

CHAPTER 8 HOMEWORK SOLUTIONS

Note: the first number corresponds to the homework problem from the 5th edition while the second number in parentheses corresponds to the homework problem in the 6th edition. If the problem does not occur in the 6th edition, the symbol (–) is present.

SECTION 8.1

$$1, (1). \quad y' = x^2 + y^2, \quad y(0) = 1$$

The Taylor series about zero has the form

$$y(x) = \sum_{n=0}^{\infty} y^n(0)x^n/n! = y(0) + y'(0)x + y''(0)x^2/2 + y'''(0)x^3/6 + \cdots.$$

To compute the first three nonzero terms we need the first three derivatives of y which do not vanish when evaluated at zero:

$$y(0) = 1$$

$$y' = x^2 + y^2 \quad \longrightarrow \quad y'(0) = 0^2 + 1^2 = 1$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(x^2 + y^2) = 2x + 2yy' \quad \longrightarrow \quad y''(0) = 2(0) + 2(1)(1) = 2.$$

Hence

$$y(x) = 1 + x + 2x^2/2 + \cdots$$

$$y(x) = 1 + x + x^2 + \cdots.$$

$$3, (3). \quad y' = \sin y + e^x, \quad y(0) = 0$$

The Taylor series about zero has the form

$$y(x) = \sum_{n=0}^{\infty} y^n(0)x^n/n! = y(0) + y'(0)x + y''(0)x^2/2 + y'''(0)x^3/6 + \cdots.$$

To compute the first three nonzero terms we need the first three derivatives of y which do not vanish when evaluated at zero:

$$y(0) = 0$$

$$y' = \sin y + e^x \longrightarrow y'(0) = \sin 0 + e^0 = 1$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(\sin y + e^x) = y' \cos y + e^x \longrightarrow y''(0) = 1 \cdot \cos 0 + e^0 = 2$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(y' \cos y + e^x) = y'' \cos y - (y')^2 \sin y + e^x \longrightarrow y'''(0) = 2 \cdot \cos 0 - (1)^2 \cdot \sin 0 + e^0 = 3.$$

Hence

$$y(x) = x + 2x^2/2 + 3x^3/6 + \dots$$

$$y(x) = x + x^2 + x^3/2 + \dots$$

5, (5). $x'' + tx = 0$, $x(0) = 1$, $x'(0) = 0$

The Taylor series about zero has the form

$$x(t) = \sum_{n=0}^{\infty} x^{(n)}(0)t^n/n! = x(0) + x'(0)t + x''(0)t^2/2 + x'''(0)t^3/6 + \dots$$

To compute the first three nonzero terms we need the first three derivatives of x which do not vanish when evaluated at zero:

$$x(0) = 1$$

$$x'(0) = 0$$

$$x'' = -tx \longrightarrow x''(0) = 0(1) = 0$$

$$x''' = \frac{d}{dt}(x'') = \frac{d}{dt}(-tx) = -x - tx' \longrightarrow x'''(0) = -1 - 0(0) = -1$$

$$x^{iv} = \frac{d}{dt}(x''') = \frac{d}{dt}(-x - tx') = -2x' - tx'' \longrightarrow x^{iv}(0) = -2(0) - 0(0) = 0$$

$$x^v = \frac{d}{dt}(x^{iv}) = \frac{d}{dt}(-2x' - tx'') = -3x'' - tx''' \longrightarrow x^v(0) = -3(0) - 0(-1) = 0$$

$$x^{vi} = \frac{d}{dt}(x^v) = \frac{d}{dt}(-3x'' - tx''') = -4x''' - tx^{iv} \longrightarrow x^{vi}(0) = -4(-1) - 0(0) = 4$$

Hence

$$x(t) = 1 + 0(t) + 0(t^2)/2 - 1(t^3)/6 + 0(t^4)/24 + 0(t^5)/120 + 4(t^6)/720 + \dots$$

$$x(t) = 1 - t^3/6 + t^6/180 + \dots$$

7, (7). $y''(\theta) + y(\theta)^3 = \sin \theta$, $y(0) = 0$, $y'(0) = 0$

The Taylor series about zero has the form

$$y(\theta) = \sum_{n=0}^{\infty} y^{(n)}(0)\theta^n/n! = y(0) + y'(0)\theta + y''(0)\theta^2/2 + y'''(0)\theta^3/6 + \dots$$

To compute the first three nonzero terms we need the first three derivatives of y which do not vanish when evaluated at zero:

$$y(0) = 0$$

$$y'(0) = 0$$

$$y(\theta)'' = -y(\theta)^3 + \sin \theta \longrightarrow y''(0) = -0^3 + 0 = 0$$

$$y(\theta)''' = \frac{d}{d\theta}(y''(\theta)) = \frac{d}{d\theta}(-y(\theta)^3 + \sin \theta) = -3y(\theta)^2 y'(\theta) + \cos(\theta) \longrightarrow y'''(0) = -3(0)(0) + 1 = 1$$

$$y^{iv}(\theta) = \frac{d}{d\theta}(y'''(\theta)) = \frac{d}{d\theta}(-3y(\theta)^2 y'(\theta) + \cos(\theta)) = -6y(\theta)(y'(\theta))^2 - 3y(\theta)^2 y''(\theta) - \sin \theta$$

$$\longrightarrow y^{iv}(0) = -6(0)(0)^2 - 3(0)^2(0) - 0 = 0$$

$$y^v(\theta) = \frac{d}{d\theta}(y^{iv}(\theta)) = \frac{d}{d\theta}(-6y(\theta)(y'(\theta))^2 - 3y(\theta)^2 y''(\theta) - \sin \theta)$$

$$\begin{aligned}
&= -6(y'(\theta))^3 - 12y(\theta)y'(\theta)y''(\theta) - 6y(\theta)y'(\theta)y''(\theta) - 3y(\theta)^2y'''(\theta) - \cos \theta \\
&= -6(y'(\theta))^3 - 18y(\theta)y'(\theta)y''(\theta) - 3y(\theta)^2y'''(\theta) - \cos \theta \\
&\longrightarrow y^v(0) = -6(0)^3 - 18(0)(0)(0) - 3(0)(1) - 1 = -1
\end{aligned}$$

$$\begin{aligned}
y^{vi}(\theta) &= \frac{d}{dt}(y^v(\theta)) = \frac{d}{d\theta}(-6(y'(\theta))^3 - 18y(\theta)y'(\theta)y''(\theta) - 3y(\theta)^2y'''(\theta) - \cos \theta) \\
&= -18(y'(\theta))^2y''(\theta) - 18(y'(\theta))^2y''(\theta) - 18y(\theta)(y''(\theta))^2 - 18y(\theta)y'(\theta)y'''(\theta) - 6y(\theta)y'(\theta)y'''(\theta) \\
&\quad - 3y(\theta)^2y^{iv}(\theta) + \sin \theta \\
&= -36(y'(\theta))^2y''(\theta) - 18y(\theta)(y''(\theta))^2 - 24y(\theta)y'(\theta)y'''(\theta) - 3y(\theta)^2y^{iv}(\theta) + \sin \theta \\
&\longrightarrow y^{vi}(0) = -36(0)^2(0) - 18(0)(0)^2 - 24(0)(0)(1) - 3(0)^2(0) + 0 = 0
\end{aligned}$$

$$\begin{aligned}
y^{vii}(\theta) &= \frac{d}{dt}(y^{vi}(\theta)) = \frac{d}{d\theta}(-36(y'(\theta))^2y''(\theta) - 18y(\theta)(y''(\theta))^2 - 24y(\theta)y'(\theta)y'''(\theta) - 3y(\theta)^2y^{iv}(\theta) + \sin \theta) \\
&= -72y'(\theta)(y''(\theta))^2 - 36(y'(\theta))^2y'''(\theta) - 18y'(\theta)(y''(\theta))^2 - 36y(\theta)y''(\theta)y'''(\theta) - 24(y'(\theta))^2y'''(\theta) \\
&\quad - 24y(\theta)y''(\theta)y'''(\theta) - 24y(\theta)y'(\theta)y^{iv}(\theta) - 6y(\theta)y'(\theta)y^{iv}(\theta) - 3y(\theta)^2y^v(\theta) + \cos \theta \\
&= -90y'(\theta)(y''(\theta))^2 - 60(y'(\theta))^2y'''(\theta) - 60y(\theta)y''(\theta)y'''(\theta) - 30y(\theta)y'(\theta)y^{iv}(\theta) - 3y(\theta)^2y^v(\theta) + \cos \theta \\
&\longrightarrow y^{vii}(0) = -90(0)(0)^2 - 60(0)^2(1) - 60(0)(0)(0) - 30(0)(0)(0) - 3(0)^2(0) + 1 = 1
\end{aligned}$$

