RESEARCH STATEMENT

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1. Introduction

One of the most ubiquitous concepts in mathematics is that of an incidence structure. Roughly speaking, this consists of two sets of objects together with a relation between them. For example, you can think of the points and lines in a geometry. Then if a particular point lies on a particular line, we say those two are "incident."

Given an incidence structure, you can encode it as a rectangular array. Let the rows of the array correspond to the first set of objects; the columns correspond to the second set of objects. Place a one in the (i, j)-position of the array if the object corresponding to row i is incident with the object corresponding to column j. Place a zero in all other positions. This array is called an incidence matrix.

By studying this incidence matrix you can learn about the given incidence structure. From linear algebra and abelian group theory, there are several numerical invariants associated to such a matrix. Two structures that are "essentially the same" (this can be made precise) will have the same invariants. In particular, if you want to show that two structures are not the same it is sufficient to show that they have different invariants. It is thus of interest to compute invariants of incidence matrices.

Some of these numerical invariants are stronger than others. One of the strongest invariants is the Smith normal form of the matrix. By "stronger" I mean that if you know the Smith normal form of a matrix, then you can immediately deduce other invariants such as rank, p-rank (for any prime p), determinant (for square matrices), etc. Recall that the Smith normal form of a (possibly nonsquare) integer matrix is a diagonal matrix of the same size, with the diagonal entries subject to certain divisibility conditions. More formally, if A is an $m \times n$ integer matrix, then there exist unimodular (invertible over the integers) matrices P and Q such that the matrix $PAQ^{-1} = (d_{i,j})$ satisfies

$$d_{i,j} = 0, \text{ for } i \neq j$$

and

$$d_{i,i} \text{ divides } d_{i+1,i+1}, \text{ for } 1 \leq i < \min\{m, n\}.$$

The diagonal entries of the Smith normal form of A are unique up to sign, and are called the invariant factors of the matrix A. The multiset of prime power factors of the invariant factors are called the elementary divisors of A.

2. Current Research

My research involves using techniques from group representation theory to compute the Smith normal form of certain incidence matrices. Specifically, consider the lines in projective 3-dimensional space over a finite field. Define two lines to be incident if and only if they are skew. Ordering the lines in some arbitrary but fixed manner, we may form the incidence matrix A of this relation.

It is useful to view this setup as a special case of a more general situation. Let V be an $(n+1)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $q = p^t$ is a prime power. Denote by $\mathcal{L}_r$ the set of r-dimensional subspaces of V. So $\mathcal{L}_1$ denotes the points, $\mathcal{L}_2$ denotes the lines, etc. of the projective geometry $\mathbb{P}(V)$. Define an r-dimensional subspace U and an s-dimensional subspace W to be incident if and only if $U \cap W = \{0\}$, and denote by $A_{r,s}$ the $|\mathcal{L}_r| \times |\mathcal{L}_s|$ incidence matrix.

These matrices $A_{r,s}$ are naturally interesting, and have been studied by a number of mathematicians. The reader is referred to the surveys [13, 12] (see also [4, Introduction]). They can be thought of as the adjacency matrices of q-analogues of the classical Kneser graphs. When $r = 1$, the incidence structure is that of a 2-design with "classical parameters," and these incidence matrices are the generator matrices of codes closely related to the Reed-Muller codes. This is what initially motivated their study, and in this case their Smith normal forms have been found [8, 10, 4]. When neither r nor s is one, the p-ranks of these matrices have been determined [11], but not their elementary divisors. Thus the particular situation I have been studying (corresponding to when $n = 3$, $r = s = 2$, and $A = A_{2,2}$) is a basic and important case to consider.

Early last year Peter Sin and I managed to compute the elementary divisors of the incidence matrix A in the case that the underlying field is of prime order [6]. Working independently, A. E. Brouwer discovered these as well. For projective space over an arbitrary finite field the problem is much more complicated. Recently, however, the three of us have managed to completely solve the problem–computing the Smith normal form of A in all cases [2].

