Counting Canonical Partitions in the Random Graph

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Abstract

Algorithms are given for computing the number of n-element diagonal sets and the number of n-element strongly diagonal sets of binary sequences of length at most $2n-2$. The first number corresponds to the number of weak embedding types of $(2n-1)$-element transversal rooted binary set theoretic subtrees of binary sequences of length at most $2n-2$. The second number corresponds to the number of parts, r_n, in the canonical partition of Laflamme, Sauer and Vuksanovic of the n element subsets of an infinite random (Rado) graph, $\mathbb{RG}$. The value r_n is the critical value for a partition relation, since $\mathbb{RG} \to (\mathbb{RG})^n_{<\omega/r_n}$ and $\mathbb{RG} \not\to (\mathbb{RG})^n_{<\omega/r_n-1}$. These results are generalized to the set of d-ary sequences to obtain polynomials $\rho_n(d)$ and $\alpha_n(d)$. The sequence $\alpha_n(1)$ is the tangent numbers, and $r_n = \alpha_n(2)$. A key ingredient of the proof is the count of special collections of finite rooted binary trees related to Joyce trees.

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1 Introduction

By definition, a Joyce tree is a rooted binary tree for which no two nodes have the same level and all levels up to that of the top leaf have a node. They were named *Joyce trees* by Ross Street [11] after the physicist William P. Joyce who was using them in his calculations. Such trees underlie the definitions of *diagonal* and *strongly diagonal* subsets of d-ary sequences, that is, finite sequences whose entries come from the set $\{0, 1, \ldots, d-1\}$.

Before defining *diagonal set*, we introduce some auxilliary notions. Say $s = \langle s_0, s_1, \ldots, s_{i-1} \rangle$ and $t = \langle t_0, t_1, \ldots, t_{j-1} \rangle$ are *incomparable* if neither is an initial segment of the other. Call a set of d-ary sequences an *antichain* if its elements are pairwise incomparable. Call a set of d-ary sequences *transversal* if no two sequences have the same length. The *meet* of two sequences s and t, denoted $s \wedge t$, is the longest initial segment common to both. It may be empty, $\langle\rangle$. Given a set S of d-ary sequences, let $S^\wedge$ denote its *meet closure*, where $S^\wedge := \{ s \wedge t : s, t \in S \}$. Note that the set S is a subset of its meet closure, $S^\wedge$.

A set D of d-ary sequences is *diagonal* if it is an antichain whose meet closure forms a transversal rooted binary tree as follows:

- the set $D^\wedge$ is transversal;
- the root is the longest initial segment common to all elements of D;
- for any $s \in D^\wedge \setminus D$, the children of s are the shortest extensions, if any, in $D^\wedge$ of $s^\frown\langle \ell \rangle$ for $\ell < d$; and
- the set of leaves of the tree is D.

Since the tree is required to be binary, for each s, there will be exactly two values of $\ell < d$ for which $s^\frown\langle \ell \rangle$ has an extension. Note that the tree described above starting from an n-element set D has n leaves, so the meet closure has a total of $2n-1$ elements, all of which have different lengths. Thus it may be drawn as a rooted binary tree with one node on each of $2n-1$ levels. The natural notion of level of a sequence is the length of the sequence. With this understanding of levels, the transversal rooted binary tree $D^\wedge$ is isomorphic to a a Joyce tree which would have nodes on levels 0, 1, $\ldots$, $2n-2$.

A *set theoretic tree* is a partially ordered set, $(T, \leq)$, with the property that for any $t \in T$, the set of *predecessors*, $\{ s \in T : s < t \}$ is well-ordered by $<$. This notion of tree generalizes nicely to the transfinite. Of particular interest in this paper is the tree ${}^{\omega>}d$ of finite d-ary sequences partially ordered by end-extension, $\subseteq$, and subtrees of this tree such as ${}^{(2n-2)\geq}d$, the collections of d-ary sequences of length at most $2n-2$.

In Theorem 9.10, we give an expression for counting the number $\rho_n(d)$ of n-element diagonal subsets of ${}^{(2n-2)\geq}d$. As a corollary, we derive the fact that the number of weak embedding types of transversal rooted binary set theoretic subtrees of $({}^{(2n-2)\geq}d, \subseteq)$ with $2n-1$ nodes is also $\rho_n(d)$. Weakly embedded trees were introduced by Milliken [5], who proved a Ramsey Theorem for them which has been applied widely. The expression for $\rho_n(d)$ is translated into Maple code for easy computation.

A set D of d-ary sequences is *strongly diagonal* if it is diagonal and satisfies the following additional properties:

- for all $s = \langle s_0, s_1, \ldots, s_{i-1} \rangle \in D^\wedge$ and $t = \langle t_0, t_1, \ldots, t_{j-1} \rangle \in D$ with $i < j$, if $t_i \neq 0$, then s is an initial segment of t;
- for all $s = \langle s_0, s_1, \ldots, s_{i-1} \rangle \in D^\wedge$, if $t = \langle t_0, t_1, \ldots, t_{j-1} \rangle \in D$ is an extension of s, then either $t_i = 0$ or $t_i = 1$.

In Theorem 9.4, we give an expression for counting the number $\alpha_n(d)$ of n-element strongly diagonal subsets of ${}^{(2n-2)\geq}d$. This expression is translated into Maple code for ease of computation.

The algorithm for counting $\alpha_n(d)$ can be used to count the number of cells in a canonical partition of the n-element subsets of universal purely binary relational structures. This application is discussed in the final section of the paper, where the notion of canonical partition is defined and universal purely binary relational structures are discussed.