Hence

$$\begin{aligned}
y(\theta) &= 0 + 0(\theta) + 0(\theta^2)/2 + 1(\theta^3)/6 + 0(\theta^4)/24 - 1(\theta^5)/120 + 0(\theta^6)/720 + 1(\theta^7)/5040 + \dots \\
y(\theta) &= \theta^3/6 - \theta^5/120 + \theta^7/5040 + \dots
\end{aligned}$$

SECTION 8.3

11, (11). $y' + (x + 2)y = 0$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + (x + 2) \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 2 a_n x^n = 0.$$

Performing a shift so that the exponents of the first and second terms are equal to n we obtain

$$\sum_{n=-1}^{\infty} (n + 1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

Explicitly writing the first two terms from the first series and the first term from the third series yields

$$0(a_0)x^{-1} + 1(a_1) + \sum_{n=1}^{\infty} (n + 1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + 2a_0 + \sum_{n=1}^{\infty} 2 a_n x^n = 0$$

$$2a_0 + a_1 + \sum_{n=1}^{\infty} [(n + 1) a_{n+1} + a_{n-1} + 2a_n] x^n = 0.$$

From this last equation we obtain the relations

$$2a_0 + a_1 = 0$$

$$a_{n+1} = -a_{n-1}/(n + 1) - 2a_n/(n + 1), \quad n \geq 0.$$

From these equations we can write

$$a_1 = -2a_0$$

$$n = 1, \quad a_2 = -a_0/2 - a_1 = 3a_0/2$$

$$n = 2, \quad a_3 = -a_1/3 - 2a_2/3 = 2a_0/3 - a_0 = -a_0/3.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$y = a_0(1 - 2x + 3x^2/2 - x^3/3 + \dots).$$

13, (13). $z'' - x^2 z = 0$

We proceed directly by writing

$$z = \sum_{n=0}^{\infty} a_n x^n$$

$$z' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$z'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Performing a shift so that the exponents are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

Explicitly writing the first four terms from the first series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + 2a_2 + 6a_3x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-2}]x^n = 0.$$

From this last equation we obtain the relations

$$a_2 = 0$$

$$a_3 = 0$$

$$a_{n+2} = a_{n-2}/[(n+2)(n+1)], \quad n \geq 2.$$

From these equations we can write

$$n = 2, \quad a_4 = a_0/12$$

$$n = 3, \quad a_5 = a_1/20.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$y = a_0(1 + x^4/12 + \dots) + a_1(x + x^5/20 + \dots).$$

15, (15). $y'' + (x-1)y' + y = 0$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + (x-1) \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponents of the first and third series are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_nx^n - \sum_{n=-1}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_nx^n = 0.$$

Explicitly writing the first two terms from the first series and the first term from the third series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_nx^n - 0(a_0)x^{-1} - \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_nx^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n - (n+1)a_{n+1} + a_n]x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + (n+1)a_n]x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = a_{n+1}/(n+2) - a_n/(n+2), \quad n \geq 0.$$

From these equations we can write

$$n = 0, \quad a_2 = a_1/2 - a_0/2$$

$$n = 1, \quad a_3 = a_2/3 - a_1/3 = -a_1/6 - a_0/6.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$y = a_0(1 - x^2/2 - x^3/6 + \dots) + a_1(x + x^2/2 - x^3/6 + \dots).$$

17, (17). $w'' - x^2w' + w = 0$

We proceed directly by writing

$$w = \sum_{n=0}^{\infty} a_nx^n$$

$$w' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$w'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponents of the first and second series are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Explicitly writing the first three terms from the first series and the first term from the third series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n-1) a_{n-1} x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$a_0 + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$a_0 + 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - (n-1) a_{n-1} + a_n] x^n = 0$$

From this last equation we obtain the relations

$$a_2 = -a_0/2$$

$$a_{n+2} = -a_n/[(n+2)(n+1)] + (n-1)a_{n-1}/[(n+2)(n+1)], \quad n \geq 1.$$

From these equations we can write

$$n = 1, \quad a_3 = -a_1/6 + 0(a_0) = -a_1/6$$

Thus we can write

$$w = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$w = a_0(1 - x^2/2 + \dots) + a_1(x - x^3/6 + \dots).$$

$$19, (19). \quad y' - 2xy = 0$$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

Performing a shift so that the exponents of the first and second series are equal to n we obtain

$$\sum_{n=-1}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} 2a_{n-1}x^n = 0.$$

Explicitly writing the first two term from the first series yields

$$0(a_0)x^{-1} + a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=1}^{\infty} 2a_{n-1}x^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - 2a_{n-1}]x^n = 0.$$

From this last equation we obtain the relations

$$a_1 = 0$$

$$a_{n+1} = 2a_{n-1}/(n+1), \quad n \geq 1.$$

From these relations we can write

$$n = 1, \quad a_2 = 2a_0/2 = a_0 = a_0/1!$$

$$n = 2, \quad a_3 = 2a_1/3 = 0$$

$$n = 3, \quad a_4 = 2a_2/4 = 2^2 a_0/[2 \cdot 4] = 2^2 a_0/[2^2(1 \cdot 2)] = a_0/2!$$

$$n = 4, \quad a_5 = 2a_3/5 = 0$$

$$n = 5, \quad a_6 = 2a_4/6 = 2^3 a_0/[2 \cdot 4 \cdot 6] = 2^3 a_0/[2^3(1 \cdot 2 \cdot 3)] = a_0/3!.$$

Hence for n odd, $n = 2k + 1$, $a_{2k+1} = 0$; for n even, $n = 2k$, $a_{2k} = a_0/k!$. Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{k=0}^{\infty} x^{2k}/k!.$$

21, (21). $y'' - xy' + 4y = 0$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0.$$

Explicitly writing the first two terms from the first series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n + 4 a_n] x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (4-n)a_n]x^n = 0$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = (n-4)a_n/[(n+2)(n+1)], \quad n \geq 0.$$

From these relations we can write

$$n = 0, \quad a_2 = -4a_0/[2 \cdot 1]$$

$$n = 1, \quad a_3 = -3a_1/[3 \cdot 2]$$

$$n = 2, \quad a_4 = -2a_2/[4 \cdot 3] = (2 \cdot 4)a_0/[4 \cdot 3 \cdot 2 \cdot 1]$$

$$n = 3, \quad a_5 = -a_3/[5 \cdot 4] = (-3 \cdot -1)a_1/[5 \cdot 4 \cdot 3 \cdot 2]$$

$$n = 4, \quad a_6 = 0a_4/[6 \cdot 5] = 0$$

$$n = 5, \quad a_7 = a_5/[7 \cdot 6] = (-3 \cdot -1 \cdot 1)a_1/[7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2]$$

$$n = 6, \quad a_8 = 2a_6/[8 \cdot 7] = 0$$

$$n = 7, \quad a_9 = 3a_7/[9 \cdot 8] = (-3 \cdot -1 \cdot 1 \cdot 3)a_1/[9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2]$$

$$n = 8, \quad a_{10} = 4a_8/[10 \cdot 9] = 0$$

$$n = 9, \quad a_{11} = 5a_9/[11 \cdot 10] = (-3 \cdot -1 \cdot 1 \cdot 3 \cdot 5)a_1/[11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2]$$

Hence, $a_0 = a_0$, $a_2 = -2a_0$, $a_4 = a_0/3$ and $a_n = 0$ for all other even subscripts;

for n odd, $n = 2k + 1$, $a_{2k+1} = (-3 \cdot -1 \cdot 1 \cdots (2k-5))a_1/(2k+1)!$.

