du = -(1/5)\int u^{-1}\,du.$$

Evaluating the new integral we find

$$-(1/5)\int u^{-1}\,du = -(1/5)[\ln|u| + C] = -(1/5)\ln|u| + C.$$

Substituting $u = 1 - x^5$ into the antiderivative yields

$$\int \frac{x^4}{1-x^5}\,dx = -(1/5)\ln|1-x^5| + C.$$

13.

$$\begin{aligned} u &= x - 2 \\ \frac{du}{dx} &= 1 \\ du &= 1\,dx \\ du &= dx. \end{aligned}$$

Substituting into the original integral gives

$$\int \frac{2}{x-2}\,dx = \int \frac{2}{u}\,du = 2\int u^{-1}\,du.$$

Evaluating the new integral we find

$$2\int u^{-1}\,du = 2[\ln|u| + C] = 2\ln|u| + C.$$

Substituting $u = x - 2$ into the antiderivative yields

$$\int \frac{2}{x-2}\,dx = 2\ln|x-2| + C.$$

15.

$$u = 0.3x^2 - 0.4x + 2$$
$$\frac{du}{dx} = 0.6x - 0.4$$
$$du = (0.6x - 0.4)\ dx$$
$$du = 2(0.3x - 0.2)\ dx$$
$$(1/2)du = (0.3x - 0.2)\ dx$$

Substituting into the original integral gives

$$\int \frac{0.3x - 0.2}{0.3x^2 - 0.4x + 2}\ dx = \int \frac{1}{0.3x^2 - 0.4x + 2}\ (0.3x - 0.2)\ dx = \int \frac{1}{u}\ (1/2)\ du = (1/2)\int u^{-1}\ du.$$

Evaluating the new integral we find

$$(1/2)\int u^{-1}\ du = (1/2)[\ln|u| + C] = (1/2)\ln|u| + C.$$

Substituting $u = 0.3x^2 - 0.4x + 2$ into the antiderivative yields

$$\int \frac{0.3x - 0.2}{0.3x^2 - 0.4x + 2}\ dx = (1/2)\ln|0.3x^2 - 0.4x + 2| + C.$$

17.

$$u = 3x^2 - 1$$
$$\frac{du}{dx} = 6x$$
$$du = 6x\ dx$$
$$(1/6)du = x\ dx.$$

Substituting into the original integral gives

$$\int \frac{x}{3x^2 - 1}\ dx = \int \frac{1}{3x^2 - 1}\ x\ dx = \int \frac{1}{u}\ (1/6)\ du = (1/6)\int u^{-1}\ du.$$

Evaluating the new integral we find

$$(1/6)\int u^{-1}\ du = (1/6)[\ln|u| + C] = (1/6)\ln|u| + C.$$

Substituting $u = 3x^2 - 1$ into the antiderivative yields

$$\int \frac{x}{3x^2-1}\ dx = (1/6)\ln|3x^2-1| + C.$$

19.

$$\begin{aligned} u &= e^{-2x} \\ \frac{du}{dx} &= -2e^{-2x} \\ du &= -2e^{-2x}\ dx \\ (-1/2)du &= e^{-2x}\ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int e^{-2x}\ dx = \int (-1/2)\ du = -(1/2)\int 1\ du.$$

Evaluating the new integral we find

$$-(1/2)\int 1\ du = -(1/2)[u + C] = -(1/2)u + C.$$

Substituting $u = e^{-2x}$ into the antiderivative yields

$$\int e^{-2x}\ dx = -(1/2)e^{-2x} + C.$$

21.

$$\begin{aligned} u &= e^{2-x} \\ \frac{du}{dx} &= -e^{2-x} \\ du &= -e^{2-x}\ dx \end{aligned}$$

$$-du = e^{2-x}\ dx.$$

Substituting into the original integral gives

$$\int e^{2-x}\ dx = \int \ -du = -\int 1\ du.$$

Evaluating the new integral we find

$$-\int 1\ du = -[u + C] = -u + C.$$

Substituting $u = e^{-2x}$ into the antiderivative yields

$$\int e^{2-x}\ dx = -e^{2-x} + C.$$

23.

$$u = e^{-x^2}$$
$$\frac{du}{dx} = -2xe^{-x^2}$$
$$du = -2xe^{-x^2}\ dx$$
$$(-1/2)du = xe^{-x^2}\ dx.$$

Substituting into the original integral gives

$$\int xe^{-x^2}\ dx = \int (-1/2)\ du = -(1/2)\int 1\ du.$$

Evaluating the new integral we find

$$-(1/2)\int 1\ du = -(1/2)[u + C] = -(1/2)u + C.$$

Substituting $u = e^{-x^2}$ into the antiderivative yields

$$\int xe^{-x^2}\ dx = -(1/2)e^{-x^2} + C.$$

25.

$$u = e^{-x}$$
$$\frac{du}{dx} = -e^{-x}$$
$$du = -e^{-x}\ dx.$$

We rewrite the original integral as

$$\int e^x - e^{-x}\ dx = \int e^x\ dx + \int -e^{-x}\ dx = e^x + C + \int -e^{-x}\ dx$$

and concentrate on the second integral. Substituting into the second integral gives

$$\int -e^{-x}\ dx = \int\ du = \int 1\ du.$$

Evaluating the new integral we find

$$\int 1\ du = u + C.$$

Substituting $u = e^{-x}$ into the antiderivative yields

$$\int -e^{-x}\ dx = e^{-x} + C.$$

Thus the antiderivative of the original integral is given by

$$\int e^x - e^{-x}\ dx = \int e^x\ dx + \int -e^{-x}\ dx = e^x + e^{-x} + C.$$

27.

$$u = 1 + e^x$$
$$\frac{du}{dx} = e^x$$
$$du = e^x\ dx.$$

Substituting into the original integral gives

$$\int \frac{e^x}{1+e^x}\ dx = \int \frac{1}{1+e^x}\ e^x\ dx = \int \frac{1}{u}\ du.$$

Evaluating the new integral we find

$$\int \frac{1}{u}\ du = \ln|u| + C.$$

Substituting $u = 1 + e^x$ into the antiderivative yields

$$\int \frac{e^x}{1+e^x}\ dx = \ln|1+e^x| + C.$$

29.

$$\begin{aligned} u &= e^{\sqrt{x}} \\ \frac{du}{dx} &= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \\ du &= \frac{e^{\sqrt{x}}}{2\sqrt{x}}\ dx \\ 2\ du &= \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx = \int 2\ du.$$

Evaluating the new integral we find

$$\int 2\ du = 2u + C.$$

Substituting $u = e^{\sqrt{x}}$ into the antiderivative yields

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx = 2e^{\sqrt{x}} + C.$$

31.

$$u = e^{3x} + x^3$$
$$\frac{du}{dx} = 3e^{3x} + 3x^2$$
$$du = 3(e^{3x} + x^2)\ dx$$
$$(1/3)\ du = (e^{3x} + x^2)\ dx.$$

Substituting into the original integral gives

$$\int \frac{e^{3x} + x^2}{(e^{3x} + x^3)^3}\ dx = \int \frac{1}{(e^{3x} + x^3)^3}\ (e^{3x} + x^2)\ dx = \int \frac{1}{u^3}\ (1/3)\ du = (1/3) \int u^{-3}\ du$$

Evaluating the new integral we find

$$(1/3) \int u^{-3}\ du = (1/3)[-(1/2)u^{-2} + C] = \frac{-1}{6u^2} + C.$$

Substituting $u = e^{3x} + x^3$ into the antiderivative yields

$$\int \frac{e^{3x} + x^2}{(e^{3x} + x^3)^3}\ dx = \frac{-1}{6(e^{3x} + x^3)^2} + C.$$

33.

$$u = e^{2x} + 1$$
$$\frac{du}{dx} = 2e^{2x}$$
$$du = 2e^{2x}\ dx$$
$$(1/2)\ du = e^{2x}\ dx.$$

Substituting into the original integral gives

$$\int e^{2x}(e^{2x} + 1)^3\ dx = \int (e^{2x} + 1)^3\ e^{2x}\ dx = \int u^3\ (1/2)\ du = (1/2) \int u^3\ du$$

Evaluating the new integral we find

$$(1/2)\int u^3\ du = (1/2)[(1/4)u^4 + C] = (1/8)u^4 + C.$$

Substituting $u = e^{2x} + 1$ into the antiderivative yields

$$\int e^{2x}(e^{2x}+1)^3\ dx = (1/8)(e^{2x}+1)^4 + C.$$

35.

$$u = \ln 5x$$
$$\frac{du}{dx} = \frac{1}{x}$$
$$du = \frac{1}{x}\ dx.$$

Substituting into the original integral gives

$$\int \frac{\ln 5x}{x}\ dx = \int \ln 5x\ \frac{1}{x}\ dx = \int u\ du.$$

Evaluating the new integral we find

$$\int u\ du = (1/2)u^2 + C.$$

Substituting $u = \ln 5x$ into the antiderivative yields

$$\int \frac{\ln 5x}{x}\ dx = (1/2)(\ln 5x)^2 + C.$$

37.

$$u = \ln x$$
$$\frac{du}{dx} = \frac{1}{x}$$
$$du = \frac{1}{x}\ dx.$$

Substituting into the original integral gives

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\,\frac{1}{x}\,dx = \int \frac{1}{u}\,du.$$

Evaluating the new integral we find

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Substituting $u = \ln x$ into the antiderivative yields

$$\int \frac{1}{x\ln x}\,dx = \ln|\ln x| + C.$$

39.

$$u = \ln x$$
$$\frac{du}{dx} = \frac{1}{x}$$
$$du = \frac{1}{x}\,dx.$$

Substituting into the original integral gives

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{\ln x}\,\frac{1}{x}\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du.$$

Evaluating the new integral we find

$$\int u^{1/2}\,du = (2/3)u^{3/2} + C.$$

Substituting $u = \ln x$ into the antiderivative yields

$$\int \frac{\sqrt{\ln x}}{x}\,dx = (2/3)(\ln x)^{3/2} + C.$$

41. To evaluate the integral we separate it into the following two integrals before proceeding since each integral requires a different substitution:

$$\int \left(xe^{x^2} - \frac{x}{x^2+2}\right)\,dx = \int\, xe^{x^2}\,dx - \int\,\frac{x}{x^2+2}\,dx.$$

For the first integral we use the substitution

$$\begin{aligned} u &= e^{x^2} \\ \frac{du}{dx} &= 2xe^{x^2} \\ du &= 2xe^{x^2}\,dx \\ (1/2)du &= xe^{x^2}\,dx. \end{aligned}$$

Substituting into the original integral gives

$$\int\, xe^{x^2}\,dx = \int\,(1/2)\,du = (1/2)\int\,1\,du.$$

Evaluating the new integral we find

$$(1/2)\int 1\,du = (1/2)[u + C] = (1/2)u + C.$$

Substituting $u = e^{x^2}$ into the antiderivative yields

$$\int\, xe^{x^2}\,dx = (1/2)e^{x^2} + C.$$

For the second integral we use the substitution

$$\begin{aligned} u &= x^2 + 1 \\ \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ (1/2)du &= x\,dx. \end{aligned}$$