The paper is organized as follows. Section 2 contains basic facts and notation for finitely branching trees. Section 3 introduces the equivalence relations of *similarity* and *shape similarity.* Section 4 explores structural properties of diagonal sets and introduces *meet indicator sequences.* Section 5 uses the structural information to give a characterization of shape similarities. In Sections 6-8, we compute, for a fixed meet indicator sequence, the sizes of three collections of $(n+1)$-element diagonal subsets of the d-ary sequences of length at most $2n$: the collection of diagonal sets with minimal passing numbers, the collection of strongly diagonal sets, and the collection of diagonal sets. In Section 9, we sum over all meet indicator sequences to obtain algorithms for computing (1) the size $\alpha_{n+1}(d)$ of the collection of $(n+1)$-element strongly diagonal subsets of the d-ary sequences of length at most $2n$, and (2) the size $\rho_{n+1}(d)$ of the collection of $(n+1)$-element diagonal subsets of the d-ary sequences of length at most $2n$. Some sample polynomials are given, as are a few facts about $\alpha_n(d)$, e.g. the degree, the leading coefficient. In Section 10 we apply the algorithms of Section 9 to weak embedding types of types of transversal rooted binary set theoretic subtrees of $({}^{(2n-2)\geq}d, \subseteq)$ with $2n-1$ and to canonical partitions of universal countable purely binary relational structures, and the partition relations derivable from them.

2 Basic information

This section collects basic notation and definitions in one place for the convenience of the reader. Some of the definitions from the introduction are repeated for consistency.

Each ordinal is the set of its predecessors, so, in particular, $0 = \emptyset$, $1 = \{0\}$, $n = \{0, 1, \ldots, n-1\}$, and $\omega = \{0, 1, \ldots, \}$ is the set of non-negative integers. Denote by ${}^{\omega>}d$ the set of all finite sequences $s = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ with entries from $d = \{0, 1, \ldots, d-1\}$. When partially ordered by end-extension, $\subseteq$, this set is the regular d-ary tree with root the empty sequence, $\emptyset$. We write ${}^{(2n)\geq}d$ for the subtree of ${}^{\omega>}d$ of sequences of length at most $2n$. The *meet*, $s \wedge t$, of $s, t \in {}^{\omega>}d$ is the longest sequence which is an initial segment of both s and t. We write $|s| = m$ to indicate that the length of the sequence s is m. We freely move between functional notation for sequences and writing them explicitly as an ordered list, such as $\langle +1, +1, -1\rangle$. We write $s^\frown t$ for the concatenation of s and t.

Definition 2.1. Call a subset $P \subseteq {}^{\omega>}d$ *transversal* if different sequences have different lengths. Call P an antichain if all pairs are incomparable, i.e. no sequence is a proper initial segment of any other sequence. Call P *binary* if for each sequence $s \in P$ there are at most two values $i < d$ for which $s^\frown\langle i\rangle$ has an extension in P, that is, if each s has at most two immediate successors. Call P *diagonal* if it is an antichain whose meet closure is binary and transversal.

Facts 2.2. For any d with $2 \leq d < \omega$, and any $P \subseteq {}^{\omega>}d$, let $P^\wedge$ denote the closure of P under pairwise meets.

1. $P^\wedge = \{x \wedge y : x, y \in P\}$; that is, every element of $P^\wedge$ is realized as the meet of a pair of elements of P;

2. $P^\wedge$ is a rooted subtree of ${}^{\omega>}d$;

3. $|P^\wedge| \leq 2|P| - 1$;

4. if $P^\wedge$ is binary, then $|P^\wedge| = 2|P| - 1$;

5. if $|P^\wedge| \leq d$, then $|P|+1 \leq |P^\wedge|$, and this value is attained if all elements of P are immediate successors of the same node of ${}^{\omega>}d$;

6. if $d < |P^\wedge| = n$, then $n + \lceil (n-1)(d-1) \rceil \leq |P^\wedge|$.

3 Similarity

The equivalence relations of *similarity* and *shape similarity* are introduced for sets of d-ary sequences to prepare for counting diagonal and strongly diagonal sets.

Definition 3.1. Two subsets $P, Q \subseteq {}^{\omega>}d$ are *shape similar*, in symbols, $P \approx Q$, if there is a *shape similarity* $f : P \to Q$, that is a bijection satisfying the following preservation conditions for all $x, y, u, v \in P$:

1. (meet containment) $x \wedge y \subseteq u \wedge v$ if and only if $f(x) \wedge f(y) \subseteq f(u) \wedge f(v)$;

2. (meet length order) $|x \wedge y| < |u \wedge v|$ if and only if $|f(x) \wedge f(y)| < |f(u) \wedge f(v)|$;

3. ($<_{\text{lex}}$ order) if $x <_{\text{lex}} y$, then $f(x) <_{\text{lex}} f(y)$.

Call P and Q *similar*, in symbols $P \sim Q$, if there is a *similarity* $f : P \to Q$, namely a shape similarity which satisfies the following preservation condition for all $x, z \in P$:

4. (leaf passing number) if $|z| > |x|$, then $z(|x|) = f(z)(|f(x)|)$;

Next consider a notion of collapse for diagonal sets.

Definition 3.2. For any sequence $s : q \to 2$ any strictly increasing function $h : p \to q$, define the *collapse of s with respect to h* by $\text{clp}_h(s) = s \circ h$, i.e. $\text{clp}_h(s)(k) = s(h(k))$ for all $k < p$.

If $B \subseteq {}^{\omega>}d$ is an n-element diagonal set whose meet closure is listed in increasing order as $b_0, b_1, \ldots, b_{2n-2}$ and $I : 2n-1 \to \omega$ is the function $i \mapsto |b_i|$, the $\text{clp}(B) = \{ \text{clp}_I(b) : b \in B \}$ is the *collapse of B*.