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0(1 - 2x^2 + x^4/3) + a_1[x + \sum_{k=1}^{\infty} (-3 \cdot -1 \cdot 1 \cdots (2k-5))x^{2k+1}/(2k+1)!].$$

$$23, (23). \quad z'' - x^2 z' - xz = 0$$

We proceed directly by writing

$$z = \sum_{n=0}^{\infty} a_n x^n$$

$$z' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$z'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

and substituting into the above equation obtain

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} na_n x^{n-1} - x \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} &= 0. \end{aligned}$$

Performing a shift so that the exponents of all the series are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n-1)a_{n-1}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

Explicitly writing the first three terms from the first series yields

$$\begin{aligned} 0(a_0)x^{-2} + 0(a_1)x^{-1} + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n-1)a_{n-1}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n &= 0 \\ 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} - a_{n-1}]x^n &= 0 \\ 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - na_{n-1}]x^n &= 0. \end{aligned}$$

From this last equation we obtain the relations

$$a_2 = 0$$

$$a_{n+2} = na_{n-1}/[(n+2)(n+1)], \quad n \geq 1.$$

From these relations we can write

$$\begin{aligned} n = 1, \quad a_3 &= 1 \cdot a_0 / [3 \cdot 2] \\ n = 2, \quad a_4 &= 2a_1 / [4 \cdot 3] \\ n = 3, \quad a_5 &= 3a_2 / [5 \cdot 4] = 0 \\ n = 4, \quad a_6 &= 4a_3 / [6 \cdot 5] = (1 \cdot 4)a_0 / [6 \cdot 5 \cdot 3 \cdot 2] \\ n = 5, \quad a_7 &= 5a_4 / [7 \cdot 6] = (2 \cdot 5)a_1 / [7 \cdot 6 \cdot 4 \cdot 3] \\ n = 6, \quad a_8 &= 6a_5 / [8 \cdot 7] = 0 \\ n = 7, \quad a_9 &= 7a_6 / [9 \cdot 8] = (1 \cdot 4 \cdot 7)a_0 / [9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2] \end{aligned}$$

$$n = 8, \quad a_{10} = 8a_7/[10 \cdot 9] = (2 \cdot 5 \cdot 8)a_1/[10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3].$$

From this table. we can subdivide the coefficients into three categories:

$$\text{for } n = 3k + 2, \quad a_{3k+2} = 0;$$

for $n = 3k$, let's examine, for example, a_9 ; if we multiply the denominator by the numerator for this term, the product is $9!$; hence we can write $a_9 = (1 \cdot 4 \cdot 7)^2 a_0/9!$; in general this pattern is followed so that we obtain $a_{3k} = (1 \cdot 4 \cdots (3k - 2))^2 a_0/(3k)!$;

for $n = 3k + 1$, let's examine, for example, a_{10} ; if we multiply the denominator by the numerator for this term, the product is $10!$; hence we can write $a_{10} = (2 \cdot 5 \cdot 8)^2 a_0/10!$; in general this pattern is followed so that we obtain $a_{3k+1} = (2 \cdot 5 \cdots (3k - 1))^2 a_1/(3k + 1)!$;

Thus we have

$$z = \sum_{n=0}^{\infty} a_n x^n = a_0 \left(1 + \sum_{k=1}^{\infty} (1 \cdot 4 \cdots (3k-2))^2 x^{3k} / (3k)! \right) + a_1 \left(x + \sum_{k=1}^{\infty} (2 \cdot 5 \cdots (3k-1))^2 x^{3k+1} / (3k+1)! \right).$$

$$25, (25). \quad w'' + 3xw' - w = 0, \quad w(0) = 2, \quad w'(0) = 0$$

We proceed directly by writing

$$w = \sum_{n=0}^{\infty} a_n x^n$$

$$w' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$w'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 3x \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 3na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

Explicitly writing the first two terms from the first series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 3na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 3na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + 3na_n - a_n]x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (3n-1)a_n]x^n = 0$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -(3n-1)a_n/[(n+2)(n+1)], \quad n \geq 0;$$

in addition the initial values give

$$a_0 = 2$$

$$a_1 = 0.$$

From these equations we can write

$$n = 0, \quad a_2 = a_0/2 = 1$$

$$n = 1, \quad a_3 = -2a_1/6 = 0$$

$$n = 2, \quad a_4 = -5a_2/12 = -5/12$$

$$n = 3, \quad a_5 = 8a_3/20 = 0$$

$$n = 4, \quad a_6 = -11a_4/30 = 11/72$$

Thus we can write

$$w = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$w = 2 + x^2 - 5x^4/12 + 11x^6/72 + \dots$$

27, (27). $(x + 1)y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$(x + 1) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponents of the first and second series are equal to n we obtain

$$\sum_{n=-1}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Explicitly writing the first two term from the first series and the first two terms from the second series yields

$$0(a_0)x^{-1} + \sum_{n=0}^{\infty} n(n+1) a_{n+1} x^n + 0(a_0) + 0(a_1)x + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [n(n+1) a_{n+1} + (n+2)(n+1) a_{n+2} - a_n] x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -n a_{n+1} / (n+2) + a_n / [(n+2)(n+1)], \quad n \geq 0;$$

in addition the initial values give

$$a_0 = 0$$

$$a_1 = 1.$$

From these equations we can write

$$n = 0, \quad a_2 = a_0/2 = 0$$

$$n = 1, \quad a_3 = -a_2/3 + a_1/6 = 1/6$$

$$n = 2, \quad a_4 = -a_3/2 + a_2/12 = -1/12$$

$$n = 3, \quad a_5 = -3a_4/5 + a_3/20 = 1/20 + 1/120 = 7/120.$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$y = x + x^3/6 - x^4/12 + 7x^5/120 + \dots$$

SECTION 8.4

$$7, (7). \quad y' + 2(x - 1)y = 0, \quad x_0 = 1$$

For these problems it is necessary that the coefficient functions are written as power series about x_0 ; the coefficient function for y is already in this form, hence we proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n(x - 1)^n$$

$$y' = \sum_{n=0}^{\infty} n a_n(x - 1)^{n-1}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n a_n(x - 1)^{n-1} + 2(x - 1) \sum_{n=0}^{\infty} a_n(x - 1)^n = 0$$

or after simplification

$$\sum_{n=0}^{\infty} n a_n(x - 1)^{n-1} + \sum_{n=0}^{\infty} 2a_n(x - 1)^{n+1} = 0.$$

Performing a shift so that the exponents of both series is equal to n we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} (n + 1)a_{n+1}(x - 1)^n + \sum_{n=1}^{\infty} 2a_{n-1}(x - 1)^n &= 0 \\ a_1 + \sum_{n=1}^{\infty} (n + 1)a_{n+1}(x - 1)^n + \sum_{n=1}^{\infty} 2a_{n-1}(x - 1)^n &= 0 \\ a_1 + \sum_{n=1}^{\infty} [(n + 1)a_{n+1} + 2a_{n-1}](x - 1)^n &= 0. \end{aligned}$$

From this last equation we obtain the recurrence relation

$$a_{n+1} = -2a_{n-1}/(n + 1), \quad n \geq 1$$

as well as the fact

$$a_1 = 0.$$

From these two pieces of information we can write

$$n = 1, \quad a_2 = -a_0$$

$$\begin{aligned}
n = 2, \quad a_3 &= -2a_1/3 = 0 \\
n = 3, \quad a_4 &= -a_2/2 = a_0/2 \\
n = 4, \quad a_5 &= -2a_3/5 = 0 \\
n = 5, \quad a_6 &= -a_4/3 = -a_0/6.
\end{aligned}$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 + a_6(x-1)^6 + \dots$$

$$y = a_0[1 - (x-1)^2 + (x-1)^4/2 - (x-1)^6/6 + \dots].$$

9, (9). $(x^2 - 2x)y'' + 2y = 0, \quad x_0 = 1$

For these problems it is necessary that the coefficient functions are written as power series about x_0 ; hence we must express the function $f(x) = x^2 - 2x$ as a Taylor series about $x_0 = 1$. Following the standard approach we write

$$f(x) = x^2 - 2x \quad f(1) = -1$$

$$f'(x) = 2x - 2 \quad f'(1) = 0$$

$$f''(x) = 2 \quad f''(1) = 2$$

$$f^n(x) = 0 \quad \text{for all } n \geq 3$$

so that

$$f(x) = \sum_{n=0}^{\infty} f^n(1)(x-1)^n/n! = -1 + 0(x-1) + 2(x-1)^2/2 = -1 + (x-1)^2.$$