Substituting into the original integral gives

$$\int \frac{x}{x^2+1}\ dx = \int \frac{1}{x^2+1}\ x\ dx = \int \frac{1}{u}\ (1/2)\ du = (1/2)\int u^{-1}\ du.$$

Evaluating the new integral we find

$$(1/2)\int u^{-1}\ du = (1/2)[\ln|u| + C] = (1/2)\ln|u| + C.$$

Substituting $u = x^2 + 1$ into the antiderivative yields

$$\int \frac{x}{x^2+1}\ dx = (1/2)\ln|x^2+1| + C.$$

Combining these results we find

$$\int \left(xe^{x^2} - \frac{x}{x^2+2}\right)\ dx = (1/2)e^{x^2} - (1/2)\ln|x^2+1| + C.$$

43.

$$\begin{aligned} u &= \sqrt{x} - 1 \\ \frac{du}{dx} &= \frac{1}{2\sqrt{x}} \\ du &= \frac{1}{2\sqrt{x}}\ dx \\ 2\sqrt{x}\ du &= \ dx. \end{aligned}$$

Before proceeding we write

$$u = \sqrt{x} - 1 \quad \rightarrow \quad \sqrt{x} = u + 1 \quad \rightarrow \quad x = (u+1)^2$$

so that

$$2\sqrt{x}\ du = 2(u+1)\ du = \ dx.$$

Substituting into the original integral gives

$$\int \frac{x+1}{\sqrt{x}-1}\ dx = \int \frac{(u+1)^2+1}{u}\ 2(u+1)\ du = \int \frac{2[(u+1)^2+1](u+1)}{u}\ du$$
$$= 2\int \frac{u^3+3u^2+4u+2}{u}\ du = 2\int\ u^2+3u+4+\frac{2}{u}\ du$$

Evaluating the new integral we find

$$2\int\ u^2+3u+4+\frac{2}{u}\ du = 2[(1/3)u^3+(3/2)u^2+4u+2\ln|u|+C] = (2/3)u^3+3u^2+8u+4\ln|u|+C.$$

Substituting $u = \sqrt{x}-1$ into the antiderivative yields

$$\int \frac{x+1}{\sqrt{x}-1}\ dx = (2/3)(\sqrt{x}-1)^3+3(\sqrt{x}-1)^2+8(\sqrt{x}-1)+4\ln|\sqrt{x}-1|+C.$$

45.

$$u = x-1$$
$$\frac{du}{dx} = 1$$
$$du = \ dx.$$

Before proceeding we write

$$u = x-1 \quad \to \quad x = u+1.$$

Substituting into the original integral gives

$$\int\ x(x-1)^5\ dx = \int (u+1)u^5\ du = \int\ u^6+u^5\ du.$$

Evaluating the new integral we find

$$\int\ u^6+u^5\ du = (1/7)u^7+(1/6)u^6+C.$$

Substituting $u = x-1$ into the antiderivative yields

$$\int\ x(x-1)^5\ dx = (1/7)(x-1)^7+(1/6)(x-1)^6+C.$$

47.

$$\begin{aligned} u &= 1+\sqrt{x} \\ \frac{du}{dx} &= \frac{1}{2\sqrt{x}} \\ du &= \frac{1}{2\sqrt{x}}\ dx \\ 2\sqrt{x}\ du &= \ dx. \end{aligned}$$

Before proceeding we write

$$u = \sqrt{x}+1 \quad \rightarrow \quad \sqrt{x} = u-1$$

so that

$$2\sqrt{x}\ du = 2(u-1)\ du = \ dx.$$

Substituting into the original integral gives

$$\begin{aligned} \int \frac{1-\sqrt{x}}{1+\sqrt{x}}\ dx &= \int \frac{1-(u-1)}{u}\ 2(u-1)\ du = \int \frac{2(2-u)(u-1)}{u}\ du \\ &= 2\int \frac{-u^2+3u-2}{u}\ du = 2\int -u+3-\frac{2}{u}\ du \end{aligned}$$

Evaluating the new integral we find

$$2\int -u+3-\frac{2}{u}\ du = 2[-(1/2)u^2+3u-2\ln|u|+C] = -u^2+6u-4\ln|u|+C.$$

Substituting $u = 1+\sqrt{x}$ into the antiderivative yields

$$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\ dx = -(1+\sqrt{x})^2+6(1+\sqrt{x})-4\ln|1+\sqrt{x}|+C.$$

49.

$$u = 1-v$$

$$\frac{du}{dv} = -1$$
$$du = -\ dv$$
$$-\ du = \ dv$$

Before proceeding we write
$$u = 1 - v \quad \rightarrow \quad v = 1 - u.$$

Substituting into the original integral gives
$$\int \ v^2(1-v)^6 \ dv = \int (1-u)^2 u^6 \ (-1) \ du = -\int \ u^6 - 2u^7 + u^8 \ du.$$

Evaluating the new integral we find
$$-\int \ u^6 - 2u^7 + u^8 \ du = -(1/7)u^7 + (1/4)u^8 - (1/9)u^9 + C.$$

Substituting $u = 1 - v$ into the antiderivative yields
$$\int \ v^2(1-v)^6 \ dv = -(1/7)(1-v)^7 + (1/4)(1-v)^8 - (1/9)(1-v)^9 + C.$$

51. $f(x) = \int \ 5(2x-1)^4 \ dx.$

$$u = 2x - 1$$
$$\frac{du}{dx} = 2$$
$$du = 2 \ dx$$
$$(1/2) \ du = \ dx.$$

Substituting into the original integral gives
$$\int \ 5(2x-1)^4 \ dx = \int \ 5(u)^4 \ (1/2) \ du = (5/2) \int \ (u)^4 \ du.$$

Evaluating the new integral we find

$$(5/2)\int u^4\ du = (5/2)[(1/5)u^5 + C] = (1/2)u^5 + C.$$

Substituting $u = 2x - 1$ into the antiderivative yields

$$f(x) = \int\ 5(2x-1)^4\ dx = (1/2)(2x-1)^5 + C.$$

We evaluate the constant C by noting that $(1,3) = (1, f(1))$ and substituting into the above equation find

$$3 = (1/2)(2(1)-1)^5 + C = (1/2) + C \quad \rightarrow \quad C = (5/2).$$

Hence, $f(x) = (1/2)(2x-1)^5 + (5/2)$.

53. $f(x) = \int\ -2xe^{-x^2+1}\ dx$.

$$\begin{aligned} u &= -x^2 + 1 \\ \frac{du}{dx} &= -2x \\ du &= -2x\ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int\ -2xe^{-x^2+1}\ dx = \int\ e^{-x^2+1}\ -2x\ dx = \int\ e^u\ du.$$

Evaluating the new integral we find

$$\int e^u\ du = e^u + C.$$

Substituting $u = -x^2 + 1$ into the antiderivative yields

$$f(x) = \int\ -2xe^{-x^2+1}\ dx = e^{-x^2+1} + C.$$

We evaluate the constant C by noting that $(1,0) = (1, f(1))$ and substituting into the above equation find

$$0 = e^{-(1)^2+1} + C = 1 + C \quad \rightarrow \quad C = -1.$$

Hence, $f(x) = e^{-x^2+1} - 1$.

59. $p(x) = \int \frac{-250x}{(16+x^2)^{3/2}} \ dx$.

$$\begin{aligned} u &= 16 + x^2 \\ \frac{du}{dx} &= 2x \\ du &= 2x \ dx. \end{aligned}$$

Substituting into the original integral gives

$$\int \frac{-250x}{(16+x^2)^{3/2}} \ dx = \int \frac{-125}{(16+x^2)^{3/2}} \ 2x \ dx = \int \frac{-125}{u^{3/2}} \ du = -125 \int u^{-3/2} \ du.$$

Evaluating the new integral we find

$$-125 \int u^{-3/2} \ du = -125[-2u^{-1/2} + C] = 250u^{-1/2} + C.$$

Substituting $u = 16 + x^2$ into the antiderivative yields

$$p(x) = \int \frac{-250x}{(16+x^2)^{3/2}} \ dx = 250(16 + x^2)^{-1/2} + C.$$

We evaluate the constant C by noting that $(3, 50) = (3, p(3))$ and substituting into the above equation find

$$50 = 250(16 + (3)^2)^{-1/2} + C = 50 + C \quad \rightarrow \quad C = 0.$$

Hence, $p(x) = 250(16 + x^2)^{-1/2}$.

61. $r(t) = \int \frac{30}{\sqrt{2t+4}} \ dt$.

$$u = 2t + 4$$

$$\frac{du}{dt} = 2$$
$$du = 2\ dt.$$

Substituting into the original integral gives

$$\int \frac{30}{\sqrt{2t+4}}\ dt = \int \frac{15}{\sqrt{2t+4}}\ 2\ dt = \int \frac{15}{\sqrt{u}}\ du = 15 \int u^{-1/2}\ du.$$

Evaluating the new integral we find

$$15 \int u^{-1/2}\ du = 15[2u^{1/2} + C] = 30u^{1/2} + C.$$

Substituting $u = 2t + 4$ into the antiderivative yields

$$r(t) = \int \frac{30}{\sqrt{2t+4}}\ dt = 30\sqrt{2t+4} + C.$$

We evaluate the constant C by noting that the radius is zero at time zero: $(0, 0) = (0, r(0))$ and substituting into the above equation find

$$0 = 30\sqrt{2t+4} + C = 60 + C \quad \rightarrow \quad C = -60.$$

Hence, $r(t) = 30\sqrt{2t+4} - 60$ and $r(16) = 30\sqrt{2(16)+4} - 60 = 120$ feet, and $A = \pi[r(16)]^2 = 14,400\ \pi$ square feet.