For the interested reader, I will end this section with a few words about how representation theory comes into play here. Set $G = GL(n+1, q)$. Upon fixing a basis of V, the group G has a natural action on the sets $\mathcal{L}_i$ which preserves incidence. The matrix $A_{r,s}$ can thus be viewed as a homomorphism of $\mathbb{Z}G$-permutation modules

$$\eta_{r,s} : \mathbb{Z}^{\mathcal{L}_r} \to \mathbb{Z}^{\mathcal{L}_s}.$$

Computing the elementary divisors of $A_{r,s}$ is equivalent to finding a cyclic decomposition of the cokernel of $\eta_{r,s}$. Since only the p-torsion is unknown in general, it is more convenient to view the matrix entries as coming from a certain p-adic local ring R. Reduction modulo the maximal ideal of R induces a homomorphism of $\mathbb{F}_qG$-permutation modules

$$\overline{\eta_{r,s}} : \mathbb{F}_q^{\mathcal{L}_r} \to \mathbb{F}_q^{\mathcal{L}_s}.$$

If we set $M_i = \{x \in R^{\mathcal{L}_r} | \eta_{r,s}(x) \in p^i R^{\mathcal{L}_s}\}$, then we have a filtration

$$R^{\mathcal{L}_r} = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

and reduction induces the p-filtration

$$\mathbb{F}_q^{\mathcal{L}_r} = \overline{M_0} \supseteq \overline{M_1} \supseteq \overline{M_2} \supseteq \cdots .$$

The multiplicity of p^i as an elementary divisor of $A_{r,s}$ is precisely the dimension over $\mathbb{F}_q$ of the $\mathbb{F}_qG$-module $\overline{M_i}/\overline{M_{i+1}}$.

In most of the recent progress made in computing these elementary divisor multiplicities [10, 4, 11, 6, 2] much use is made of the $\mathbb{F}_qG$-submodule structure of the permutation module on projective points, $\mathbb{F}_q^{\mathcal{L}_1}$, which was described in great detail in [1]. The submodule lattice turns out to be isomorphic to the lattice of order ideals of a particular poset, and in all cases thus far the dimensions we are trying to compute can be expressed in terms of the dimensions of the $\mathbb{F}_q G$-composition factors of $\mathbb{F}_q^{\mathcal{L}_1}$. These papers are great examples of how representation theory can be applied to the solution of combinatorial problems, and also of how it can shed light on the nature of solutions already found by other means (for example, Hamada's p-rank formula [7]).

3. Future Projects

It is a goal of mine to determine the Smith normal forms of the zero-intersection incidence matrices $A_{r,s}$, for all parameters r, s, n, p, t (as described in the previous section). As already mentioned, this problem is still far from being solved. However, in our attack on skew lines in $PG(3, q)$ [2] we were in fact able to determine some of the p-elementary divisors for all of the matrices $A_{r,s}$. More precisely, we have determined the multiplicity of p^i as an elementary divisor of $A_{r,s}$, for $0 \leq i < t$. This corollary is a nice extension of the p-rank results in [11], and may find future application. An obvious application is to distinguish these incidence structures from others having the same p-rank, but there have been more interesting applications of results of this type in finite geometry and design theory [5, 3, 12].

The reader may be wondering why the incidence relations here seem to reflect a notion of "far apart" rather than the usual notion of "close together." This is really a matter of taste, as the Smith normal form results for this zero-intersection incidence relation can be readily translated to corresponding results for non-zero intersection (i.e., the complementary relation where two subspaces are incident if and only if their intersection is non-trivial). However, for some reason the problem seems less difficult, and the solution is cleaner, when it is formulated in this way. I am interested generally in these relations of "oppositeness."

When neither r nor s is one, there are other choices for the notion of incidence besides zero-intersection. For example, we can define two subspaces to be incident if and only if the smaller space is contained in the larger one. The corresponding problem for this inclusion relation appears to be orders of magnitude harder, and in this situation not even the p-ranks of the incidence matrices are known in general. One thing that I find interesting about this inclusion relation is that products of incidence matrices (or composition of incidence maps) behave well, in some situations. The most general results we proved in [2] give information about the elementary divisors of a matrix product, using information about the elementary divisors of the factor matrices. I hope to find application of these techniques here.

Many of the results about Smith normal forms of matrix products are difficult to use. They either apply in too special a situation (for example, when factor matrices are square and have relatively prime determinant), or are "interlacing" type theorems. The general lemmas proved in [2] can be viewed as an extension of some of the results of [9], where the author excellently plays the matrix viewpoint against the module viewpoint, and vice versa. Our contribution to the discussion is the definition and development of the notions of *right SNF basis* and *left SNF basis*. I think these concepts will be important in future applications of the Smith normal form. I also think there are still very basic results to be proved about Smith normal forms and elementary divisors, and I will be considering this beautiful topic in the long term.

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