Lemma 3.3. *If $D \subseteq {}^{\omega>}d$ is an n-element diagonal set, then $D \sim \mathrm{clp}(D)$ and $D \approx \mathrm{clp}(D)$.*

Proof. Let $d_0, d_1, \ldots, d_{2n-2}$ enumerate the meet closure of an n-element diagonal set in increasing order of length, and let $I : (2n-1) \to \omega$ be defined by $I(j) = |d_j|$. Then $\mathrm{clp}(d_0) = \emptyset$. By recursion, $|\mathrm{clp}(d_j)| = j$. Thus for $i < j$, $\mathrm{clp}(d_j)(|\mathrm{clp}(d_i)|) = \mathrm{clp}(d_j)(i) = d_j(|d_i|)$. Therefore, the function $\mathrm{clp}_I : B \to \mathrm{clp}(B)$ is a similarity and a shape similarity. □

Lemma 3.4. *For any n, d with $2 \leq n, d < \omega$, the relations of shape similarity and similarity are equivalence relations on n-element diagonal subsets of ${}^{\omega>}d$, and each has finitely many equivalence classes.*

Proof. To see that these are equivalence relations, check that the identity is both a shape similarity and a similarity, that inverses of shape similarities are shape similarities, and the same is true for similarities, and both shape similarities and similarities are closed under composition. To see that there are only finitely many shape similarity classes, and only finitely many similarity classes, note that every diagonal set of size $n \geq 2$ is shape similar and similar to one which is a subtree of the d-ary tree of sequences of length at most $2n-2$, by Lemma 3.3 □

Let $[P]_\approx$ denote the shape similarity class of a diagonal set P, and let $[P]_\sim$ denote its similarity class.

We restate the definition of *strongly diagonal* using the compact notation introduced in the previous section.

Definition 3.5. A subset $D \subseteq {}^{\omega>}d$ is *strongly diagonal* if it is an antichain whose meet closure, $D^\wedge$, is transversal, and if, for all $x, y, z \in D$ with $x \neq y$:

1. $|x \wedge y| < |z|$ and $x \wedge y \not\subseteq z$ implies $z(|x \wedge y|) = 0$.

2. $x(|x \wedge y|) = 0$ or $x(|x \wedge y|) = 1$.

Lemma 3.6. *Every subset $D \subseteq {}^{\omega>}d$ which is strongly diagonal is also diagonal. Moreover, the property of being strongly diagonal is hereditary. That is, every subset of a strongly diagonal set is strongly diagonal.*

Lemma 3.7. *Suppose D and E are both strongly diagonal. If $D \sim E$, then $\mathrm{clp}(D) = \mathrm{clp}(E)$.*

Proof. Let $d_0, d_1, \ldots, d_{2n-2}$ enumerate $D^\wedge$, and $e_0, e_1, \ldots, e_{2n-2}$ enumerate $E^\wedge$, both in increasing order of length. Let $I, J : (2n-1) \to \omega$ be defined by $I(j) = |d_j|$ and $J(k) = |e_k|$. Now suppose $g : D \to E$ is a similarity. Then for all $d_\ell \in D$, every value $\mathrm{clp}_I(d_\ell)(k)$ is either a leaf passing number or meet passing number of D. Since g preserves length order, meet length order, lexicographic order and leaf passing numbers, and since both D and E are strongly diagonal, it follows that for all k, $\mathrm{clp}_I(d_\ell)(k) = \mathrm{clp}_J(e_\ell)(k)$, so $\mathrm{clp}_I(d_\ell) = \mathrm{clp}_J(e_\ell)$. ☐

Lemma 3.8. *Every diagonal set D is similar to a strongly diagonal set.*

Proof. Without loss of generality, assume that $\mathrm{clp}(D) = D$. Enumerate $D^\wedge$ in increasing order of length as $d_0, d_1, \ldots, d_{2n-2}$. Let e_j for $j < n$ be the sequence of length $|j| = |d_j|$ such that for all $\ell < j$,

$$e_j(\ell) = \begin{cases} d_j(\ell) & \text{if } d_\ell \in D, \\ |\{ d_k(\ell) : d_\ell \subsetneq d_k <_{\mathrm{lex}} d_j \}|, & \text{if } d_\ell \subsetneq d_j \text{ and } d_\ell \notin D \\ 0, & \text{otherwise.} \end{cases}$$

Let $E = \{ e_j : d_j \in D \}$. Since D has binary meet closure, the cardinality of $\{ d_k(\ell) : d_\ell \subsetneq d_k <_{\mathrm{lex}} d_j \}$ is either zero or one. Thus by construction, E is strongly diagonal. The reader may check that $f : D \to E$ defined by $f(d_j) = e_j$ is the required similarity. ☐

Lemma 3.9. *The number of similarity classes of n-element diagonal subsets of the complete d-ary tree equals the number of strongly diagonal subsets of ${}^{(2n-2)\geq}d$.*

Proof. By Lemma 3.8, every similarity class includes a representative which is strongly diagonal. By Lemma 3.7, strongly diagonal sets which are similar have the same collapse. By Definition 3.2, the collapse of an n-element diagonal set is a subset of ${}^{(2n-2)\geq}d$. Hence the mapping that takes a similarity class to the collapse of a strongly diagonal representative is a bijection. ☐

One more collection is of interest for the purposes of counting.

Definition 3.10. A diagonal set $A \subseteq {}^{\omega>}d$ has *minimal passing numbers* if for all $a, b \in A^\wedge$,

1. if $a \subsetneq b$, then $b(|a|) < 2$; and

2. if $a \not\subseteq b$ and $|a| < |b|$, then $b(|a|) = 0$.

Note that diagonal sets with minimal passing numbers have collapses which are subsets of ${}^{\omega>}2$.