Section 6.3

1. The width of each of the rectangles is (1/3) so that the sum of the areas of the rectangles is given by

$$(1.9)(1/3) + (1.5)(1/3) + (1.8)(1/3) + (2.4)(1/3) + (2.7)(1/3) + (2.5)(1/3)$$

$$= (1/3)[1.9 + 1.5 + 1.8 + 2.4 + 2.7 + 2.5] = (1/3)[12.8] \approx 4.27.$$

3. The area under the curve is a triangle with a base length of 2 and a height of 6 so that the area is given by

$$A = (1/2)bh = (1/2)(2)(6) = 6.$$

The interval $[0, 2]$ is divided into the four subintervals $[0, 0.5]$, $[0.5, 1.0]$, $[1.0, 1.5]$, $[1.5, 2.0]$, each of length $\Delta x = \frac{2-0}{4} = 0.5$ and with left endpoints given by 0, 0.5, 1.0, and 1.5. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.5)\Delta x + f(1.0)\Delta x + f(1.5)\Delta x$$

$$A \approx (0)(0.5) + (1.5)(0.5) + (3)(0.5) + (4.5)(0.5) = 4.5.$$

The interval $[0, 2]$ is divided into the eight subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1.0]$, $[1.0, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, $[1.75, 2.0]$, each of length $\Delta x = \frac{2-0}{8} = 0.25$ and with left endpoints given by 0, 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, and 1.75. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.25)\Delta x + f(0.5)\Delta x + f(0.75)\Delta x + f(1.0)\Delta x + f(1.25)\Delta x + f(1.5)\Delta x + f(1.75)\Delta x$$

$$A \approx (0)(0.25) + (0.75)(0.25) + (1.5)(0.25) + (2.25)(0.25) + (3)(0.25) + (3.75)(0.25) + (4.5)(0.25) + (5.25)(0.25)$$

$$A \approx 5.25.$$

5. The area under the curve is a triangle with a base length of 2 and a height of 4 so that the area is given by

$$A = (1/2)bh = (1/2)(2)(4) = 4.$$

The interval $[0, 2]$ is divided into the five subintervals $[0, 0.4]$, $[0.4, 0.8]$, $[0.8, 1.2]$, $[1.2, 1.6]$, $[1.6, 2.0]$, each of length $\Delta x = \frac{2-0}{5} = 0.4$ and with left endpoints given by 0, 0.4, 0.8, 1.2, and 1.6. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.4)\Delta x + f(0.8)\Delta x + f(1.2)\Delta x + f(1.6)\Delta x$$

$$A \approx (4)(0.4) + (3.2)(0.4) + (2.4)(0.4) + (1.6)(0.4) + (0.8)(0.4) = 4.8.$$

The interval $[0, 2]$ is divided into the ten subintervals $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, $[0.8, 1.0]$, $[1.0, 1.2]$, $[1.2, 1.4]$, $[1.4, 1.6]$, $[1.6, 1.8]$, $[1.8, 2.0]$, each of length $\Delta x = \frac{2-0}{10} = 0.22$ and with left endpoints given by 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, and 1.8. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.2)\Delta x + f(0.4)\Delta x + f(0.6)\Delta x + f(0.8)\Delta x + f(1.0)\Delta x + f(1.2)\Delta x$$

$$+f(1.4)\Delta x + f(1.6)\Delta x + f(1.8)\Delta x$$

$$A \approx (4)(0.2) + (3.6)(0.2) + (3.2)(0.2) + (2.8)(0.2) + (2.4)(0.2) + (2)(0.2) + (1.6)(0.2)$$

$$+(1.2)(0.2) + (0.8)(0.2) + (0.4)(0.2)$$

$$A \approx 4.4.$$

7. The interval $[2, 4]$ is divided into the two subintervals $[2, 3]$, $[3, 4]$, each of length $\Delta x = \frac{4-2}{2} = 1$ and with midpoints given by 2.5 and 3.5. Hence an approximation of the area under the curve is

$$A \approx f(2.5)\Delta x + f(3.5)\Delta x = (6.25)(1) + (12.25)(1) = 18.5.$$

The interval $[2, 4]$ is divided into the five subintervals $[2, 2.4]$, $[2.4, 2.8]$, $[2.8, 3.2]$, $[3.2, 3.6]$, $[3.6, 4.0]$, each of length $\Delta x = \frac{4-2}{5} = 0.4$ and with midpoints given by 2.2, 2.6, 3.0, 3.4, and 3.8. Hence an approximation of the area under the curve is

$$A \approx f(2.2)\Delta x + f(2.6)\Delta x + f(3.0)\Delta x + f(3.4)\Delta x + f(3.8)\Delta x$$

$$A \approx (4.84)(0.4) + (6.76)(0.4) + (9)(0.4) + (11.56)(0.4) + (14.44)(0.4) = 18.64.$$

The interval $[2, 4]$ is divided into the ten subintervals $[2, 2.2]$, $[2.2, 2.4]$, $[2.4, 2.6]$, $[2.6, 2.8]$, $[2.8, 3.0]$, $[3.0, 3.2]$, $[3.2, 3.4]$, $[3.4, 3.6]$, $[3.6, 3.8]$, $[3.8, 4.0]$ each of length $\Delta x = \frac{4-2}{10} = 0.2$ and with midpoints given by 2.1, 2.3, 2.5, 2.7, 2.9, 3.1, 3.3, 3.5, 3.7, and 3.8. Hence an approximation of the area under the curve is

$$A \approx f(2.1)\Delta x + f(2.3)\Delta x + f(2.5)\Delta x + f(2.7)\Delta x + f(2.9)\Delta x + f(3.1)\Delta x + f(3.3)\Delta x$$
$$+f(3.5)\Delta x + f(3.7)\Delta x + f(3.9)\Delta x$$

$$A \approx (4.41)(0.2) + (5.29)(0.2) + (6.25)(0.2) + (7.29)(0.2) + (8.41)(0.2) + (9.61)(0.2) + (10.89)(0.2)$$
$$+(12.25)(0.2) + (13.69)(0.2) + (15.21)(0.2)$$
$$A \approx 18.66.$$

9. The interval $[2, 4]$ is divided into the two subintervals $[2, 3]$, $[3, 4]$, each of length $\Delta x = \frac{4-2}{2} = 1$ and with right endpoints given by 3.0 and 4.0. Hence an approximation of the area under the curve is

$$A \approx f(3)\Delta x + f(4)\Delta x = (9)(1) + (16)(1) = 25.$$

The interval $[2, 4]$ is divided into the five subintervals $[2, 2.4]$, $[2.4, 2.8]$, $[2.8, 3.2]$, $[3.2, 3.6]$, $[3.6, 4.0]$, each of length $\Delta x = \frac{4-2}{5} = 0.4$ and with right endpoints given by 2.4, 2.8, 3.3, 3.6, and 4.0. Hence an approximation of the area under the curve is

$$A \approx f(2.4)\Delta x + f(2.8)\Delta x + f(3.2)\Delta x + f(3.6)\Delta x + f(4.0)\Delta x$$

$$A \approx (5.76)(0.4) + (7.84)(0.4) + (10.24)(0.4) + (12.96)(0.4) + (16.0)(0.4) = 21.12.$$

The interval $[2, 4]$ is divided into the ten subintervals $[2, 2.2]$, $[2.2, 2.4]$, $[2.4, 2.6]$, $[2.6, 2.8]$, $[2.8, 3.0]$, $[3.0, 3.2]$, $[3.2, 3.4]$, $[3.4, 3.6]$, $[3.6, 3.8]$, $[3.8, 4.0]$ each of length $\Delta x = \frac{4-2}{10} = 0.2$ and with right endpoints given by 2.2, 2.4, 2.6, 2.8, 3.0, 3.2, 3.4, 3.6, 3.8, and 4.0. Hence an approximation of the area under the curve is

$$A \approx f(2.2)\Delta x + f(2.4)\Delta x + f(2.6)\Delta x + f(2.8)\Delta x + f(3.0)\Delta x + f(3.2)\Delta x + f(3.4)\Delta x$$
$$+f(3.6)\Delta x + f(3.8)\Delta x + f(4.0)\Delta x$$

$$A \approx (4.84)(0.2) + (5.76)(0.2) + (6.76)(0.2) + (7.84)(0.2) + (9.0)(0.2) + (10.24)(0.2) + (11.56)(0.2)$$
$$+(12.96)(0.2) + (14.44)(0.2) + (16.0)(0.2)$$
$$A \approx 19.88.$$

11. The interval $[0, 1]$ is divided into the two subintervals $[0, 0.5]$, $[0.5, 1]$, each of length $\Delta x = \frac{1-0}{2} = 0.5$ and with left endpoints given by 0.0 and 0.5. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.5)\Delta x = (0)(0.5) + (0.125)(0.5) = 0.0625.$$

The interval $[0, 1]$ is divided into the five subintervals $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, $[0.8, 1.0]$, each of length $\Delta x = \frac{1-0}{5} = 0.2$ and with left endpoints given by 0, 0.2, 0.4, 0.6, and 0.8. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.2)\Delta x + f(0.4)\Delta x + f(0.6)\Delta x + f(0.8)\Delta x$$
$$A \approx (0)(0.2) + (0.008)(0.2) + (0.064)(0.2) + (0.216)(0.2) + (0.512)(0.2) = 0.16.$$

The interval $[0, 1]$ is divided into the ten subintervals $[0, 0.1]$, $[0.1, 0.2]$, $[0.2, 0.3]$, $[0.3, 0.4]$, $[0.4, 0.5]$, $[0.5, 0.6]$, $[0.6, 0.7]$, $[0.7, 0.8]$, $[0.8, 0.9]$, $[0.9, 1.0]$ each of length $\Delta x = \frac{1-0}{10} = 0.1$ and with left endpoints given by 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9. Hence an approximation of the area under the curve is

$$A \approx f(0)\Delta x + f(0.1)\Delta x + f(0.2)\Delta x + f(0.3)\Delta x + f(0.4)\Delta x + f(0.5)\Delta x + f(0.6)\Delta x$$

$$+f(0.7)\Delta x + f(0.8)\Delta x + f(0.9)\Delta x$$

$$A \approx (0)(0.1) + (0.001)(0.1) + (0.008)(0.1) + (0.027)(0.1) + (0.064)(0.1) + (0.125)(0.1) + (0.216)(0.1)$$

$$+(0.343)(0.1) + (0.512)(0.1) + (0.729)(0.1)$$

$$A \approx 0.2025.$$

13. The interval $[0, 2]$ is divided into the five subintervals $[0, 0.4]$, $[0.4, 0.8]$, $[0.8, 1.2]$, $[1.2, 1.6]$, $[1.6, 2.0]$, each of length $\Delta x = \frac{2-0}{5} = 0.4$ and with midpoints given by 0.2, 0.6, 1.0, 1.4, and 1.8. Hence an approximation of the area under the curve is

$$A \approx f(0.2)\Delta x + f(0.6)\Delta x + f(1.0)\Delta x + f(1.4)\Delta x + f(1.8)\Delta x$$

$$A \approx (1.04)(0.4) + (1.36)(0.4) + (2.0)(0.4) + (2.96)(0.4) + (4.24)(0.4) = 4.64.$$