Next we proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n$$

$$y' = \sum_{n=0}^{\infty} n a_n(x-1)^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

and substituting into the above equation obtain

$$[-1 + (x - 1)^2] \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + 2 \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

or after simplification

$$-\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we have

$$-\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} [-(n+2)(n+1)a_{n+2} + n(n-1)a_n + 2a_n](x-1)^n = 0$$

$$\sum_{n=0}^{\infty} [-(n+2)(n+1)a_{n+2} + (n^2 - n + 2)a_n](x-1)^n = 0$$

From this last equation we obtain the recurrence relation

$$a_{n+2} = (n^2 - n + 2)a_n / [(n+2)(n+1)], \quad n \geq 0.$$

From this relation we can write

$$n = 0, \quad a_2 = a_0$$

$$n = 1, \quad a_3 = a_1/3$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \dots$$

$$y = a_0[1 + (x-1)^2 + \dots] + a_1[(x-1) + (x-1)^3/3 + \dots].$$

11, (11). $x^2y'' - y' + y = 0, \quad x_0 = 2$

For these problems it is necessary that the coefficient functions are written as power series about x_0 ; hence we must express the function $f(x) = x^2$ as a Taylor series about $x_0 = 2$. Following the standard approach we write

$$f(x) = x^2 \quad f(2) = 4$$

$$f'(x) = 2x \quad f'(2) = 4$$

$$f''(x) = 2 \quad f''(2) = 2$$

$$f^n(x) = 0 \quad \text{for all } n \geq 3$$

so that

$$f(x) = \sum_{n=0}^{\infty} f^n(1)(x-2)^n/n! = 4+4(x-2)+2(x-2)^2/2 = 4+4(x-2)+(x-2)^2.$$

Next we proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n(x-2)^n$$

$$y' = \sum_{n=0}^{\infty} na_n(x-2)^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2}$$

and substituting into the above equation obtain

$$[4+4(x-2)+(x-2)^2] \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} - \sum_{n=0}^{\infty} na_n(x-2)^{n-1} + \sum_{n=0}^{\infty} a_n(x-2)^n = 0$$

or after simplification

$$\begin{aligned} & \sum_{n=0}^{\infty} 4n(n-1)a_n(x-2)^{n-2} + \sum_{n=0}^{\infty} 4n(n-1)a_n(x-2)^{n-1} + \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^n \\ & - \sum_{n=0}^{\infty} na_n(x-2)^{n-1} + \sum_{n=0}^{\infty} a_n(x-2)^n = 0. \end{aligned}$$

Performing a shift so that the exponents of the first, second, and fourth series are equal to n we have

$$\begin{aligned} & \sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2}(x-2)^n + \sum_{n=0}^{\infty} 4n(n+1)a_{n+1}(x-2)^n + \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}(x-2)^n \\ & + \sum_{n=0}^{\infty} a_n(x-2)^n = 0. \end{aligned}$$

$$\sum_{n=0}^{\infty} [4(n+2)(n+1)a_{n+2} + 4n(n+1)a_{n+1} + n(n-1)a_n - (n+1)a_{n+1} + a_n](x-2)^n = 0$$

$$\sum_{n=0}^{\infty} [4(n+2)(n+1)a_{n+2} + (4n-1)(n+1)a_{n+1} + (n^2 - n + 1)a_n](x-2)^n = 0$$

From this last equation we obtain the recurrence relation

$$a_{n+2} = -(4n-1)a_{n+1}/[4(n+2)] - (n^2 - n + 1)a_n/[4(n+2)(n+1)], \quad n \geq 0.$$

From this relation we can write

$$n = 0, \quad a_2 = a_1/8 - a_0/8$$

$$n = 1, \quad a_3 = -a_2/4 - a_1/24 = -7a_1/96 + a_0/32.$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n(x-2)^n = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \dots$$

$$y = a_0[1 - (x-2)^2/8 + (x-2)^3/32 + \dots] + a_1[(x-2) + (x-2)^2/8 - 7(x-2)^3/96 + \dots].$$

13, (13). $x' + (\sin t)x = 0, \quad x(0) = 1$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about zero by explicitly writing out the first several terms in the expansion:

$$x = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \dots.$$

The equation $x(0) = 1$ tells us that $a_0 = 1$ so we can write

$$x = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \dots \quad (1)$$

and after differentiating with respect to t

$$x' = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 + \dots.$$

The Taylor expansion about zero for $\sin t$ is given by

$$\sin t = \sum_{n=0}^{\infty} (-1)^n t^{2n+1} / (2n+1)! = t - t^3/6 + t^5/120 + \dots \quad (2)$$

hence multiplying (1) by (2) we obtain after simplification

$$(\sin t)x = t + a_1t^2 + (a_2 - \frac{1}{6})t^3 + (a_3 - \frac{a_1}{6})t^4 + (a_4 - \frac{a_2}{6} + \frac{1}{120})t^5 + \dots$$

Thus we can write

$$\begin{aligned} x' + (\sin t)x &= a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 + \dots \\ &+ t + a_1t^2 + (a_2 - \frac{1}{6})t^3 + (a_3 - \frac{a_1}{6})t^4 + (a_4 - \frac{a_2}{6} + \frac{1}{120})t^5 + \dots \end{aligned}$$

or after collecting similar terms

$$\begin{aligned} x' + (\sin t)x &= a_1 + (2a_2 + 1)t + (3a_3 + a_1)t^2 + (4a_4 + a_2 - \frac{1}{6})t^3 \\ &+ (5a_5 + a_3 - \frac{a_1}{6})t^4 + (6a_6 + a_4 - \frac{a_2}{6} + \frac{1}{120})t^5 + \dots \end{aligned}$$

Setting coefficients equal to zero yields

$$\begin{aligned} a_1 &= 0 \\ 2a_2 + 1 &= 0 \longrightarrow a_2 = -1/2 \\ 3a_3 + a_1 &= 0 \longrightarrow a_3 = 0 \\ 4a_4 + a_2 - \frac{1}{6} &= 0 \longrightarrow a_4 = 1/6 \\ 5a_5 + a_3 - \frac{a_1}{6} &= 0 \longrightarrow a_5 = 0 \\ 6a_6 + a_4 - \frac{a_2}{6} + \frac{1}{120} &= 0 \longrightarrow a_6 = -31/120. \end{aligned}$$

Placing the value of these coefficients into (1) gives

$$x = 1 - t^2/2 + t^4/6 - 31t^6/120 + \dots$$

15, (15). $(x^2 + 1)y'' - e^x y' + y = 0$, $y(0) = 1$, $y'(0) = 1$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about zero by explicitly writing out the first several terms in the expansion:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

The equation $y(0) = 1$ tells us that $a_0 = 1$ and $y'(0) = 1$ implies $a_1 = 1$ so we can write

$$y = 1 + x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

and after differentiating with respect to x

$$y' = 1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots \quad (1),$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$$

The Taylor expansion about zero for e^x is given by

$$e^x = \sum_{n=0}^{\infty} x^n/n! = 1 + x + x^2/2 + x^3/6 + x^4/24 + \dots \quad (2)$$

hence multiplying (1) by (2) we obtain after simplification

$$e^x y' = 1 + (2a_2 + 1)x + (3a_3 + 2a_2 + \frac{1}{2})x^2 + (4a_4 + 3a_3 + a_2 + \frac{1}{6})x^3 + \dots$$

In addition multiplying x^2 by y'' yields

$$x^2 y'' = 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + \dots$$

Thus we can write

$$\begin{aligned} (x^2 + 1)y'' - e^x y' + y &= 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + \dots \\ &+ 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots \\ - 1 - (2a_2 + 1)x - (3a_3 + 2a_2 + \frac{1}{2})x^2 - (4a_4 + 3a_3 + a_2 + \frac{1}{6})x^3 + \dots \\ &+ 1 + x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots \end{aligned}$$

or after collecting similar terms up to the third power of x

$$\begin{aligned} (x^2 + 1)y'' - e^x y' + y &= 2a_2 + (6a_3 - 2a_2)x + (12a_4 - 3a_3 - a_2 - \frac{1}{2})x^2 \\ &+ (20a_5 - 4a_4 + 4a_3 - a_2 - \frac{1}{6})x^3 + \dots \end{aligned}$$

Setting coefficients equal to zero yields

$$2a_2 = 0 \longrightarrow a_2 = 0$$

$$6a_3 - 2a_2 = 0 \longrightarrow a_3 = 0$$

$$12a_4 - 3a_3 - a_2 - \frac{1}{2} = 0 \longrightarrow a_4 = 1/24$$

$$20a_5 - 4a_4 + 4a_3 - a_2 - \frac{1}{6} = 0 \longrightarrow a_5 = 1/60$$

Placing the value of these coefficients into the expansion for $y(x)$ gives

$$y = 1 + x + x^4/24 + x^5/60 + \dots$$

17, (17). $y'' - (\sin x)y = 0$, $y(\pi) = 1$, $y'(\pi) = 0$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about π by explicitly writing out the first several terms in the expansion:

$$y = a_0 + a_1(x-\pi) + a_2(x-\pi)^2 + a_3(x-\pi)^3 + a_4(x-\pi)^4 + a_5(x-\pi)^5 + a_6(x-\pi)^6 + \dots$$