Lemma 3.11. *Every n-element diagonal set $A \subseteq {}^{\omega>}d$ is shape similar to a diagonal set with minimal passing numbers.*

Proof. For $a \in A$, define $b_a : |a| \to d$ by changing only those values necessary to make it have minimal passing numbers. Let $B = \{ b_a : a \in A \}$ and check that $A \approx B$. ☐

Lemma 3.12. *Suppose D and E are n-element diagonal sets with minimal passing numbers. If $D \approx E$, then $\mathrm{clp}(D) = \mathrm{clp}(E)$.*

Proof. The proof is like that for Lemma 3.3 with a shape similarity in place of a similarity, and minimal passing numbers making up the difference. ☐

The two previous lemmas combine to give the next result.

Lemma 3.13. *The number shape similarity types of n-element diagonal subsets of ${}^{\omega>}d$ equals the number of diagonal subsets of ${}^{(2n-2)\geq}d$ with minimal passing numbers.*

4 Structural properties of diagonal sets

Level preserving embeddings play an important role in the Ramsey theory of finite subtrees of regular finite branching trees. In discussions of them, it is useful to be able to talk about immediate successors relative to a subtree as well as relative to the original tree. The notation below was introduced by Milliken [5].

Definition 4.1. For any tree T of sequences under end extension, and any node $s \in T$, define the *set of immediate successors of s in T* as

$$\mathrm{IS}(s, T) := \{ t \in T : s \subset t \wedge (\forall u)(s \subset u \subseteq t \implies u = t) \}.$$

Here is a parallel definition that generalizes the concept of level of a tree.

Definition 4.2. For any tree T of sequences under end extension, and any node $s \in T$, define the *successor level of s in T* as

$$\mathrm{S}\ell(s, T) := \{ t \in T : |s| < |t| \wedge (\forall u)(|s| < |u| \wedge u \subseteq t \implies u = t) \}.$$

Facts 4.3. Suppose $P \subseteq {}^{\omega>}d$ for some d with $2 \leq d < \omega$ is a finite set.

1. For all $p \in P$, $\mathrm{IS}(p, P) \subseteq \mathrm{S}\ell(p, P)$.
2. For all $p, q \in P$, if $p \neq q$, then $\mathrm{IS}(p, P) \neq \mathrm{IS}(q, P)$.
3. If $r \in P^\wedge$ is the root of $P^\wedge$, then the non-empty sets $\mathrm{IS}(p, P)$ for $p \in P^\wedge$ form a partition of $P^\wedge \setminus \{r\}$.

Lemma 4.4. *Let P be a subset of ${}^{\omega>}d$. For all p and q in $P^\wedge$, if $|p| < |q|$, then there is a unique $s \in S\ell(p, P^\wedge)$ with $s \subseteq q$. Moreover, if $IS(p, P) = S\ell(p, P)$, then $p \subseteq q$ for all $q \in P$ with $|q| > |p|$.*

Lemma 4.5. *Suppose that A is a diagonal set. Further suppose that $a, b \in A^\wedge$ are such that $|a| < |b|$, and $|b|$ is minimal among lengths larger than $|a|$. That is, suppose that for all $c \in A^\wedge$, if $|a| < |c| \leq |b|$, then $|c| = |b|$. Then for all $c \in A^\wedge$ with $|c| = |b|$, one has $IS(c, A^\wedge) \subseteq S\ell(b, A^\wedge)$. In addition,*

$$S\ell(a, A^\wedge) = S\ell(b, A^\wedge) \cup \{ b \} \setminus IS(b, A^\wedge).$$

Definition 4.6. Suppose that $A \subseteq {}^{\omega>}d$ is an n-element diagonal set with $A^\wedge$ enumerated in decreasing length order as $x_0, \ldots, x_{2n-2}$. The *meet indicator sequence of A*, σ_A, is the sequence of $+1$'s and -1's of length $2n-1$ defined by $\sigma_A(j) := 1$ if and only if $x_j \in A$. Given any $\sigma : 2n-1 \to \{-1, +1\}$, define the *tally sequence of σ* $\tau_\sigma : (2n-1) \to \omega$ by $\tau_\sigma(0) = 0$ and for positive j, $\tau_\sigma(j) := \sum_{i<j} \sigma(j)$. For notational convenience, if $\sigma = \sigma_A$, write τ_A for τ_{σ_A}.

Lemma 4.7. *Suppose that $A \subseteq {}^{\omega>}d$ is an $(n+1)$-element diagonal set Then for all $j < 2n+1$,*

$$|S\ell(x_j, A^\wedge)| = \tau_A(j).$$

Proof. Let $x_0, x_1, \ldots, x_{2n}$ enumerate $A^\wedge$ in decreasing order of length. Use recursion, Lemma 4.5, and the fact that x_j has no immediate successors if $x_j \in A$, and otherwise has exactly two. $\square$

Corollary 4.8. *Suppose that $A \subseteq {}^{\omega>}d$ is an $(n+1)$-element diagonal set. Then for all positive $k \leq 2n+1$, the partial sum $\sum_{j<k} \sigma_A(j)$ is positive, and $\sum_{j<2n+1} \sigma_A(j) = 1$.*

Proof. The value of $\sum_{j<k} \sigma_A(j)$, for positive $k < 2n+1$, is $\tau_A(k)$, by definition of τ_A. By the previous lemma, the value of τ_A is also $|\mathrm{S}\ell(x_k, A^\wedge)|$. Since $\mathrm{S}\ell(x_k, A^\wedge)$ is non-empty, for positive $k < 2n+1$, it follows that the proper partial sums are all positive. Since σ_A is a sequence of $n+1$ $(+1)$'s and n (-1)'s, the sum of the entire sequence is $+1$, as desired. $\square$

5 Ordering closures of diagonal sets

Definition 5.1. For sequences a and b write $a \lessdot b$ if $|a| < |b|$.