15. The interval $[1, 3]$ is divided into the four subintervals $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$, each of length $\Delta x = \frac{3-1}{4} = 0.5$ and with right endpoints given by 1.5, 2, 2.5, and 3. Hence an approximation of the area under the curve is

$$A \approx f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x + f(3)\Delta x$$

$$A \approx (2/3)(0.5) + (1/2)(0.5) + (2/5)(0.5) + (1/3)(0.5) = 0.95.$$

Section 6.4

1. Since $f(x) = 2 \geq 0$ on $[1, 4]$,

$$A = \int_1^4 2\ dx = [2x]|_1^4 = [2(4)] - [2(1)] = 6.$$

3. $f(x) = 2x = 0 \ \rightarrow \ x = 0$; $f(-1) < 0$ and $f(1) > 0$ so that $f(x) \geq 0$ on $[1, 3]$,

$$A = \int_1^3 2x\ dx = [x^2]|_1^3 = [(3)^2] - [(1)^2] = 8.$$

5. $f(x) = 2x + 3 = 0 \ \rightarrow \ x = -1.5$; $f(-2) < 0$ and $f(-1) > 0$ so that $f(x) \geq 0$ on $[-1, 2]$,

$$A = \int_{-1}^2 2x + 3\ dx = [x^2 + 3x]|_{-1}^2 = [(2)^2 + 3(2)] - [(-1)^2 + 3(-1)] = 12.$$

7. $f(x) = -x^2 + 4 = 0 \ \rightarrow \ x = \pm 2$; $f(-3) < 0$, $f(0) > 0$, and $f(3) < 0$ so that $f(x) \geq 0$ on $[-1, 2]$,

$$A = \int_{-1}^2 -x^2 + 4\ dx = [-(1/3)x^3 + 4x]|_{-1}^2 = [-(1/3)(2)^3 + 4(2)] - [-(1/3)(-1)^3 + 4(-1)]$$

$$[-(8/3) + 8] - [(1/3) - 4] = [16/3] - [-11/3] = 9.$$

9. Since $f(x) = \frac{1}{x} \geq 0$ on $[1, 2]$,

$$A = \int_1^2 \frac{1}{x}\ dx = [\ln|x|]|_1^4 = [\ln 4] - [\ln 1] = \ln 4.$$

11. Since $f(x) = \sqrt{x} \geq 0$ on $[1, 9]$,

$$A = \int_1^9 \sqrt{x}\, dx = \int_1^9 x^{1/2}\, dx = [(2/3)x^{3/2}]|_1^9 = (2/3)(9)^{3/2} - (2/3)(1)^{3/2} = (2/3)(27) - (2/3)(1) = (52/3).$$

13. $f(x) = 1 - \sqrt[3]{x} = 0 \;\rightarrow\; x = 1$; $f(0) > 0$ and $f(2) < 0$ so that $f(x) \geq 0$ on $[-8, -1]$,

$$A = \int_{-8}^{-1} 1 - \sqrt[3]{x}\, dx = \int_{-8}^{-1} 1 - x^{1/3}\, dx = [x - (3/4)x^{4/3}]|_{-8}^{-1} = [(-1) - (3/4)(-1)^{4/3}] - [(-8) - (3/4)(-8)^{4/3}]$$

$$= [(-1) - (3/4)(1)] - [(-8) - (3/4)(16)] = [-(7/4)] - [-20] = (73/4).$$

15. Since $f(x) = e^x \geq 0$ on $[0, 2]$,

$$A = \int_0^2 e^x\ dx = \int_0^2 e^x\ dx = [e^x]|_0^2 = e^2 - e^0 = e^2 - 1.$$

17.

$$\int_2^4 3\ dx = [3x]|_2^4 = [3(4)] - [3(2)] = [12] - [6] = 6.$$

19.

$$\int_1^3 (2x + 3)\ dx = [x^2 + 3x]|_1^3 = [(3)^2 + 3(3)] - [(1)^2 + 3(1)] = [18] - [4] = 14.$$

21.

$$\int_{-1}^3 2x^2\ dx = [(2/3)x^3]|_{-1}^3 = [(2/3)(3)^3] - [(2/3)(-1)^3] = [18] - [-2/3] = (56/3).$$

23.

$$\int_{-2}^2 (x^2 - 1)\ dx = [(1/3)x^3 - x]|_{-2}^2 = [(1/3)(2)^3 - (2)] - [(1/3)(-2)^3 - (-2)] = [2/3] - [-2/3] = (4/3).$$

25.

$$\int_1^8 4x^{1/3}\ dx = [3x^{4/3}|_1^8 = [3(8)^{4/3}] - [3(1)^{4/3}] = [3(16)] - [3(1)] = 45.$$

27.

$$\int_0^1 (x^3-2x^2+1)\ dx = [(1/4)x^4-(2/3)x^3+x]|_0^1 = [(1/4)(1)^4-(2/3)(1)^3+(1)]-[(1/4)(0)^4-(2/3)(0)^3+(0)]$$

$$= [(1/4) - (2/3) + 1] - [0] = (7/12).$$

29.

$$\int_2^4 \frac{1}{x}\ dx = [\ln|x|]|_2^4 = [\ln 4] - [\ln 2] = \ln(4/2) = \ln 2.$$

31.

$$\int_0^4 x(x^2-1)\ dx = \int_0^4 x^3-x\ dx = [(1/4)x^4-(1/2)x^2]|_0^4 = [(1/4)(4)^4-(1/2)(4)^2]-[(1/4)(0)^4-(1/2)(0)^2]$$

$$= [(1/4)(256) - (1/2)(16)] - [0] = 56.$$

33.

$$\int_1^3 (t^2-t)^2\ dt = \int_1^3 t^4 - 2t^3 + t^2\ dt = [(1/5)t^5 - (1/2)t^4 + (1/3)t^3]|_1^3$$

$$= [(1/5)(3)^5 - (1/2)(3)^4 + (1/3)(3)^3] - [(1/5)(1)^5 - (1/2)(1)^4 + (1/3)(1)^3]$$

$$= [(1/5)(243) - (1/2)(81) + (1/3)(27)] - [(1/5) - (1/2) + (1/3)] = [171/10] - [1/30] = (256/15).$$

35.

$$\int_{-3}^{-1} \frac{1}{x^2}\ dx = \int_{-3}^{-1} x^{-2}\ dx = [-x^{-1}]|_{-3}^{-1} = [-(-1)^{-1}] - [-(-3)^{-1}] = [1] - [(1/3)] = (2/3).$$

37.

$$\int_1^4 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\, dx = \int_1^4 (x^{1/2} - x^{-1/2})\, dx = [(2/3)x^{3/2} - 2x^{1/2}]|_1^4$$

$$= [(2/3)(4)^{3/2} - 2(4)^{1/2}] - [(2/3)(1)^{3/2} - 2(1)^{1/2}] = [(2/3)(8) - 2(2)] - [(2/3) - 2]$$

$$= [4/3] - [-4/3] = (8/3).$$

39.

$$\int_1^4 \frac{3x^3 - 2x^2 + 4}{x^2}\, dx = \int_1^4 \frac{3x^3}{x^2} + \frac{-2x^2}{x^2} + \frac{4}{x^2}\, dx = \int_1^4 3x - 2 + 4x^{-2}\, dx = [(3/2)x^2 - 2x - 4x^{-1}]|_1^4$$

$$= [(3/2)(4)^2 - 2(4) - 4(4)^{-1}] - [(3/2)(1)^2 - 2(1) - 4(1)^{-1}] = [(3/2)(16) - 2(4) - 4(1/4)] - [(3/2) - 2 - 4]$$

$$= [15] - [-9/2] = (39/2).$$

41.

$$C(x) = \int 0.0003x^2 - 0.12x + 20\, dx = 0.0001x^3 - 0.06x^2 + 20x + C.$$

Since the daily fixed cost is 800 dollars,

$$800 = C(0) = 0.0001(0)^3 - 0.06(0)^2 + 20(0) + C = C \quad \rightarrow \quad C = 800$$

so that the cost function is given by

$$0.0001x^3 - 0.06x^2 + 20x + 800.$$

The cost for producing the first 300 units is therefore given by

$$C(300) = 0.0001(300)^3 - 0.06(300)^2 + 20(300) + 800 = 4,100 \text{ dollars}.$$

We can calculate the cost of producing the 201st thru the 300th toaster in two ways: by the first technique, the cost is given by the difference $C(300) - C(200)$ (we leave the calculation to the student). In the second technique we evaluate the integral

$$\int_{200}^{300} C'(x)\ dx = \int_{200}^{300} 0.0003x^2 - 0.12x + 20\ dx = [0.0001x^3 - 0.06x^2 + 20x]|_{200}^{300}$$
$$= [3,300] - [2,400] = 900 \text{ dollars.}$$

The two techniques are equivalent since $C(300) - C(200) = \int_{200}^{300} C'(x)\ dx$ and naturally give the same answer.

43. The profit realized from the manufacture and sale of 200 units per day is

$$P(200) - P(0) = \int_{0}^{200} P'(x)\ dx = \int_{0}^{200} -0.0003x^2 + 0.02x + 20\ dx = [-0.0001x^3 + 0.01x^2 + 20x]|_{0}^{200}$$
$$= [3,600] - [0] = 3,600 \text{ dollars}$$

so that

$$P(200) = 3,600 + P(0) \text{ dollars.}$$

We know that that for the revenue function $R(x)$, $R(0) = 0$ (if you sell zero items you have zero revenue); additionally from problem 41 we know that $C(0) = 800$ dollars so that

$$P(0) = R(0) - C(0) = 0 - 800 = -800 \text{ dollars.}$$

Thus

$$P(200) = 3,600 + P(0) = 3,600 - 800 = 2,800 \text{ dollars.}$$

The additional profit realized by selling an additional 20 units is given by

$$P(220) - P(200) = \int_{200}^{220} P'(x)\ dx = \int_{200}^{220} -0.0003x^2 + 0.02x + 20\ dx = [-0.0001x^3 + 0.01x^2 + 20x]|_{200}^{220}$$
$$= [3,819.20] - [3,600] = 219.20 \text{ dollars}$$

Section 6.5

1. We begin with the following u substitution:

$$u = x^2 - 1$$
$$\frac{du}{dx} = 2x$$
$$du = 2x\ dx$$
$$(1/2)\ du = x\ dx$$
$$x = 0 \quad \rightarrow \quad u = (0)^2 - 1 = -1$$
$$x = 2 \quad \rightarrow \quad u = (2)^2 - 1 = 3$$

and rewrite the integral as

$$\int_0^2 x(x^2-1)^3\ dx = \int_0^2 (x^2-1)^3\ x\ dx = \int_{-1}^3 u^3\ (1/2)\ du = (1/2)\int_{-1}^3 u^3\ du.$$

Evaluating this integral we find

$$(1/2)\int_{-1}^3 u^3\ du = (1/2)[(1/4)u^4]|_{-1}^3 = (1/2)[(1/4)(3)^4] - (1/2)[(1/4)(-1)^4] = (1/2)[81/4] - (1/2)[1/4] = 10.$$

3. We begin with the following u substitution:

$$u = 5x^2 + 4$$
$$\frac{du}{dx} = 10x$$
$$du = 10x\ dx$$
$$(1/10)\ du = x\ dx$$
$$x = 0 \quad \rightarrow \quad u = 5(0)^2 + 4 = 4$$
$$x = 1 \quad \rightarrow \quad u = 5(1)^2 + 4 = 9$$

and rewrite the integral as

$$\int_0^1 x\sqrt{5x^2+4}\ dx = \int_0^1 \sqrt{5x^2+4}\ x\ dx = \int_4^9 u^{1/2}\ (1/10)\ du = (1/10)\int_4^9 u^{1/2}\ du.$$