The equation $y(\pi) = 1$ tells us that $a_0 = 1$ and $y'(\pi) = 0$ implies $a_1 = 0$ so we can write

$$y = 1 + a_2(x-\pi)^2 + a_3(x-\pi)^3 + a_4(x-\pi)^4 + a_5(x-\pi)^5 + a_6(x-\pi)^6 + \dots \quad (1)$$

and after differentiating with respect to x

$$y' = 2a_2(x-\pi) + 3a_3(x-\pi)^2 + 4a_4(x-\pi)^3 + 5a_5(x-\pi)^4 + 6a_6(x-\pi)^5 + \dots ,$$

$$y'' = 2a_2 + 6a_3(x-\pi) + 12a_4(x-\pi)^2 + 20a_5(x-\pi)^3 + 30a_6(x-\pi)^4 + \dots$$

The Taylor expansion about π for $\sin x$ is given by

$$\sin x = \sum_{n=0}^{\infty} (-1)^{n+1} (x-\pi)^{2n+1} / (2n+1)! = -(x-\pi) + (x-\pi)^3/6 - (x-\pi)^5/120 + \dots \quad (2)$$

hence multiplying (1) by (2) we obtain after simplification

$$(\sin x)y = -(x-\pi) + (-a_2 + \frac{1}{6})(x-\pi)^3 - a_3(x-\pi)^4 + (-a_4 + \frac{a_2}{6} - \frac{1}{120})(x-\pi)^5 + \dots$$

Thus we can write

$$y'' - (\sin x)y = 2a_2 + 6a_3(x-\pi) + 12a_4(x-\pi)^2 + 20a_5(x-\pi)^3 + 30a_6(x-\pi)^4 + \dots$$

$$+ (x-\pi) - (-a_2 + \frac{1}{6})(x-\pi)^3 + a_3(x-\pi)^4 - (-a_4 + \frac{a_2}{6} - \frac{1}{120})(x-\pi)^5 + \dots$$

or after collecting similar terms up to the fourth power of $(x - \pi)$

$$y'' - (\sin x)y = 2a_2 + (6a_3 + 1)(x - \pi) + (12a_4)(x - \pi)^2 + (20a_5 + a_2 - \frac{1}{6})(x - \pi)^3 \\ + (30a_6 + a_3)(x - \pi)^4 + \dots$$

Setting coefficients equal to zero yields

$$2a_2 = 0 \longrightarrow a_2 = 0$$

$$6a_3 + 1 = 0 \longrightarrow a_3 = 1/6$$

$$12a_4 = 0 \longrightarrow a_4 = 0$$

$$20a_5 + a_2 - \frac{1}{6} = 0 \longrightarrow a_5 = 1/120$$

$$30a_6 + a_3 = 0 \longrightarrow a_6 = 1/180$$

Placing the value of these coefficients into the expansion for $y(x)$ gives

$$y = 1 - (x - \pi)^3/6 + (x - \pi)^5/120 + (x - \pi)^6/(180) + \dots$$

19, (19). $y'' - e^{2x}y' + (\cos x)y = 0$, $y(0) = -1$, $y'(0) = 1$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about zero by explicitly writing out the first several terms in the expansion:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

The equation $y(0) = -1$ tells us that $a_0 = -1$ and $y'(0) = 1$ implies $a_1 = 1$ so we can write

$$y = -1 + x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots \quad (1)$$

and after differentiating with respect to x

$$y' = 1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots \quad (2),$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$$

The Taylor expansion about zero for e^{2x} is given by

$$e^{2x} = \sum_{n=0}^{\infty} (2^n x^n)/n! = 1 + 2x + 2x^2 + 4x^3/3 + 2x^4/3 + \dots \quad (3)$$

hence multiplying (2) by (3) we obtain after simplification

$$e^{2x}y' = 1 + (2a_2 + 2)x + (3a_2 + 4a_2 + 2)x^2 + (4a_4 + 6a_3 + 4a_2 + \frac{4}{3})x^3 + \dots$$

The Taylor expansion about zero for $\cos x$ is given by

$$\cos x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)! = 1 - x^2/2 + x^4/24 + \dots \quad (4)$$

hence multiplying (1) by (4) we obtain after simplification

$$(\cos x)y = -1 + x + (a_2 + \frac{1}{2})x^2 + (a_3 - \frac{1}{2})x^3 + \dots$$

Thus we can write

$$\begin{aligned} y'' - e^{2x}y' + (\cos x)y &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots \\ &- 1 - (2a_2 + 2)x - (3a_2 + 4a_2 + 2)x^2 - (4a_4 + 6a_3 + 4a_2 + \frac{4}{3})x^3 + \dots \\ &+ -1 + x + (a_2 + \frac{1}{2})x^2 + (a_3 - \frac{1}{2})x^3 + \dots \end{aligned}$$

or after collecting similar terms up to the first power of x

$$y'' - e^{2x}y' + (\cos x)y = (2a_2 - 2) + (6a_3 - 2a_2 - 1)x + \dots$$

Setting coefficients equal to zero yields

$$2a_2 - 2 = 0 \longrightarrow a_2 = 1$$

$$6a_3 - 2a_2 - 1 = 0 \longrightarrow a_3 = 1/2$$

Placing the value of these coefficients into the expansion for $y(x)$ gives

$$y = -1 + x + x^2 + x^3/2 + \dots$$

21, (21). $y' - xy = \sin x$.

We begin with the expansions for y , y' , and $\sin x$:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \dots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - (1/6)x^3 + (1/120)x^5 - (1/4200)x^7 + \dots$$

and

$$-xy = -x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \dots) = -a_0 x - a_1 x^2 - a_2 x^3 - a_3 x^4 - a_4 x^5 - a_5 x^6 - a_6 x^7 + \dots$$

Substituting these expansions back into the ODE yields

$$\begin{aligned} & (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \dots) + (-a_0 x - a_1 x^2 - a_2 x^3 - a_3 x^4 - a_4 x^5 - a_5 x^6 - a_6 x^7 + \dots) \\ & = x - (1/6)x^3 + (1/120)x^5 - (1/4200)x^7 + \dots \end{aligned}$$

Equating coefficients for like powers of x gives

$$x^0 : a_1 = 0$$

$$x^1 : 2a_2 - a_0 = 1 \quad \rightarrow \quad a_2 = (1/2)(a_0 + 1)$$

$$x^2 : 3a_3 - a_1 = 0 \quad \rightarrow \quad a_3 = 0$$

$$x^3 : 4a_4 - a_2 = -(1/6) \quad \rightarrow \quad a_4 = (1/4)(a_2 - (1/6))$$

$$a_4 = (1/4)[(1/2)(a_0 + 1) - (1/6)] = (1/8)a_0 + (1/12)$$

$$x^4 : 5a_5 - a_3 = 0 \quad \rightarrow \quad a_5 = 0$$

$$x^5 : 6a_6 - a_4 = (1/120) \quad \rightarrow \quad a_6 = (1/6)(a_4 + (1/120))$$

$$a_6 = (1/6)[(1/8)a_0 + (1/12) + (1/120)] = (1/48)a_0 + (11/720).$$

Thus

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$= a_0 + (0)x + [(1/2)(a_0+1)]x^2 + (0)x^3 + [(1/8)a_0 + (1/12)]x^4 + (0)x^5 + [(1/48)a_0 + (11/720)]x^6$$

or after regrouping

$$= a_0[1 + (1/2)x^2 + (1/8)x^4 + (1/48)x^6 + \dots] + [(1/2)x^2 + (1/12)x^4 + (11/720)x^6].$$

23, (23). $z'' + xz' + z = x^2 + 2x + 1$

We begin with the expansions for z , z' , and z'' :

$$z = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$z' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots$$

$$z'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$$

and

$$xz' = x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots) = a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + 5a_5x^5 + 6a_6x^6 + \dots$$