Definition 5.2. Suppose that $A, B \subseteq {}^{\omega >}d$ are diagonal. Say that bijection $g : A^\wedge \to B^\wedge$ *preserves successors* if for all $a \in A^\wedge$, $\mathrm{IS}(g(a), B^\wedge) = \{ g(c) : c \in \mathrm{IS}(a, A^\wedge) \}$, and g restricted to $\mathrm{IS}(a, A^\wedge)$ preserves lexicographic order.

Lemma 5.3. *Suppose that $A, B \subseteq {}^{\omega >}d$ are diagonal. If $f : A \to B$ is a shape similarity, then the extension $f^* : A^\wedge \to B^\wedge$ of f defined by $f^*(x \wedge y) = f(x) \wedge f(y)$ preserves length order, length equality, and successors. Moreover, f^* is the unique $\lessdot$ preserving map.*

Proof. Suppose $f : A \to B$ is a shape similarity. Then f^* preserves length order and length equality by definition of shape similarity, It preserves successors by basic facts about shape similarities. It is the unique $\lessdot$ preserving map because it preserves length order and $A^\wedge$, $B^\wedge$ are both transversal. □

Lemma 5.4. *Suppose that $A, B \subseteq {}^{\omega >}d$ are diagonal, and $g : A^\wedge \to B^\wedge$ is a bijection that preserves length order, length equality, and successors. Then the restriction of g to A is a shape similarity of A to B.*

Proof. List $A^\wedge$ in decreasing order of length as $x_0, x_1, \ldots, x_{2n-2}$, and list $B^\wedge$ in decreasing order of length as $y_0, y_1, \ldots, y_{2n-2}$.

Use induction on $\ell < 2n - 1$ to show that for all $i < j \le \ell$, $g(x_i) = y_i$, $g(x_j) = y_j$, $x_i \subseteq x_j$ if and only if $g(x_i) \subseteq g(x_j)$, and $x_i \in \mathrm{S}\ell(x_j, A^\wedge)$ if and only if $g(x_i) \in \mathrm{S}\ell(g(x_j), B^\wedge)$.

To start the induction, note that $g(x_0) = y_0$ and $g(x_1) = y_1$, since these are the two longest elements of $A^\wedge$ and $B^\wedge$, and g preserves length order. Also, $\mathrm{S}\ell(x_0, A^\wedge) = \emptyset = \mathrm{S}\ell(y_0, B^\wedge)$, and $\mathrm{S}\ell(x_1, A^\wedge) = \{x_0\}$, $\mathrm{S}\ell(y_1, B^\wedge) = \{y_0\} = \{g(x_0\}$.

Use the facts that g preserves length order, length equality, and successors, together with Lemma 4.5 in the interesting case of the induction step. □

6 Minimal passing numbers

This section includes the computation of the number of shape similarity classes of $(n+1)$-element diagonal subsets of ${}^{2n \geq}d$ with the same meet indicator sequence as a given diagonal set with minimal passing numbers. To simplify the statements, the following notation is introduced.

Notation. Let $\Delta(n, d)$ denote the n-element diagonal subsets of ${}^{2n\geq}d$, and let $\mathcal{M}_d^n$ be the elements of $\Delta(n, d)$ with minimal passing numbers.

Recall that in Lemma 4.7, the meet indicator sequence σ_A was used to compute the size of $\mathrm{S}\ell(x_j, A^\wedge) = \tau_A(j)$ for j positive.

Lemma 6.1. *Suppose that $A \in \Delta(n+1, d)$ is an $(n+1)$-element diagonal set. Then*

$$\left|\left\{ B \in \mathcal{M}_d^{n+1} : \sigma_A = \sigma_B \right\}\right| = \prod_{\substack{j<2n \\ \sigma(j)<0}} \left(\sum_{i<j} \sigma(i)\right)\left(-1 + \sum_{i<j} \sigma(i)\right)$$
$$= \prod_{\substack{j<2n \\ \sigma(j)<0}} \tau_A(j)\,(\tau_A(j) - 1)\,.$$

Proof. For each $B \in \mathcal{M}^{n+1}$, with $B^\wedge$ enumerated in decreasing length order as $y_0, y_1, \ldots, y_{2n+1}$, define $C_B : (2n+1) \to d \times d$ by $C_B(j) = \langle 0, 0\rangle$ if $\sigma_B(j) = +1$, and otherwise, $C_B(j) = \langle \ell, m\rangle$ where y_p and y_q, the ℓth and mth elements in the length order of $\mathrm{S}\ell(y_j, B^\wedge)$, satisfy the following conditions: $y_p{}^\frown\langle 0\rangle \subseteq y_j$, $y_q{}^\frown\langle 1\rangle \subseteq y_j$, and $y_p, y_q \in \mathrm{IS}(y_j, B^\wedge)$.

If B and D in $\Delta(n+1, d)$ have minimal passing numbers, and $C_B = C_D$, then by induction using Lemma 4.5 one can show that the unique length order preserving map g from $B^\wedge$ to $D^\wedge$ sends $\mathrm{IS}(b, B^\wedge)$ to $\mathrm{IS}(g(b), D^\wedge)$ and $\mathrm{S}\ell(b, B^\wedge)$ to $\mathrm{S}\ell(g(b), D^\wedge)$. Thus by Lemmas 5.4 and 3.12, if B and D in $\Delta(n+1, d)$ have minimal passing numbers, and $C_B = C_D$, then $B \approx D$ and $B = D$.

The value declared in the lemma is the number of functions $C : (2n+1) \to d \times d$ such that $C(j) = \langle 0, 0\rangle$ if $\sigma_A(j) = +1$ and $C(j) = \langle \ell, m\rangle$ for $\ell \neq m$ with both less than $\tau_A(j)$. For every $B \in \Delta(n+1, d)$ with minimal passing numbers, C_B is such a function. Thus to complete the proof, it is enough to show that for each such function C, there is some $D \in \mathcal{M}_d^{n+1}$ with $\sigma_D = \sigma_A$ and $C_D = C$.