Evaluating this integral we find

$$(1/10)\int_4^9 u^{1/2}\ du = (1/10)[(2/3)u^{3/2}]|_{-1}^{3} = (1/10)[(2/3)(9)^{3/2}] - (1/10)[(2/3)(4)^{3/2}]$$
$$= (1/10)[18] - (1/10)[16/3] = (38/30) = (19/15).$$

5. We begin with the following u substitution:

$$u = x^3 + 1$$
$$\frac{du}{dx} = 3x^2$$
$$du = 3x^2\ dx$$
$$(1/3)\ du = x^2\ dx$$
$$x = 0 \quad \rightarrow \quad u = (0)^3 + 1 = 1$$
$$x = 2 \quad \rightarrow \quad u = (2)^3 + 1 = 9$$

and rewrite the integral as

$$\int_0^2 x^2(x^3+1)^{3/2}\ dx = \int_0^2 (x^3+1)^{3/2}\ x^2\ dx = \int_1^9 u^{3/2}\ (1/3)\ du = (1/3)\int_1^9 u^{3/2}\ du.$$

Evaluating this integral we find

$$(1/3)\int_1^9 u^{3/2}\ du = (1/3)[(2/5)u^{5/2}]|_{-1}^{3} = (1/3)[(2/5)(9)^{5/2}] - (1/3)[(2/5)(1)^{5/2}]$$
$$= (1/3)[486/5] - (1/3)[2/5] = (484/15).$$

7. We begin with the following u substitution:

$$u = 2x + 1$$
$$\frac{du}{dx} = 2$$
$$du = 2\ dx$$

$$(1/2)\ du = \ dx$$

$$x = 0 \quad \rightarrow \quad u = 2(0) + 1 = 1$$

$$x = 1 \quad \rightarrow \quad u = 2(1) + 1 = 3$$

and rewrite the integral as

$$\int_0^1 \frac{1}{\sqrt{2x+1}}\ dx = \int_1^3 \frac{1}{u^{1/2}}\ (1/2)\ du = \int_1^3 u^{-1/2}\ (1/2)\ du = (1/2)\int_1^3 u^{-1/2}\ du.$$

Evaluating this integral we find

$$(1/2)\int_1^3 u^{-1/2}\ du = (1/2)[2u^{1/2}]|_1^3 = (1/2)[2(3)^{1/2}] - (1/2)[2(1)^{1/2}] = \sqrt{3} - 1.$$

9. We begin with the following u substitution:

$$u = 2x - 1$$

$$\frac{du}{dx} = 2$$

$$du = 2\ dx$$

$$(1/2)\ du = \ dx$$

$$x = 1 \quad \rightarrow \quad u = 2(1) - 1 = 1$$

$$x = 2 \quad \rightarrow \quad u = 2(2) - 1 = 3$$

and rewrite the integral as

$$\int_1^2 (2x-1)^4\ dx = \int_1^3 u^4\ (1/2)\ du = (1/2)\int_1^3 u^4\ du.$$

Evaluating this integral we find

$$(1/2)\int_1^3 u^4\ du = (1/2)[(1/5)u^5]|_1^3 = (1/2)[(1/5)(3)^5] - (1/2)[(1/5)(1)^5] = (1/2)[243/5] - (1/2)[1/5] = (121/5).$$

11. We begin with the following u substitution:

$$u = x^3 + 1$$
$$\frac{du}{dx} = 3x^2$$
$$du = 3x^2\ dx$$
$$(1/3)\ du = x^2\ dx$$
$$x = -1 \quad \rightarrow \quad u = (-1)^3 + 1 = 0$$
$$x = 1 \quad \rightarrow \quad u = (1)^3 + 1 = 2$$

and rewrite the integral as

$$\int_{-1}^{1} x^2(x^3+1)^4\ dx = \int_{-1}^{1} (x^3+1)^4\ x^2\ dx = \int_0^2 u^4\ (1/3)\ du = (1/3)\int_0^2 u^4\ du.$$

Evaluating this integral we find

$$(1/3)\int_0^2 u^4\ du = (1/3)[(1/5)u^5]|_0^2 = (1/3)[(1/5)(2)^5] - (1/3)[(1/5)(0)^4] = (1/3)[32/5] - (1/3)[0] = (32/15).$$

13. We begin with the following u substitution:

$$u = x - 1$$
$$\frac{du}{dx} = 1$$
$$du = dx$$
$$x = 1 \quad \rightarrow \quad u = (1) - 1 = 0$$
$$x = 5 \quad \rightarrow \quad u = (5) - 1 = 4.$$

Additionally, from the equation $u = x - 1$ we solve for x and obtain

$$x = u + 1.$$

Next we rewrite the integral as

$$\int_1^5 x\sqrt{x-1}\ dx = \int_0^4 (u+1)u^{1/2}\ du = \int_0^4 u^{3/2} + u^{1/2}\ du.$$

Evaluating this integral we find

$$\int_0^4 u^{3/2}+u^{1/2}\ du = [(2/5)u^{5/2}+(2/3)u^{3/2}]|_0^4 = [(2/5)(4)^{5/2}+(2/3)(4)^{3/2}]-[(2/5)(0)^{5/2}+(2/3)(0)^{3/2}]$$

$$= [(2/5)(32) + (2/3)(8)] - [0] = (64/5) + (16/3) = (272/15).$$

15. We begin with the following u substitution:

$$u = x^2$$
$$\frac{du}{dx} = 2x$$
$$du = 2x\ dx$$
$$(1/2)\ du = x\ dx$$
$$x = 0 \quad \rightarrow \quad u = (0)^2 = 0$$
$$x = 2 \quad \rightarrow \quad u = (2)^2 = 4$$

and rewrite the integral as

$$\int_0^2 xe^{x^2}\ dx = \int_0^2 e^{x^2}\ x\ dx = \int_0^4 e^u\ (1/2)\ du = (1/2)\int_0^4 e^u\ du$$

Evaluating this integral we find

$$(1/2)\int_0^4 e^u\ du = (1/2)[e^u]|_0^4 = (1/2)[e^4] - (1/2)[e^0] = (1/2)[e^4] - (1/2)[1] = (1/2)(e^4 - 1).$$

17. We begin with the following u substitution:

$$u = 2x$$
$$\frac{du}{dx} = 2$$

$$du = 2\ dx$$
$$(1/2)\ du = \ dx$$
$$x = 0 \quad \rightarrow \quad u = 2(0) = 0$$
$$x = 1 \quad \rightarrow \quad u = 2(1) = 2.$$

Additionally, from the equation $u = 2x$ we solve for x and obtain

$$x = (1/2)u.$$

Next we rewrite the integral as

$$\int_0^1 (e^{2x} + x^2 + 1)\ dx = \int_0^2 (e^u + [(1/2)u]^2 + 1)\ (1/2)\ du = (1/2)\int_0^2 (e^u + (1/4)u^2 + 1)\ du.$$

Evaluating this integral we find

$$(1/2)\int_0^2 (e^u + (1/4)u^2 + 1)\ du = (1/2)[e^u + (1/12)u^3 + u]|_0^2$$
$$= (1/2)[e^2 + (1/12)(2)^3 + (2)] - (1/2)[e^0 + (1/12)(0)^3 + (0)] = (1/2)[e^2 + (1/12)(8) + (2)] - (1/2)[e^0]$$
$$= (1/2)[e^2 + (2/3) + (2)] - (1/2)[1] = (1/2)[e^2 + (8/3)] - (1/2)[1] = (1/2)(e^2 + (5/3)).$$

19. We begin with the following u substitution:

$$u = x^2 + 1$$
$$\frac{du}{dx} = 2x$$
$$du = 2x\ dx$$
$$(1/2)\ du = x\ dx$$
$$x = -1 \quad \rightarrow \quad u = (-1)^2 + 1 = 2$$
$$x = 1 \quad \rightarrow \quad u = (1)^2 + 1 = 2$$

and rewrite the integral as

$$\int_{-1}^{1} xe^{x^2+1}\ dx = \int_{-1}^{1} e^{x^2+1}\ x\ dx = \int_{2}^{2} e^{u}\ (1/2)\ du = (1/2)\int_{2}^{2} e^{u}\ du = 0.$$

We note that the integral is equal to zero by Property 1 on page 442.

21. We begin with the following u substitution:

$$\begin{aligned} u &= x - 2 \\ \frac{du}{dx} &= 1 \\ du &= \ dx \\ x = 3 \quad \rightarrow \quad u &= (3) - 2 = 1 \\ x = 6 \quad \rightarrow \quad u &= (6) - 2 = 4 \end{aligned}$$

and rewrite the integral as

$$\int_{3}^{6} \frac{2}{x-2}\ dx = \int_{1}^{4} \frac{2}{u}\ du = \int_{1}^{4}\ 2u^{-1}\ du.$$

Evaluating this integral we find

$$\int_{1}^{4} 2u^{-1}\ du = [2\ln|u|]|_{1}^{4} = [2\ln 4] - [2\ln 1] = [2\ln 4] - [0] = 2\ln 4.$$

23. We begin with the following u substitution:

$$\begin{aligned} u &= x^3 + 3x^2 - 1 \\ \frac{du}{dx} &= 3x^2 + 6x \\ du &= 3x^2 + 6x\ dx \\ du &= 3(x^2 + 2x)\ dx \\ (1/3)du &= (x^2 + 2x)\ dx \\ x = 1 \quad \rightarrow \quad u &= (1)^3 + 3(1)^2 - 1 = 3 \end{aligned}$$

$$x = 2 \quad \rightarrow \quad u = (2)^3 + 3(2)^2 - 1 = 19$$

and rewrite the integral as

$$\int_1^2 \frac{x^2 + 2x}{x^3 + 3x^2 - 1}\, dx = \int_3^{19} \frac{1}{u}\, (1/3)\; du = (1/3)\int_3^{19} u^{-1}\; du.$$

Evaluating this integral we find

$$(1/3)\int_3^{19} u^{-1}\; du = (1/3)[\ln|u|]|_{-5}^{19} = (1/3)[\ln 19] - (1/3)[\ln 3] = (1/3)[\ln 19 - \ln 3].$$

25. We begin with the following u substitution:

$$v = 2u$$
$$\frac{dv}{du} = 2$$
$$dv = 2\; du$$
$$(1/2)\; dv = \; du$$
$$u = 1 \quad \rightarrow \quad v = 2(1) = 2$$
$$u = 2 \quad \rightarrow \quad v = 2(2) = 4.$$

Additionally, from the equation $v = 2u$ we solve for u and obtain

$$u = (1/2)v.$$

Next we rewrite the integral as

$$\int_1^2 \left(4e^{2u} - \frac{1}{u}\right) du = \int_2^4 \left(4e^v - \frac{1}{(1/2)v}\right) (1/2)\; dv = (1/2)\int_2^4 \left(4e^v - \frac{2}{v}\right) du = (1/2)\int_2^4 (4e^v - 2v^{-1})\; dv.$$