Substituting these expansions back into the ODE yields

$$(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots) + (a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + 5a_5x^5 + 6a_6x^6 + \dots)$$

$$+ (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots) = x^2 + 2x + 1.$$

Equating coefficients for like powers of x gives

$$x^0 : \quad 2a_2 + a_0 = 1 \quad \rightarrow \quad a_2 = -(1/2)a_0 + (1/2)$$

$$x^1 : \quad 6a_3 + a_1 + a_1 = 2 \quad \rightarrow \quad a_3 = (1/6)(-2a_1 + 2) = -(1/3)a_1 + (1/3).$$

Thus

$$z = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = a_0 + a_1x + [-(1/2)a_0 + (1/2)]x^2 + [-(1/3)a_1 + (1/3)]x^3 + \dots$$

or after regrouping

$$z = a_0[1 - (1/2)x^2 + \dots] + a_1[x - (1/3)x^3 + \dots] + [(1/2)x^2 + (1/3)x^3 + \dots].$$

$$25, (25). \quad (1 + x^2)y'' - xy' + y = e^{-x}$$

We begin with the expansions for y , y' , y'' , and e^{-x} :

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n x^n / (n!) = 1 - x + (1/2)x^2 - (1/6)x^3 + (1/24)x^4 - (1/120)x^5 + \dots$$

and

$$-xy' = -x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots) = -a_1x - 2a_2x^2 - 3a_3x^3 - 4a_4x^4 - 5a_5x^5 - 6a_6x^6 + \dots$$

$$x^2 y'' = x^2(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots) = 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + 30a_6x^6 + \dots.$$

Substituting these expansions back into the ODE yields

$$\begin{aligned} & (2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots) + (2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + 30a_6x^6 + \dots) \\ & + (-a_1x - 2a_2x^2 - 3a_3x^3 - 4a_4x^4 - 5a_5x^5 - 6a_6x^6 + \dots) + (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots) \\ & = 1 - x + (1/2)x^2 - (1/6)x^3 + (1/24)x^4 - (1/120)x^5 + \dots \end{aligned}$$

Equating coefficients for like powers of x gives

$$x^0 : \quad 2a_2 + a_0 = 1 \quad \rightarrow \quad a_2 = -(1/2)a_0 + (1/2)$$

$$x^1 : \quad 6a_3 - a_1 + a_1 = -1 \quad \rightarrow \quad a_3 = -(1/6).$$

Thus

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = a_0 + a_1x + [-(1/2)a_0 + (1/2)]x^2 + [-(1/6)]x^3 + \dots$$

or after regrouping

$$y = a_0[1 - (1/2)x^2 + \dots] + a_1[x + \dots] + [(1/2)x^2 - (1/6)x^3 + \dots].$$

$$27, (27). \quad (1 - x^2)y'' - y' + y = \tan x$$

We begin with the expansions for y , y' , y'' , and e^{-x} :

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$$

$$\tan x = x + (1/3)x^3 + (2/15)x^5 + \dots$$

and

$$-y' = -(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \dots) = -a_1 - 2a_2x - 3a_3x^2 - 4a_4x^3 - 5a_5x^4 - 6a_6x^5 + \dots$$

$$-x^2y'' = -x^2(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots) = -2a_2x^2 - 6a_3x^3 - 12a_4x^4 - 20a_5x^5 - 30a_6x^6 + \dots$$

Substituting these expansions back into the ODE yields

$$\begin{aligned} & (2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots) + (-2a_2x^2 - 6a_3x^3 - 12a_4x^4 - 20a_5x^5 - 30a_6x^6 + \dots) \\ & + (-a_1 - 2a_2x - 3a_3x^2 - 4a_4x^3 - 5a_5x^4 - 6a_6x^5 + \dots) + (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots) \\ & = x + (1/3)x^3 + (2/15)x^5 + \dots \end{aligned}$$

Equating coefficients for like powers of x gives

$$x^0: \quad 2a_2 - a_1 + a_0 = 0 \quad \rightarrow \quad a_2 = (1/2)a_1 - (1/2)a_0$$

$$x^1: \quad 6a_3 - 2a_2 + a_1 = 1 \quad \rightarrow \quad a_3 = (1/3)a_2 - (1/6)a_1 + (1/6)$$

$$a_3 = (1/3)[(1/2)a_1 - (1/2)a_0] - (1/6)a_1 + (1/6) = -(1/6)a_0 + (1/6).$$

Thus

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = a_0 + a_1x + [(1/2)a_1 - (1/2)a_0]x^2 + [-(1/6)a_0 + (1/6)]x^3 + \dots$$

or after regrouping

$$y = a_0[1 - (1/2)x^2 - (1/6)x^3 + \dots] + a_1[x + (1/2)x^2 + \dots] + [(1/6)x^3 + \dots].$$

SECTION 8.6

19. (19). $9x^2y'' + 9x^2y' + 2y = 0$

Placing the problem in standard form yields

$$y'' + y' + 2y/(9x^2) = 0$$

so that

$$p_0 = \lim_{x \rightarrow 0} x(1) = 0$$

$$q_0 = \lim_{x \rightarrow 0} x^2[2/(9x^2)] = 2/9$$

and the indicial equation is given by

$$r(r - 1) + 0(r) + 2/9 = 0$$

$$r^2 - r + 2/9 = 0$$

$$9r^2 - 9r + 2 = 0$$

$$(3r - 1)(3r - 2) = 0$$

with exponents of singularity $r = 1/3, 2/3$. We choose the larger of the two values and write

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}}$$

$$y' = \sum_{n=0}^{\infty} (n + \frac{2}{3}) a_n x^{n-\frac{1}{3}}$$

$$y'' = \sum_{n=0}^{\infty} (n - \frac{1}{3})(n + \frac{2}{3}) a_n x^{n-\frac{4}{3}}$$

and substituting into the original equation obtain

$$9x^2 \sum_{n=0}^{\infty} (n - \frac{1}{3})(n + \frac{2}{3}) a_n x^{n-\frac{4}{3}} + 9x^2 \sum_{n=0}^{\infty} (n + \frac{2}{3}) a_n x^{n-\frac{1}{3}} + 2 \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}} = 0$$

or after simplification

$$\sum_{n=0}^{\infty} 9(n - \frac{1}{3})(n + \frac{2}{3}) a_n x^{n+\frac{2}{3}} + \sum_{n=0}^{\infty} 9(n + \frac{2}{3}) a_n x^{n+\frac{5}{3}} + \sum_{n=0}^{\infty} 2a_n x^{n+\frac{2}{3}} = 0$$

$$\sum_{n=0}^{\infty} (3n - 1)(3n + 2) a_n x^{n+\frac{2}{3}} + \sum_{n=0}^{\infty} 3(3n + 2) a_n x^{n+\frac{5}{3}} + \sum_{n=0}^{\infty} 2a_n x^{n+\frac{2}{3}} = 0.$$

Performing a shift so that the exponent of the second series is equal to $n + \frac{2}{3}$ we obtain

$$\sum_{n=0}^{\infty} (3n-1)(3n+2)a_n x^{n+\frac{2}{3}} + \sum_{n=1}^{\infty} 3(3(n-1)+2)a_{n-1} x^{n+\frac{2}{3}} + \sum_{n=0}^{\infty} 2a_n x^{n+\frac{2}{3}} = 0$$

$$\sum_{n=0}^{\infty} (3n-1)(3n+2)a_n x^{n+\frac{2}{3}} + \sum_{n=1}^{\infty} 3(3n-1)a_{n-1} x^{n+\frac{2}{3}} + \sum_{n=0}^{\infty} 2a_n x^{n+\frac{2}{3}} = 0.$$