Let $C : (2n+1) \to d \times d$ be such a function. We plan to define z_j for $j < 2n+1$ by recursion, but first we need to create sets of indices that mimic level sets and sets of immediate successors. Define L_k, S_k for $k < 2n+1$

by recursion. To start the recursion, let $S_0 = L_0 = \emptyset$. If L_h, S_h have been defined for $h < k$, let $L_k = L_{k-1} \cup \{k-1\} \setminus S_{k-1}$. If $\sigma_A(k) = 1$, then let $S_k = \emptyset$; otherwise, for $C(k) = \langle \ell, m \rangle$, set $S_k = \{p, q\}$ where p and q are the ℓth and mth elements of L_k listed in increasing order. Note that the sets S_k partition $0, \dots, 2n-1$ into two element sets. Next define z_k by recursion on $k \le 2n$ in reverse order. Let $z_{2n} = \emptyset$. Suppose that z_h has been defined for $h > k$ with $|z_h| = h$. Let r be the unique value with $k \in S_r$. Let z_k be the sequence of length k extending z_r to length k by padding with zeros except possibly at r where $z_k(r) = 1$ if r is the mth element of L_k and $z_k(r) = 0$ otherwise. Let $D = \{\, z_k : \sigma_A(k) > 0 \,\}$. By construction, $D^\wedge = \{\, z_k : k \le 2n \,\}$. Thus D is diagonal. Furthermore, D has minimal passing numbers, and $C_D = C$. □

7 Small strongly diagonal sets

This section includes the computation of the number of strongly diagonal sets in $\Delta(n+1, d)$ with a given meet indicator sequence.

Notation. Let $\mathcal{A}_d^n$ denote the strongly diagonal sets in $\Delta(n, d)$.

Lemma 7.1. *Suppose that $A \in \mathcal{A}_d^{n+1}$ is an $(n+1)$-element strongly diagonal set and $\sigma = \sigma_A$. Set $Q(\sigma) = \sum_{0<j<2n \wedge \sigma(j)=1} \tau_\sigma(j)$. The number of $(n+1)$-element strongly diagonal subsets of ${}^{2n\geq}d$ with meet indicator sequence σ is*

$$\left|\left\{\, B \in \mathcal{A}_d^{n+1} : \sigma_B = \sigma \,\right\}\right| = d^{Q(\sigma)} \prod_{\substack{j \le 2n \\ \sigma(j) < 0}} \tau_\sigma(j) \left(\tau_\sigma(j) - 1\right).$$

Proof. By Lemma 3.11, every diagonal set is shape similar to one with minimal passing numbers.

Suppose $B \in \mathcal{A}_d^{n+1}$ with $\sigma_B = \sigma$ and $D \in [B]_\approx$ has minimal passing numbers. Then $\sigma_D = \sigma$ also. Enumerate $B^\wedge$ and $D^\wedge$ in decreasing order of length as $y_0, y_1, \dots, y_{2n}$ and $z_0, z_1, \dots, z_{2n}$. Since B and D are shape similar, B is strongly diagonal and D has minimal passing numbers, $y_n(|y_i|) = z_n(|z_i|)$ except possibly when $y_i \in B$ (and $z_i \in D$). To determine the number of possibly places for change without repetition, sum up the sizes of $\mathrm{S}\ell(y_i, B^\wedge)$ for $y_i \in B$. Notice that $Q(\sigma)$ is this sum, since the cardinality of the successor level may be computed from $\sigma_B = \sigma = \sigma_D$, by Lemma 4.7.

Thus for a fixed diagonal set $D \in \Delta(n+1, d)$ with minimal passing numbers, the number of sets $B \in \mathcal{A}_d^{n+1}$ with $\sigma_B = \sigma$ and $B \approx D$ is $d^{Q(\sigma)}$. Every diagonal set with minimal passing numbers is strongly diagonal. Moreover, since elements of $\Delta(n+1,d)$ are their own collapses, by Lemma 3.12, it is enough to multiply the number of diagonal sets in $\Delta(n+1,d)$ by $d^{Q(\sigma)}$. Thus the lemma follows from Lemma 6.1. $\square$

8 Small Diagonal sets

This section includes the computation of the number of $(n+1)$-element diagonal subsets of ${}^{2n\geq}d$ with a given meet indicator sequence.

Lemma 8.1. *Suppose $A \subseteq {}^{2n\geq}d$ is an $(n+1)$-element diagonal set and $\sigma = \sigma_A$. Let $R(\sigma) := \sum_{j<2n} [\tau_\sigma(j) + \sigma(j) - 1]$. Then the number of $(n+1)$-element diagonal subsets of ${}^{2n\geq}d$ with meet indicator sequence σ is*

$$|\{ B \in \Delta(n+1,d) : \sigma_B = \sigma \}| = d^{R(\sigma)} \left[\frac{d(d-1)}{2}\right]^n \prod_{\substack{j\leq 2n \\ \sigma(j)<0}} \tau_\sigma(j)\,(\tau_\sigma(j)-1)\,.$$

Proof. By Lemma 3.11, every diagonal set is shape similar to one with minimal passing numbers.