Evaluating this integral we find

$$(1/2)\int_2^4 (4e^v - 2v^{-1})\ dv = (1/2)[4e^v - 2\ln|v|]|_2^4 = (1/2)[4e^4 - 2\ln 4] - (1/2)[4e^2 - 2\ln 2]$$

$$= 2e^4 - \ln 4 - 2e^2 + \ln 2 = 2e^4 - 2e^2 + \ln(2/4) = 2e^4 - 2e^2 + \ln(1/2) = 2e^4 - 2e^2 + \ln 2^{-1} = 2e^4 - 2e^2 - \ln 2.$$

27. We begin with the following u substitution:

$$u = -4x$$
$$\frac{du}{dx} = -4$$
$$du = -4\ dx$$
$$(-1/4)\ du = \ dx$$
$$x = 1 \quad \rightarrow \quad u = -4(1) = -4$$
$$x = 2 \quad \rightarrow \quad u = -4(2) = -8.$$

Additionally, from the equation $u = -4x$ we solve for x and obtain

$$x = -(1/4)u.$$

Next we rewrite the integral as

$$\int_1^2 \left(2e^{-4x} - \frac{1}{x^2}\right)\ dx = \int_{-4}^{-8} \left(2e^u - \frac{1}{[-(1/4)u]^2}\right)(-1/4)\ du$$

$$= -(1/4)\int_{-4}^{-8} \left(2e^u - \frac{16}{u^2}\right)\ du = (1/4)\int_{-8}^{-4} (2e^u - 16u^{-2})\ du.$$

Evaluating this integral we find

$$(1/4)\int_{-8}^{-4} (2e^u - 16u^{-2})\ du = (1/4)[2e^u + 16u^{-1}]|_{-8}^{-4} = (1/4)[2e^{-4} + 16(-4)^{-1}] - (1/4)[2e^{-8} + 16(-8)^{-1}]$$

$$= (1/4)[2e^{-4} - 4] - (1/4)[2e^{-8} - 2] = (1/2)e^{-4} - (1/2)e^{-8} - (1/2).$$

29. $f(x) = x^2 - 2x + 2 = 0 \;\rightarrow\; x = \frac{2 \pm \sqrt{-4}}{2}$ where use was made of the quadratic equation to find the roots; since these roots are complex numbers, the function is never zero and so it must be either positive or negative for all x. Choosing an arbitrary test point we find $f(0) > 0$ so that $f(x) \geq 0$ on $[-1, 2]$ and the area is given by

$$A = \int_{-1}^{2} x^2 - 2x + 2\; dx = [(1/3)x^3 - x^2 + 2x]|_{-1}^{2} = [(1/3)(2)^3 - (2)^2 + 2(2)] - [(1/3)(-1)^3 - (-1)^2 + 2(-1)]$$

$$= [(1/3)(8) - (4) + 2(2)] - [(1/3)(-1) - (1) + 2(-1)] = [(8/3)] - [-(10/3)] = (18/3) = 6.$$

31. $f(x) = \frac{1}{x^2}$ is clearly positive for values of x on $(-\infty, 0) \cup (0, +\infty)$ so that $f(x) \geq 0$ on $[1, 2]$ and the area is given by

$$A = \int_{1}^{2} \frac{1}{x^2}\; dx = \int_{1}^{2} x^{-2}\; dx = [-x^{-1}]|_{1}^{2} = [-(2)^{-1}] - [-(1)^{-1}]$$
$$= [-(1/2)] - [-(1)] = (1/2).$$

33. $f(x) = e^{-x/2}$ is clearly positive for values of x on $(-\infty, 0) \cup (0, +\infty)$ so that $f(x) \geq 0$ on $[-1, 2]$ and the area is given by

$$A = \int_{-1}^{2} e^{-x/2}\; dx.$$

To evaluate this integral we begin with the following u substitution:

$$u = -x/2$$
$$\frac{du}{dx} = -(1/2)$$
$$du = -(1/2)\; dx$$
$$-2\; du = \; dx$$

$$x = -1 \quad \rightarrow \quad u = -(-1)/2 = (1/2)$$
$$x = 2 \quad \rightarrow \quad u = -(2)/2 = -1.$$

We rewrite the integral as

$$\int_{-1}^{2} e^{-x/2}\ dx = \int_{1/2}^{-1} e^{u}\ (-2)\ du = -2\int_{1/2}^{-1} e^{u}\ du = 2\int_{-1}^{1/2} e^{u}\ du.$$

Evaluating this integral we find

$$2\int_{-1}^{1/2} e^{u}\ du = 2[e^{u}]|_{-1}^{1/2} = 2[e^{1/2}] - 2[e^{-1}] = 2e^{1/2} - 2e^{-1} = 2\sqrt{e} - \frac{2}{e}.$$

35. The average value is given by

$$\frac{1}{2-0}\int_{0}^{2} 2x + 3\ dx = \frac{1}{2}[x^2 + 3x]|_0^2 = \frac{1}{2}[(2)^2 + 3(2)] - \frac{1}{2}[(0)^2 + 3(0)] = \frac{1}{2}[10] - \frac{1}{2}[0] = 5.$$

37. The average value is given by

$$\frac{1}{3-1}\int_{1}^{3} 2x^2 - 3\ dx = \frac{1}{2}[(2/3)x^3 - 3x]|_1^3 = \frac{1}{2}[(2/3)(3)^3 - 3(3)] - \frac{1}{2}[(2/3)(1)^3 - 3(1)]$$
$$= \frac{1}{2}[9] - \frac{1}{2}[-(7/3)] = (17/3).$$

39. The average value is given by

$$\frac{1}{2-(-1)}\int_{-1}^{2} x^2 + 2x - 3\ dx = \frac{1}{3}[(1/3)x^3 + x^2 - 3x]|_{-1}^{2}$$

$$= \frac{1}{3}[(1/3)(2)^3 + (2)^2 - 3(2)] - \frac{1}{3}[(1/3)(-1)^3 + (-1)^2 - 3(-1)] = \frac{1}{3}[2/3] - \frac{1}{3}[11/3] = -1.$$

41. The average value is given by

$$\frac{1}{4-0}\int_0^4 \sqrt{2x+1}\ dx = \frac{1}{4}\int_0^4 \sqrt{2x+1}\ dx.$$

To evaluate this integral we begin with the following u substitution:

$$u = 2x+1$$
$$\frac{du}{dx} = 2$$
$$du = 2\ dx$$
$$(1/2)\ du = \ dx$$
$$x = 0 \quad \to \quad u = 2(0)+1 = 1$$
$$x = 4 \quad \to \quad u = 2(4)+1 = 9$$

and rewrite the integral as

$$(1/4)\int_0^4 \sqrt{2x+1}\ dx = (1/4)\int_1^9 u^{1/2}\ (1/2)\ du = (1/8)\int_1^9 u^{1/2}\ du.$$

Evaluating this integral we find

$$(1/8)\int_1^9 u^{1/2}\ du = (1/8)[(2/3)u^{3/2}]|_1^9 = (1/8)[(2/3)(9)^{3/2}] - (1/8)[(2/3)(1)^{3/2}]$$

$$= (1/8)[18] - (1/8)[(2/3)] = (13/6).$$

43. The average value is given by

$$\frac{1}{2-0}\int_0^2 xe^{x^2}\ dx = \frac{1}{2}\int_0^2 xe^{x^2}\ dx.$$

To evaluate this integral we begin with the following u substitution:

$$u = x^2$$

$$\frac{du}{dx} = 2x$$
$$du = 2x\ dx$$
$$(1/2)\ du = x\ dx$$
$$x = 0 \quad \rightarrow \quad u = (0)^2 = 0$$
$$x = 2 \quad \rightarrow \quad u = (2)^2 = 4$$

and rewrite the integral as

$$(1/2)\int_0^2 xe^{x^2}\ dx = (1/2)\int_0^2 e^{x^2}\ x\ dx = (1/2)\int_0^4 e^u\ (1/2)\ du = (1/4)\int_0^4 e^u\ du$$

Evaluating this integral we find

$$(1/4)\int_0^4 e^u\ du = (1/4)[e^u]|_0^4 = (1/4)[e^4] - (1/4)[e^0] = (1/4)[e^4] - (1/4)[1] = (1/4)(e^4 - 1).$$

51. The projected average spending is given by

$$\frac{1}{7-1}\int_1^7 0.86t^{0.96}\ dt = \frac{1}{6}[(0.86/1.96)t^{1.96}]|_1^7 = (1/6)[(0.86/1.96)(7)^{1.96}] - (1/6)[(0.86/1.96)(1)^{1.96}]$$
$$= (1/6)[9.8900] - (1/6)[0.4388] = 3.24 \text{ billion dollars per year.}$$

53. The year 2017 corresponds to $t = 10$ so the gasoline consumption in 2017 is given by

$$A(1) = 0.014(10)^2 + 1.93(10) + 140 = 160.7 \text{ billions of gallons per year.}$$

The average consumption between 2007 ($t = 0$) and 2017 ($t = 10$) is given by

$$\frac{1}{10-0}\int_0^{10} 0.014t^2 + 1.93t + 140\ dt = (1/10)[(1/3)(0.014)t^3 + (1/2)(1.93)t^2 + 140t]|_0^{10}$$
$$= (1/10)[(1/3)(0.014)(10)^3 + (1/2)(1.93)(10)^2 + 140(10)] - (1/10)[(1/3)(0.014)(0)^3 + (1/2)(1.93)(0)^2 + 140(0)]$$
$$= (1/10)[1501] = 150.1 \text{ billions of gallons per year per year.}$$

Section 6.6

1. On the interval $[0, 6]$ the function $f(x) = x^3 - 6x^2$ is less than zero so that the area of the shaded region is given by

$$A = \int_0^6 -(x^3 - 6x^2)\ dx = [-(1/4)x^4 + 2x^3]|_0^6 = [-(1/4)(6)^4 + 2(6)^3] - [-(1/4)(0)^4 + 2(0)^3]$$

$$= [108] - [0] = 108.$$

3. On the interval $[-1, 0]$ the function $f(x) = x\sqrt{1-x^2}$ is less than zero and on the interval $[0, 1]$ the function $f(x) = x\sqrt{1-x^2}$ is greater than zero so that the area of the shaded region is given by

$$A = \int_{-1}^0 -x\sqrt{1-x^2}\ dx + \int_0^1 x\sqrt{1-x^2}\ dx = -\int_{-1}^0 x\sqrt{1-x^2}\ dx + \int_0^1 x\sqrt{1-x^2}\ dx.$$