Explicitly writing the first term from the first and third series yields

$$-2a_0 x^{\frac{2}{3}} + \sum_{n=1}^{\infty} (3n-1)(3n+2)a_n x^{n+\frac{2}{3}} + \sum_{n=1}^{\infty} 3(3n-1)a_{n-1} x^{n+\frac{2}{3}} + 2a_0 x^{\frac{2}{3}} + \sum_{n=1}^{\infty} 2a_n x^{n+\frac{2}{3}} = 0$$

$$\sum_{n=1}^{\infty} (3n-1)(3n+2)a_n x^{n+\frac{2}{3}} + \sum_{n=1}^{\infty} 3(3n-1)a_{n-1} x^{n+\frac{2}{3}} + \sum_{n=1}^{\infty} 2a_n x^{n+\frac{2}{3}} = 0$$

$$\sum_{n=1}^{\infty} [(3n-1)(3n+2)a_n + 3(3n-1)a_{n-1} + 2a_n] x^{n+\frac{2}{3}} = 0$$

$$\sum_{n=1}^{\infty} [3n(3n+1)a_n + 3(3n-1)a_{n-1}] x^{n+\frac{2}{3}} = 0.$$

From this last equation we obtain the recurrence relation

$$a_n = -(3n-1)a_{n-1}/[n(3n+1)], \quad n \geq 1.$$

From this relation we can write

$$n = 1, \quad a_1 = -a_0/2$$

$$n = 2, \quad a_2 = -5a_1/14 = 5a_0/28$$

$$n = 3, \quad a_3 = -8a_2/30 = -a_0/21.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{2}{3}} = a_0 x^{\frac{2}{3}} + a_1 x^{\frac{5}{3}} + a_2 x^{\frac{8}{3}} + a_3 x^{\frac{11}{3}} + \dots$$

$$y = a_0 [x^{\frac{2}{3}} - x^{\frac{5}{3}}/2 + 5x^{\frac{8}{3}}/28 - x^{\frac{11}{3}}/21 + \dots].$$

$$21, (21). \quad x^2 y'' + xy' + x^2 y = 0$$

Placing the problem in standard form yields

$$y'' + y'/x + y = 0$$

so that

$$p_0 = \lim_{x \rightarrow 0} x[1/x] = 1$$

$$q_0 = \lim_{x \rightarrow 0} x^2[1] = 0$$

and the indicial equation is given by

$$r(r-1) + 1(r) + 0 = 0$$

$$r^2 = 0$$

with exponents of singularity $r = 0, 0$. We naturally choose $r = 0$ and write

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the original equation obtain

$$x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

or after simplification

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Performing a shift so that the exponent of the third series is equal to n we obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

Explicitly writing the first two terms from the first and second series yields

$$0(a_0) + 0(a_1 x) + \sum_{n=2}^{\infty} n(n-1) a_n x^n + 0(a_0) + a_1 x + \sum_{n=2}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$a_1x + \sum_{n=2}^{\infty} [n(n-1)a_n + na_n + a_{n-2}]x^n = 0$$

$$a_1x + \sum_{n=2}^{\infty} [n^2a_n + a_{n-2}]x^n = 0.$$

From this last equation we obtain

$$a_1 = 0$$

$$a_n = -a_{n-2}/n^2, \quad n \geq 2.$$

From these relations we can write

$$n = 2, \quad a_2 = -a_0/4$$

$$n = 3, \quad a_3 = -a_1/9 = 0$$

$$n = 4, \quad a_4 = -a_2/16 = a_0/64$$

$$n = 5, \quad a_5 = -a_3/25 = 0$$

$$n = 6, \quad a_6 = -a_4/36 = -a_0/2304$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$y = a_0[1 - x^2/4 + x^4/64 - x^6/2304 + \dots].$$

23, (23). $x^2z'' + (x^2 + x)z' - z = 0$

Placing the problem in standard form yields

$$z'' + (1 + 1/x)z' - z/x^2 = 0$$

so that

$$p_0 = \lim_{x \rightarrow 0} x(1 + 1/x) = 1$$

$$q_0 = \lim_{x \rightarrow 0} x^2[-/(x^2)] = -1$$

and the indicial equation is given by

$$r(r-1) + 1(r) - 1 = 0$$

$$r^2 - 1 = 0$$

with exponents of singularity $r = -1, 1$. We choose the larger of the two values and write

$$z = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$z' = \sum_{n=0}^{\infty} (n+1)a_n x^n$$

$$z'' = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$$

and substituting into the original equation obtain

$$x^2 \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} (n+1)a_n x^n + x \sum_{n=0}^{\infty} (n+1)a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

or after simplification

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} (n+1)a_n x^{n+2} + \sum_{n=0}^{\infty} (n+1)a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Performing a shift so that the exponent of the second series is equal to $n+1$ we obtain

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n+1} + \sum_{n=1}^{\infty} n a_{n-1} x^{n+1} + \sum_{n=0}^{\infty} (n+1)a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

or since the coefficient of the second series is n we can write

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n+1} + \sum_{n=1}^{\infty} n a_{n-1} x^{n+1} + \sum_{n=0}^{\infty} (n+1)a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

which becomes after combining series

$$\sum_{n=0}^{\infty} [n(n+1)a_n + n a_{n-1} + (n+1)a_n - a_n] x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} [n(n+2)a_n + n a_{n-1}] x^{n+1} = 0.$$

From this last equation we obtain the recurrence relation

$$a_n = -a_{n-1}/(n+2), \quad n \geq 1.$$

From this relation we can write

$$n = 1, \quad a_1 = -a_0/3$$

$$n = 2, \quad a_2 = -a_1/4 = a_0/12$$

$$n = 3, \quad a_3 = -a_2/5 = -a_0/60.$$

Thus we can write

$$z = \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 x^1 + a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots$$

$$z = a_0 [x^1 - x^2/3 + x^3/12 - x^4/60 + \dots].$$

25, (25). $4x^2 y'' + 2x^2 y' - (x + 3)y = 0$

Placing the problem in standard form yields

$$y'' + y'/2 - [1/(4x) + 3/(4x^2)]y = 0$$

so that

$$p_0 = \lim_{x \rightarrow 0} x(1/2) = 0$$

$$q_0 = \lim_{x \rightarrow 0} x^2 [-1/(4x) - 3/(4x^2)] = -3/4$$

and the indicial equation is given by

$$r(r - 1) + 0(r) - 3/4 = 0$$

$$r^2 - r - 3/4 = 0$$

$$4r^2 - 4r - 3 = 0$$

$$(2r + 1)(2r - 3) = 0$$

with exponents of singularity $r = 3/2, -1/2$. We choose the larger of the two values and write

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$

$$y' = \sum_{n=0}^{\infty} (n + \frac{3}{2}) a_n x^{n+\frac{1}{2}}$$

$$y'' = \sum_{n=0}^{\infty} (n + \frac{1}{2})(n + \frac{3}{2}) a_n x^{n-\frac{1}{2}}$$

and substituting into the original equation obtain

$$4x^2 \sum_{n=0}^{\infty} (n + \frac{1}{2})(n + \frac{3}{2}) a_n x^{n-\frac{1}{2}} + 2x^2 \sum_{n=0}^{\infty} (n + \frac{3}{2}) a_n x^{n+\frac{1}{2}} - x \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} - 3 \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} = 0$$

or after simplification

$$\sum_{n=0}^{\infty} 4(n+\frac{1}{2})(n+\frac{3}{2})a_n x^{n+\frac{3}{2}} + \sum_{n=0}^{\infty} 2(n+\frac{3}{2})a_n x^{n+\frac{5}{2}} - \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}} - \sum_{n=0}^{\infty} 3a_n x^{n+\frac{3}{2}} = 0.$$

Performing a shift so that the exponents of the second and third series is equal to $n + \frac{3}{2}$ we obtain

$$\sum_{n=0}^{\infty} 4(n+\frac{1}{2})(n+\frac{3}{2})a_n x^{n+\frac{3}{2}} + \sum_{n=1}^{\infty} 2(n+\frac{1}{2})a_{n-1} x^{n+\frac{3}{2}} - \sum_{n=1}^{\infty} a_{n-1} x^{n+\frac{3}{2}} - \sum_{n=0}^{\infty} 3a_n x^{n+\frac{3}{2}} = 0.$$