Suppose $B \in \Delta(n+1,d)$ with $\sigma_B = \sigma$ and $D \in [B]_\approx$ has minimal passing numbers. Then $\sigma_D = \sigma$ also. Enumerate $B^\wedge$ and $D^\wedge$ in decreasing order of length as $y_0, y_1, \dots, y_{2n}$ and $z_0, z_1, \dots, z_{2n}$. For each $i < j \leq 2n$, let $K(i,j)$ be the unique k with $y_k \in \mathrm{S}\ell(y_j, B^\wedge)$ such that $y_k \subseteq y_i$. By Lemma 4.4, such a value exists. Notice that $y_i(j) = y_k(j)$ is the passing number of y_i at y_j and of y_k at y_j. This association allows us to uniquely identify the passing numbers at y_j. Since $B^\wedge$ and $D^\wedge$ are shape similar, by Lemma 5.3, the unique length order preserving map $f^* : B^\wedge \to D^\wedge$ preserves successors. Use Lemma 4.5 and induction as in the proof of Lemma 5.4 to show that $\mathrm{S}\ell(f^*(a), B^\wedge) = \{ f^*(c) : c \in \mathrm{S}\ell(d, D^\wedge) \}$. It follows that $K(i,j)$ is the unique k with $z_k \in \mathrm{S}\ell(z_j, D^\wedge)$ with $z_k \subseteq z_i$. Consider three cases, with $k = K(i,j)$:

1. $x_k \in \mathrm{S}\ell(x_j, B^\wedge)$ and $x_j \in B$, in which case one also has $y_k \in \mathrm{S}\ell(y_j, D^\wedge)$ and $y_j \in D$;

2. $x_k \in \mathrm{S}\ell(x_j, B^\wedge) \setminus \mathrm{IS}(x_j, B^\wedge)$ and $x_j \notin B$, in which case one also has $y_k \in \mathrm{S}\ell(y_j, D^\wedge) \setminus \mathrm{IS}(y_j, D^\wedge)$ and $y_j \notin B$; and

3. $x_k \in \mathrm{IS}(x_j, B^\wedge)$, in which case one also has $y_k \in \mathrm{IS}(y_j, D^\wedge)$.

Given D, one can construct a diagonal set E shape similar to D with any passing numbers one likes as long as one maintains the appropriate lexicographic order for immediate successors. There are $\sum_{j<2n} [\tau_D(j) + \sigma_D(j) - 1]$ total many choices covered in the first cases. There are n meets, and each gives rise to $(d)(d-1)/2$ choices for the pair of passing numbers of its immediate successors that respect the appropriate lexicographic order.

It follows that for each diagonal set $D \in \Delta(n+1, d)$ with minimal passing numbers and $\sigma_D = \sigma$, there are $d^{R(\sigma)}[(d)(d-1)/2]^n$ many diagonal sets $B \in \Delta(n+1, d)$ with $B \approx D$ and $\sigma_B = \sigma$.

Thus the lemma follows from Lemma 4.7. □

9 The algorithms

The previous sections included computations of certain collections of $(n+1)$-element diagonal subsets of ${}^{2n\geq}d$ with a fixed meet indicator sequence σ_A: those with minimal passing numbers, the strongly diagonal ones, and the entire set of such diagonal subsets. The sequence σ_A was a key factor in all these computations. The sequences that occur in this fashion have been studied (see *Concrete Mathematics* [4] pages 345-347). They are closely related to the *ballot sequences* discussed in *Enumerative Combinatorics, volume 2* [13] (see page 173 for a definition).

Definition 9.1. A sequence $\sigma : (2n+1) \to \{-1, +1\}$ is a 2-Raney sequence of length $2n+1$ if all of its partial sums are positive and its total sum is $+1$. Let $\mathcal{R}(n)$ denote the set of 2-Raney sequences of length $2n+1$.

Lemma 9.2. *For all $n < \omega$, the number of sequences in $\mathcal{R}(n)$ is a Catalan number:*

$$|\mathcal{R}(n)| = C(n) = \binom{2n}{n}\frac{1}{n+1}.$$

Proof. George Raney [7] showed in 1959 that if $\langle x_0, x_1, \ldots, x_m\rangle$ is any sequence of integers whose sum is $+1$, then exactly one of the cyclic shifts

$$\langle x_0, x_1, \ldots, x_m\rangle, \langle x_1, x_3, \ldots, x_m, x_0\rangle, \ \ldots, \langle x_m, x_0, \ldots, x_{m-1}\rangle$$

has all of its partial sums positive. In *Concrete Mathematics*, Graham, Knuth and Patashnik [4] (see pages 345-6) show that the number of these sequences is the Catalan number above. □

Proposition 9.3. *For any positive $n < \omega$ and any $\sigma : (2n+1) \to \{-1, 1\}$, there is some diagonal set $D \subseteq {}^{\omega>}d$ with $\sigma_D = \sigma$ if and only if $\sigma \in \mathcal{R}(n)$.*

Proof. For the first direction, note that by Corollary 4.8, every meet indicator sequence for an $(n+1)$-element antichain is a 2-Raney sequence in $\mathcal{R}(n)$.

For the other direction, suppose $\sigma \in \mathcal{R}(n)$. Since the sum of the entire sequence is $+1$, there are $n+1$ indices for which σ takes the value $+1$ and n indices for which σ takes the value -1. Let $i_0, i_1, \ldots, i_{n-1}$ list the indices with $\sigma(i) = -1$ in increasing order, and let $k_0, k_1, \ldots, k_n$ list the indices with $\sigma(k) = +1$.

Since all partial sums are positive, $k_0 = 0$ and $k_1 = 1$. Moreover, $i_{n-1} = 2n$. Also $k_{j+1} < i_j$ for all $j < n-1$, else for the least ℓ with $k_{\ell+1} > i_\ell$ one has $k_\ell < i_{\ell-1} < i_\ell < k_{\ell+1}$ which leads to the contradiction that $\sum_{i \leq i_\ell} \sigma(i) = 0$.