To evaluate this integral we begin with the following u substitution:

$$u = 1 - x^2$$

$$\frac{du}{dx} = -2x$$

$$du = -2x\ dx$$

$$-(1/2)\ du = x\ dx$$

$$x = -1 \quad \rightarrow \quad u = 1 - (-1)^2 = 0$$

$$x = 0 \quad \rightarrow \quad u = 1 - (0)^2 = 1$$

$$x = 1 \quad \rightarrow \quad u = 1 - (1)^2 = 0$$

and rewrite the integral as

$$-\int_{-1}^0 x\sqrt{1-x^2}\ dx + \int_0^1 x\sqrt{1-x^2}\ dx = -\int_0^1 \sqrt{u}\ (-1/2)\ du + \int_1^0 \sqrt{u}\ (-1/2)\ du$$

$$= (1/2)\int_0^1 \sqrt{u}\ du + (-1/2)\int_1^0 \sqrt{u}\ du = (1/2)\int_0^1 \sqrt{u}\ du + (1/2)\int_0^1 \sqrt{u}\ du$$

$$= \int_0^1 \sqrt{u}\ du = \int_0^1 u^{1/2}\ du.$$

Evaluating this integral we find

$$\int_0^1 u^{1/2}\ du = [(2/3)u^{3/2}]|_0^1 = [(2/3)(1)^{3/2}] - [(2/3)(0)^{3/2}] = (2/3).$$

5. On the interval $[0, 4]$ the function $f(x) = x - 2\sqrt{x}$ is less than zero so that the area of the shaded region is given by

$$A = \int_0^4 -(x - 2\sqrt{x})\ dx = -\int_0^4 (x - 2x^{1/2})\ dx = -[(1/2)x^2 - (4/3)x^{3/2}]|_0^4$$

$$= -[(1/2)(4)^2 - (4/3)(4)^{3/2}] + -[(1/2)(0)^2 - (4/3)(0)^{3/2}]$$

$$= -[(1/2)(4)^2 - (4/3)(4)^{3/2}] + -[(1/2)(0)^2 - (4/3)(0)^{3/2}] = -[-8/3] + [0] = (8/3).$$

7. On the interval $[-1, 0]$, $x^2 \geq x^{1/3}$ and on the interval $[0, 1]$, $x^{1/3} \geq x^2$ so that the area of the shaded region is given by

$$A = \int_{-1}^0 x^2 - x^{1/3}\ dx + \int_0^1 x^{1/3} - x^2\ dx = [(1/3)x^3 - (3/4)x^{4/3}]|_{-1}^0 + [(3/4)x^{4/3} - (1/3)x^3]|_0^1$$

$$= [(1/3)(0)^3 - (3/4)(0)^{4/3}] - [(1/3)(-1)^3 - (3/4)(-1)^{4/3}] + [(3/4)(1)^{4/3} - (1/3)(1)^3] - [(3/4)(0)^{4/3} - (1/3)(0)^3]$$

$$= [0] - [-13/12] + [5/12] - [0] = (18/12) = (3/2).$$

9. On the interval $[-1, 2]$ the function $f(x) = -x^2$ is less than zero so that the area bounded by the curve and the x axis on the interval is given by

$$A = \int_{-1}^2 -(-x^2)\ dx = \int_{-1}^2 x^2\ dx = [(1/3)x^3]|_{-1}^2 = [(1/3)(2)^3] - [(1/3)(-1)^3] = [8/3] - [-1/3] = 3.$$

11. $f(x) = x^2 - 5x + 4 = (x-4)(x-1) = 0 \;\rightarrow\; x = 1, 4$ are the roots. Since the interval $[1, 3]$ is contained in the interval $[1, 4]$, the sign of $f(x)$ will be constant on $[1, 3]$. Choosing the test point $x = 2$, we find $f(2) = -2 < 0$ so that the area bounded by the curve and the x axis on the interval is given by

$$A = \int_1^3 -(x^2 - 5x + 4)\ dx = -\int_1^3 x^2 - 5x + 4\ dx = -[(1/3)x^3 - (5/2)x^2 + 4x]|_1^3$$

$$= -[(1/3)(3)^3 - (5/2)(3)^2 + 4(3)] + [(1/3)(1)^3 - (5/2)(1)^2 + 4(1)] = -[-3/2] + [11/6] = (20/6) = (10/3).$$

13. $f(x) = -1 - \sqrt{x} < 0$ for all x. Thus the area bounded by the curve and the x axis on the interval is given by

$$A = \int_0^9 -(-1 - \sqrt{x})\ dx = \int_0^9 1 + x^{1/2}\ dx = [x + (2/3)x^{3/2}]|_0^9$$

$$= [(9) + (2/3)(9)^{3/2}] - [(0) + (2/3)(0)^{3/2}] = [27] - [0] = 27.$$

15. $f(x) = -e^{(1/2)x} < 0$ for all x. Thus the area bounded by the curve and the x axis on the interval is given by

$$A = \int_{-2}^4 -(-e^{(1/2)x})\ dx = \int_{-2}^4 e^{(1/2)x}\ dx.$$

To evaluate this integral we begin with the following u substitution:

$$u = (1/2)x$$

$$\frac{du}{dx} = (1/2)$$

$$du = (1/2)\ dx$$

$$2\ du = \ dx$$

$$x = -2 \quad \rightarrow \quad u = (1/2)(-2) = -1$$

$$x = 4 \quad \rightarrow \quad u = (1/2)(4) = 2$$

and rewrite the integral as

$$\int_{-2}^{4} e^{(1/2)x}\ dx = \int_{-1}^{2} e^{u}\ (2)\ du = 2\int_{-1}^{2} e^{u}\ du.$$

Evaluating this integral we find

$$2\int_{-1}^{2} e^{u}\ du = 2[e^{u}]|_{-1}^{2} = 2[e^{2}] - 2[e^{-1}] = 2(e^{2} - e^{-1}).$$

17. $f(x) - g(x) = x^2 + 2 > 0$ for all x. Thus the area bounded by the curves on $[1, 3]$ is given by

$$A = \int_{1}^{3} f(x) - g(x)\ dx = \int_{1}^{3} x^2 + 2\ dx = [(1/3)x^3 + 2x]|_{1}^{3}$$
$$= [(1/3)(3)^3 + 2(3)] - [(1/3)(1)^3 + 2(1)] = [15] - [7/3] = (38/3).$$

19. $f(x) - g(x) = -x^2 + 3x = -x(x - 3) = 0 \ \rightarrow\ x = 0, 3$ are the roots. Since the interval $[0, 2]$ is contained in the interval $[0, 3]$, the sign of $f(x) - g(x)$ will be constant on $[0, 2]$. Choosing the test point $x = 2$, we find $f(2) - g(2) = 2 > 0$ so that the area bounded by the curves on $[0, 2]$ is given by

$$A = \int_{0}^{2} f(x) - g(x)\ dx = \int_{1}^{3} -x^2 + 3x\ dx = [-(1/3)x^3 + (3/2)x^2]|_{0}^{2}$$
$$= [-(1/3)(2)^3 + (3/2)(2)^2] - [-(1/3)(0)^3 + (3/2)(0)^2] = [15] - [0] = (10/3).$$

21. $f(x) - g(x) = x^2 + 1 - (1/3)x^3$. Unfortunately, this third degree polynomial is not readily factored so a different approach is required to determine the sign of the function on $[-1, 2]$. We form a new function $h(x) = f(x) - g(x)$ and note

$$h(x) = -(1/3)x^3 + x^2 + 1$$
$$h'(x) = -x^2 + 2x = -x(x - 2)$$
$$h''(x) = -2x + 2 = -2(x + 1).$$

For $x < 0$, $f(x) > 0$ and $g(x) < 0$ so that $h(x) = (f(x) - g(x)) > 0$ on $(\infty, 0]$. The function $h(x)$ has two critical numbers at $x = 0, 2$ with $h''(0) > 0$ and $h''(2) < 0$ so that $h(0) = 1$ is a local min and $h(2) = (7/3)$ is a local max. This information implies that on the interval $(-\infty, 2]$, $h(x) \geq 1 > 0$. Thus the area bounded by the curves on $[-1, 2]$ is given by

$$A = \int_{-1}^{2} f(x) - g(x)\ dx = \int_{-1}^{2} -(1/3)x^3 + x^2 + 1\ dx = [-(1/12)x^4 + (1/3)x^3 + x]|_{-1}^{2}$$

$$= [-(1/12)(2)^4 + (1/3)(2)^3 + (2)] - [-(1/12)(-1)^4 + (1/3)(-1)^3 + (-1)] = [10/3] - [-17/12] = (19/4).$$

23. $f(x) - g(x) = \frac{1}{x} - 2x + 1 = \frac{1}{x} - \frac{2x^2}{x} + \frac{x}{x} = \frac{-2x^2+x+1}{x} = \frac{-(2x+1)(x-1)}{x} = 0 \rightarrow$ $x = -(1/2), 1$ are the roots. Since the interval $[1, 4]$ is contained in the interval $[1, +\infty)$, the sign of $f(x)-g(x)$ will be constant on $[1, 4]$. Choosing the test point $x = 2$, we find $f(2)-g(2) = -(5/2) < 0$ so that the area bounded by the curves on $[1, 4]$ is given by

$$A = \int_{1}^{4} g(x) - f(x)\ dx = \int_{1}^{4} 2x - 1 - \frac{1}{x}\ dx = [x^2 - x - \ln|x|]|_{1}^{4}$$

$$= [(4)^2 - (4) - \ln|4|] - [(1)^2 - (1) - \ln|1|] = [12 - \ln 4] - [0] = 12 - \ln 4.$$

25. $f(x) - g(x) = e^x - \frac{1}{x}$. For $x > 1$, $e^x > 1$ and $\frac{1}{x} < 1$ so that $f(x) - g(x) > 0$ on the interval $[1, 2]$. Thus the area bounded by the curves on $[1, 2]$ is given by

$$A = \int_{1}^{2} f(x) - g(x)\ dx = \int_{1}^{2} e^x - \frac{1}{x}\ dx = [e^x - \ln|x|]|_{1}^{2}$$

$$= [e^2 - \ln|2|] - [e^1 - \ln|1|] = [e^2 - \ln 2] - [e] = e^2 - e - \ln 2.$$

27. $f(x) = x = 0 \rightarrow x = 0$ is the only root. Since $0 \in (-1, 2)$, we choose test points in the interval on either side of the root and determine the sign of $f(x)$ at these points. We note $f(-1) < 0$ and $f(1) > 0$ so that the area bounded by the curve and the x axis on the interval is given by

$$A = \int_{-1}^{0} -f(x)\ dx + \int_{0}^{2} f(x)\ dx = \int_{-1}^{0} -x\ dx + \int_{0}^{2} x\ dx = [-(1/2)x^2]|_{-1}^{0} + [(1/2)x^2]|_{0}^{2}$$

$$= ([-(1/2)(0)^2] - [-(1/2)(-1)^2]) + ([(1/2)(2)^2] - [(1/2)(0)^2]) = ([0] - [-(1/2)]) + ([2] - [0]) = (5/2).$$