Explicitly writing the first term from the first and fourth series yields

$$3a_0 x^{\frac{1}{2}} + \sum_{n=1}^{\infty} 4(n+\frac{1}{2})(n+\frac{3}{2})a_n x^{n+\frac{3}{2}} + \sum_{n=1}^{\infty} 2(n+\frac{1}{2})a_{n-1} x^{n+\frac{3}{2}} - \sum_{n=1}^{\infty} a_{n-1} x^{n+\frac{3}{2}} - 3a_0 x^{\frac{1}{2}} - \sum_{n=1}^{\infty} 3a_n x^{n+\frac{3}{2}} = 0$$

$$\sum_{n=1}^{\infty} 4(n+\frac{1}{2})(n+\frac{3}{2})a_n x^{n+\frac{3}{2}} + \sum_{n=1}^{\infty} 2(n+\frac{1}{2})a_{n-1} x^{n+\frac{3}{2}} - \sum_{n=1}^{\infty} a_{n-1} x^{n+\frac{3}{2}} - \sum_{n=1}^{\infty} 3a_n x^{n+\frac{3}{2}} = 0$$

$$\sum_{n=1}^{\infty} (2n+1)(2n+3)a_n x^{n+\frac{3}{2}} + \sum_{n=1}^{\infty} (2n+1)a_{n-1} x^{n+\frac{3}{2}} - \sum_{n=1}^{\infty} a_{n-1} x^{n+\frac{3}{2}} - \sum_{n=1}^{\infty} 3a_n x^{n+\frac{3}{2}} = 0$$

$$\sum_{n=1}^{\infty} [(2n+1)(2n+3)a_n + (2n+1)a_{n-1} - a_{n-1} - 3a_n] x^{n+\frac{3}{2}} = 0$$

$$\sum_{n=1}^{\infty} [4n(n+2)a_n + 2na_{n-1}] x^{n+\frac{3}{2}} = 0.$$

From this last equation we obtain the recurrence relation

$$a_n = -a_{n-1}/[2(n+2)], \quad n \geq 1.$$

From this relation we can write

$$n = 1, \quad a_1 = -a_0/[2 \cdot 3]$$

$$n = 2, \quad a_2 = -a_1/[2 \cdot 4] = a_0/[2^2 \cdot 3 \cdot 4]$$

$$n = 3, \quad a_3 = -a_2/[2 \cdot 5] = -a_0/[2^3 \cdot 3 \cdot 4 \cdot 5]$$

so in general

$$n = k, \quad a_k = (-1)^k a_0/[2^{k-1}(k+2)!].$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} = a_0 \sum_{n=0}^{\infty} (-1)^n x^{n+\frac{3}{2}}/[2^{n-1}(n+2)!].$$

$$27, (27). \quad xw'' - w' - xw = 0$$

Placing the problem in standard form yields

$$w'' - w'/x - w = 0$$

so that

$$p_0 = \lim_{x \rightarrow 0} x(-1/x) = -1$$

$$q_0 = \lim_{x \rightarrow 0} x^2[-1] = 0$$

and the indicial equation is given by

$$r(r-1) - (r) = 0$$

$$r^2 - 2r = 0$$

$$r(r-2) = 0$$

with exponents of singularity $r = 0, 2$. We choose the larger of the two values and write

$$w = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$w' = \sum_{n=0}^{\infty} (n+2)a_n x^{n+1}$$

$$w'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_n x^n$$

and substituting into the original equation obtain

$$x \sum_{n=0}^{\infty} (n+2)(n+1)a_n x^n - \sum_{n=0}^{\infty} (n+2)a_n x^{n+1} - x \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

or after simplification

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_n x^{n+1} - \sum_{n=0}^{\infty} (n+2)a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+3} = 0.$$

Performing a shift so that the exponents are equal to $n+2$ we obtain

$$\sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+1}x^{n+2} - \sum_{n=-1}^{\infty} (n+3)a_{n+1}x^{n+2} - \sum_{n=1}^{\infty} a_{n-1}x^{n+2} = 0.$$

Explicitly writing the first two terms from the first and second series yields

$$2a_0x + 6a_1x^2 + \sum_{n=1}^{\infty} (n+3)(n+2)a_{n+1}x^{n+2} - 2a_0x - 3a_1x^2 - \sum_{n=1}^{\infty} (n+3)a_{n+1}x^{n+2} - \sum_{n=1}^{\infty} a_{n-1}x^{n+2} = 0$$

$$3a_1x^2 + \sum_{n=1}^{\infty} [(n+3)(n+2)a_{n+1} - (n+3)a_{n+1} - a_{n-1}]x^{n+2} = 0$$

$$3a_1x^2 + \sum_{n=1}^{\infty} [(n+3)(n+1)a_{n+1} - a_{n-1}]x^{n+2} = 0$$

From this last equation we obtain

$$a_1 = 0$$

$$a_{n+1} = a_{n-1}/[(n+3)(n+1)], \quad n \geq 1.$$

From these equations we can write

$$n = 1, \quad a_2 = a_0/[4 \cdot 2] = a_0/[2^2 \cdot 2 \cdot 1] = a_0/[2^2 \cdot 2! \cdot 1!]$$

$$n = 2, \quad a_3 = a_1/[5 \cdot 3] = 0$$

$$n = 3, \quad a_4 = a_2/[6 \cdot 4] = a_0/[6 \cdot 4 \cdot 4 \cdot 2] = a_0/[2^4 \cdot 3 \cdot 2 \cdot 2 \cdot 1] = a_0/[2^4 \cdot 3! \cdot 2!]$$

$$n = 4, \quad a_5 = a_3/[7 \cdot 5] = 0$$

$$n = 5, \quad a_6 = a_4/[8 \cdot 6] = a_0/[8 \cdot 6 \cdot 6 \cdot 4 \cdot 4 \cdot 2] = a_0/[2^6 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1] = a_0/[2^6 \cdot 4! \cdot 3!]$$

$$n = 6, \quad a_7 = a_5/[9 \cdot 7] = 0$$

$$n = 7, \quad a_8 = a_6/[10 \cdot 8] = a_0/[10 \cdot 8 \cdot 8 \cdot 6 \cdot 6 \cdot 4 \cdot 4 \cdot 2] = a_0/[2^8 \cdot 5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1] = a_0/[2^8 \cdot 5! \cdot 4!]$$

so in general, for an odd subscript, $2k + 1$

$$a_{2k+1} = 0$$

while for an even subscript, $2k$

$$a_{2k} = a_0/[2^{2k}(k+1)!k!]$$

Thus we can write

$$w = \sum_{n=0}^{\infty} a_n x^{n+2} = a_0 \sum_{k=0}^{\infty} x^{2k+2}/[2^{2k}(k+1)!k!].$$

$$29, (29). \quad xy'' + (x-1)y' - 2y = 0$$

Placing the problem in standard form yields

$$y'' + (1 - 1/x)y' - 2y/x = 0$$

so that

$$p_0 = \lim_{x \rightarrow 0} x(1 - 1/x) = -1$$

$$q_0 = \lim_{x \rightarrow 0} x^2[-2/x] = 0$$

and the indicial equation is given by

$$r(r - 1) - (r) = 0$$

$$r^2 - 2r = 0$$

$$r(r - 2) = 0$$

with exponents of singularity $r = 0, 2$. We choose the larger of the two values and write

$$y = \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$y' = \sum_{n=0}^{\infty} (n+2)a_n x^{n+1}$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_n x^n$$

and substituting into the original equation obtain

$$x \sum_{n=0}^{\infty} (n+2)(n+1)a_n x^n + x \sum_{n=0}^{\infty} (n+2)a_n x^{n+1} - \sum_{n=0}^{\infty} (n+2)a_n x^{n+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

or after simplification

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} (n+2)a_n x^{n+2} - \sum_{n=0}^{\infty} (n+2)a_n x^{n+1} - \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0.$$

Performing a shift so that the exponents of the first and third terms are equal to $n+2$ we obtain

$$\sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+1} x^{n+2} + \sum_{n=0}^{\infty} (n+2)a_n x^{n+2} - \sum_{n=-1}^{\infty} (n+3)a_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0.$$

Explicitly writing the first term from the first and third series yields

$$2a_0 x^2 + \sum_{n=0}^{\infty} (n+3)(n+2)a_{n+1} x^{n+2} + \sum_{n=0}^{\infty} (n+2)a_n x^{n+2} - 2a_0 x^2 - \sum_{n=0}^{\infty} (n+3)a_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+3)(n+2)a_{n+1} x^{n+2} + \sum_{n=0}^{\infty} (n+2)a_n x^{n+2} - \sum_{n=0}^{\infty} (n+3)a_{n+1} x^{n+2} - \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+1} + (n+2)a_n - (n+3)a_{n+1} - 2a_n]x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} [(n+3)(n+1)a_{n+1} + na_n]x^{n+2} = 0.$$

From this last equation we obtain the recurrence relation

$$a_{n+1} = na_n/[(n+3)(n+1)], \quad n \geq 0.$$

From these equations we can write

$$n = 0, \quad a_1 = 0(a_0)/[3 \cdot 1] = 0$$

$$n = 1, \quad a_2 = 1(a_1)/[4 \cdot 2] = 0$$

$$n = 2, \quad a_3 = 2(a_2)/[5 \cdot 3] = 0$$

so in general, for subscripts greater than zero,

$$a_n = 0.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^{n+2} = a_0 x^2.$$