29. $f(x) = -x^2+4x-3 = -(x-1)(x-3) = 0 \;\rightarrow\; x = 1, 3$ are the roots. Since only $1 \in (-1, 2)$, we choose test points in the interval on either side of this root and determine the sign of $f(x)$ at these points. We note $f(-1) < 0$ and $f(2) > 0$ so that the area bounded by the curve and the x axis on the interval is given by

$$\begin{aligned}
A = \int_{-1}^{1} -f(x)\ dx + \int_{1}^{2} f(x)\ dx &= \int_{-1}^{1} x^2 - 4x + 3\ dx + \int_{1}^{2} -x^2 + 4x - 3\ dx \\
&= [(1/3)x^3 - 2x^2 + 3x]|_{-1}^{1} + [-(1/3)x^3 + 2x^2 - 3x]|_{1}^{2} \\
&= ([(1/3)(1)^3 - 2(1)^2 + 3(1)] - [(1/3)(-1)^3 - 2(-1)^2 + 3(-1)]) \\
&\quad +([-(1/3)(2)^3 + 2(2)^2 - 3(2)] - [-(1/3)(1)^3 + 2(1)^2 - 3(1)]) \\
&= ([4/3] - [-(16/3)]) + ([-2/3] - [-4/3]) = (22/3).
\end{aligned}$$

31. $f(x) = x^3 - 4x^2 + 3x = x(x-1)(x-3) = 0 \;\rightarrow\; x = 0, 1, 3$ are the roots. Since only $1 \in (0, 2)$, we choose test points in the interval on either side of this root and determine the sign of $f(x)$ at these points. We note $f(1/2) > 0$ and $f(3/2) < 0$ so that the area bounded by the curve and the x axis on the interval is given by

$$\begin{aligned}
A = \int_{0}^{1} f(x)\ dx + \int_{1}^{2} -f(x)\ dx &= \int_{0}^{1} x^3 - 4x^2 + 3x\ dx + \int_{1}^{2} -x^3 + 4x^2 - 3x\ dx \\
&= [(1/4)x^4 - (4/3)x^3 + (3/2)x^2]|_{0}^{1} + [-(1/4)x^4 + (4/3)x^3 - (3/2)x^2]|_{1}^{2} \\
&= ([(1/4)(1)^4 - (4/3)(1)^3 + (3/2)(1)^2] - [(1/4)(0)^4 - (4/3)(0)^3 + (3/2)(0)^2]) \\
&\quad +([-(1/4)(2)^4 + (4/3)(2)^3 - (3/2)(2)^2] - [-(1/4)(1)^4 + (4/3)(1)^3 - (3/2)(1)^2]) \\
&= ([4/3] - [-(16/3)]) + ([-2/3] - [-4/3]) = (3/2).
\end{aligned}$$

33. $f(x) = e^x - 1 = 0 \;\rightarrow\; x = 0$ is the only root. Since $0 \in (-1, 3)$, we choose test points in the interval on either side of this root and determine the sign of $f(x)$ at these points. We note $f(-1) < 0$ and $f(2) > 0$ so that the area bounded by the curve and the x axis on the interval is given by

$$A=\int_{-1}^{0} -f(x)\ dx+\int_{0}^{3} f(x)\ dx=\int_{-1}^{0} -e^x+1\ dx+\int_{0}^{3} e^x-1\ dx$$

$$=[-e^x+x]|_{-1}^{0}+[e^x-x]|_0^3=([-e^0+(0)]-[-e^{-1}+(-1)])+([e^3-(3)]-[e^0-(0)])$$

$$=([-1]-[-e^{-1}-1])+([e^3-3]-[1])=e^3+e^{-1}-4.$$

35. $f(x)-g(x)=-x^2+x+6=-(x-3)(x+2)=0 \ \rightarrow\ x=-2,3$ are the roots; hence the bounded region will occur for x values in the interval $[-2,3]$. We evaluate $(f(x)-g(x))$ at a test point on this interval and find $(f(0)-g(0))>0$ so that the area bounded by the curves is given by

$$A=\int_{-2}^{3} f(x)-g(x)\ dx=\int_{-2}^{3} -x^2+x+6\ dx=[-(1/3)x^3+(1/2)x^2+6x]|_{-2}^{3}$$

$$=[-(1/3)(3)^3+(1/2)(3)^2+6(3)]-[-(1/3)(-2)^3+(1/2)(-2)^2+6(-2)]=[27/2]-[-22/3]=(125/6).$$

37. $f(x)-g(x)=x^2-x^3=-x^2(x-1)=0 \ \rightarrow\ x=0,1$ are the roots; hence the bounded region will occur for x values in the interval $[0,1]$. We evaluate $(f(x)-g(x))$ at a test point on this interval and find $(f(1/2)-g(1/2))>0$ so that the area bounded by the curves is given by

$$A=\int_{0}^{1} f(x)-g(x)\ dx=\int_{0}^{1} x^2-x^3\ dx=[(1/3)x^3-(1/4)x^4]|_0^1$$

$$=[(1/3)(1)^3-(1/4)(1)^4]-[(1/3)(0)^3-(1/4)(0)^4]=[1/12]-[0]=(1/12).$$

39. $f(x)-g(x)=x^3-7x^2+12x=x(x-4)(x-3)=0 \ \rightarrow\ x=0,3,4$ are the roots; hence the bounded region will occur for x values in the interval $[0,4]$. We evaluate $(f(x)-g(x))$ at a test points on this interval on either side of $x=3$ and find $(f(1)-g(1))>0$ and $(f(3.5)-g(3.5))<0$ so that the area bounded by the curves is given by

$$A=\int_{0}^{3} f(x)-g(x)\ dx+\int_{3}^{4} -(f(x)-g(x))\ dx=\int_{0}^{3} x^3-7x^2+12x\ dx+\int_{3}^{4} -x^3+7x^2-12x\ dx$$

$$=[(1/4)x^4-(7/3)x^3+6x^2]|_0^3+[-(1/4)x^4+(7/3)x^3-6x^2]|_3^4$$

$$=([(1/4)(3)^4-(7/3)(3)^3+6(3)^2]-[(1/4)(0)^4-(7/3)(0)^3+6(0)^2])$$

$$+([-(1/4)(4)^4+(7/3)(4)^3-6(4)^2]-[-(1/4)(3)^4+(7/3)(3)^3-6(3)^2])$$
$$=([45/4]-[0])+([45/4]-[32/3])=(71/6).$$

41. $f(x)-g(x)=x\sqrt{9-x^2}=x\sqrt{(3-x)(3+x)}=0 \ \rightarrow \ x=-3,0,3$ are the roots; hence the bounded region will occur for x values in the interval $[-3,3]$ (Note that this is also the domain of the function $(f(x)-g(x))$). We evaluate $(f(x)-g(x))$ at a test points on this interval on either side of $x=0$ and find $(f(-1)-g(-1))<0$ and $(f(1)-g(1))>0$ so that the area bounded by the curves is given by

$$A=\int_{-3}^{0} -(f(x)-g(x))\ dx+\int_{0}^{3} f(x)-g(x)\ dx=\int_{-3}^{0} -x\sqrt{9-x^2}\ dx+\int_{0}^{3} x\sqrt{9-x^2}\ dx.$$

To evaluate this integral we begin with the following u substitution:

$$u=9-x^2$$
$$\frac{du}{dx}=-2x$$
$$du=-2x\ dx$$
$$-(1/2)\ du=x\ dx$$
$$x=-3 \quad \rightarrow \quad u=9-(-3)^2=0$$
$$x=0 \quad \rightarrow \quad u=9-(0)^2=9$$
$$x=3 \quad \rightarrow \quad u=9-(3)^2=0$$

and rewrite the integral as

$$\int_{-3}^{0} -x\sqrt{9-x^2}\ dx+\int_{0}^{3} x\sqrt{9-x^2}\ dx=\int_{0}^{9} -\sqrt{u}\ (-1/2)\ du+\int_{9}^{0} \sqrt{u}\ (-1/2)\ du.$$

$$=(1/2)\int_{0}^{9} \sqrt{u}\ du+(1/2)\int_{0}^{9} \sqrt{u}\ du=\int_{0}^{9} \sqrt{u}\ du.$$

Evaluating this integral we find

$$\int_0^9 \sqrt{u}\ du = \int_0^9 u^{1/2}\ du = [(2/3)u^{3/2}]|_0^9 = [(2/3)(9)^{3/2}] - [(2/3)(0)^{3/2}] = [18] - [0] = 18.$$

43. The area S can be thought of mathematically as the area between the curves $g(x)$ and $f(x)$ on the interval $[0, b]$. This is given by

$$S = \int_0^b (g(x) - f(x))\ dx = \int_0^b g(x)\ dx - \int_0^b f(x)\ dx.$$

The integral $\int_0^b g(x)\ dx$ gives the company's revenue under the new advertising agency (when b dollars is spent on advertising) while the integral $\int_0^b f(x)\ dx$ gives the company's revenue under the old advertising agency (when b dollars is spent on advertising). Thus S is the net change in the company's revenue when switching to the new agency.

51. The increase in cars sales over the next five years is given by the integral

$$\int_0^5 5e^{0.3t} - (5 + 0.5t^{3/2})\ dt = \int_0^5 5e^{0.3t}\ dt - \int_0^5 (5 + 0.5t^{3/2})\ dt.$$

We evaluate these integrals separately.

For the first integral we begin with following u substitution:

$$\begin{aligned} u &= 0.3t \\ \frac{du}{dt} &= 0.3 \\ du &= 0.3\ dt \\ (10/3)\ du &= \ dt \\ t = 0 \quad &\rightarrow \quad u = 0.3(0) = 0 \\ t = 5 \quad &\rightarrow \quad u = 0.3(5) = 1.5 \end{aligned}$$

and rewrite the integral as

$$\int_0^5 5e^{0.3t}\ dt = \int_0^{1.5} 5e^u\ (10/3)\ du = (50/3)\int_0^{1.5} e^u\ du.$$

Evaluating this integral we find

$$(50/3)\int_0^{1.5} e^u\ du = (50/3)[e^u]|_0^{1.5} = (50/3)[e^{1.5}] - (50/3)[e^0] = (50/3)(e^{1.5} - 1).$$

The second integral is evaluated directly

$$\int_0^5 (5 + 0.5t^{3/2})\ dt = [5t + 0.2t^{5/2}]|_0^5 = [5(5) + 0.2(5)^{5/2}] - [5(0) + 0.2(0)^{5/2}]$$
$$= [25 + 0.2(5)^{5/2}] - [0] = 25 + 0.2(5)^{5/2}.$$

Thus the increase in cars sales is given by

$$\int_0^5 5e^{0.3t} - (5 + 0.5t^{3/2})\ dt = (50/3)(e^{1.5} - 1) - (25 + 0.2(5)^{5/2}) \approx 21.848 \text{ thousand cars.}$$