Let $x_0 : 2n \to 2$ be the constantly 0 sequence. For $j < n-1$, let x_{i_j} be the constantly 0 sequence of length $(2n-2) - i_j$, and let $x_{k_{j+1}}$ be the extension of $x_{i_j}{}^\frown\langle 1 \rangle$ by all zeros to a sequence of length $2n - k_{j+1}$. Notice that $x_{i_{n-1}} = x_{2n} = \emptyset$ under this definition, and that all lengths from 0 to $2n$ occur.

Finally, let E be the set of x_k with $\sigma(k) = +1$. Then $E \subseteq {}^{2n\geq}2$ is strongly diagonal and $\sigma_E = \sigma$. ∎

Theorem 9.4. *For positive $n < \omega$, d with $2 \leq d < \omega$ and $Q(\sigma) := \sum_{j<2n \wedge \sigma(j)>0} \tau_\sigma(j)$, the number of $(n+1)$-element strongly diagonal subsets of ${}^{2n\geq}d$ is*

$$\alpha_{n+1}(d) = \sum_{\sigma \in \mathcal{R}(n)} d^{Q(\sigma)} \prod_{\substack{j \leq 2n \\ \sigma(j)<0}} \tau_\sigma(j)(\tau_\sigma(j) - 1)$$
$$= \sum_{\sigma \in \mathcal{R}(n)} \prod_{j \leq 2n} \theta_\sigma(j),$$

where $\theta_\sigma(j) = d^{\tau_\sigma(j)}$ if $\sigma(j) > 0$ and $\theta_\sigma(j) = \tau_\sigma(j)(\tau_\sigma(j) - 1)$ if $\sigma(j) < 0$.

Proof. Sum the results of Lemma 7.1. ∎

In order to take the above theorem and turn it into a Maple procedure, it is useful to introduce some small values that fit in. A single node is may be considered an antichain with transversal binary meet closure. There are no leaf or meet passing numbers to consider. Therefore, we set $\alpha_1(d) = 1$ for all d. For convenience, set $\alpha_n(d) = 0$ for all $n < 1$.

Note that $\tau_\sigma(j)$ is the difference between the number of indices smaller than j with $\sigma(j) > 0$ and the number of indices smaller than j with $\sigma(j) > 0$. Hence $\tau_\sigma(j) = j - 2k$ where k is the number of indices smaller than j with $\sigma(j) > 0$. We introduce a two parameter family of polynomials, $p(k, m, x)$, where $p(k, m, x)$ computes the sum over all initial segments μ of length k of 2-Raney sequences with m entries of (-1) and $k - m$ entries of $(+1)$ of the product $\prod_{j \le k} \theta_\mu(j)$. Then $\alpha_n(x) = p(2n - 1, n - 1, x)$. Being an initial segment of a 2-Raney sequence is something that can be easily tested: μ must satisfy $2m < k$. Thus we set $p(k, m, x) = 0$ if $2m \geq k$ or if $m < 0$. For $m = 0$, we can directly compute the value of $p(k, 0, x) = x^{[k(k-1)]/2}$. Since the recurrence relation in terms of initial segments of length one less uses both $p(k-1, m, x)$ and $p(k-1, m-1, x)$, this initial data is needed for the Maple procedure.

```
p := proc(k,m,x) option remember;
  if (k < 1 or m < 0 or 2*m >= k) then
    0
  elif (m = 0) then
    x^((k)*(k-1)/2)
  else
    p(k-1,m-1,x)*(k-1-2*(m-1))*(k-2-2*(m-1))  # ends in -1
    +p(k-1,m,x)*(x^(k-1-2*(m)))               # ends in +1
  end if end proc:
P := n -> p(2*n-1,n-1,x);

for n from 1 to 10 do
  sort(expand(P(n)));
end do;
```

Corollary 9.5. *The value $\alpha_n(1)$ is the cardinality of $\mathcal{M}_n^d$, the set of diagonal subsets of ${}^{(2n-2)\geq}d$ with minimal passing numbers, and it is also equal to the number of shape similarity classes of n-element diagonal subsets of ${}^{\omega>}d$. Moreover, $\langle\, \alpha_n(1) : 1 \leq n < \omega \,\rangle$ is the sequence of tangent numbers, so $\alpha_n(1) = t_n$ may also be computed using the generating function*

$$\tan(x) = \sum_{1}^{\infty} t_n \frac{x^{2n-1}}{(2n-1)!}.$$

Proof. Compare Lemma 6.1 and Lemma 7.1. By Lemma 3.13, the number of classes for the two collections is the same. To see that this sequence is the tangent numbers, observe that Vuksanovic computes this quantity in [14] (for a version in the language of category theory, see [1]), or check Sloane's On-Line Encyclopedia of Integer Sequences [12] where the tangent numbers appear as the number of Joyce trees on $2n-1$ nodes. □

Corollary 9.6. *For positive $n < \omega$ and d with $2 \le d < \omega$, the number of similarity classes of $(n+1)$-element diagonal subsets of ${}^{\omega>}d$ is $\alpha_{n+1}(d)$.*

Proof. Use Theorem 9.4 and Lemma 3.9. □

Examples 9.7. Here are the values of $\alpha_n(2)$ for $n = 1, \ldots, 10$:

1 :	1
2 :	4
3 :	112
4 :	12352
5 :	4437760
6 :	4686103552
7 :	13624250626048
8 :	104218697796173824
9 :	2028257407393613676544
10 :	97849915247810309454561280

Examples 9.8. The following are a few examples of the polynomials $\alpha_n(d)$:

- $\alpha_2(d) = 2d$;
- $\alpha_3(d) = 12d^3 + 4d^2$;
- $\alpha_4(d) = 144d^6 + 72d^5 + 48d^4 + 8d^3$;
- $\alpha_5(d) = 2880d^{10} + 1728d^9 + 1728d^8 + 1008d^7 + 432d^6 + 144d^5 + 16d^4$.

Corollary 9.9. *The polynomial $\alpha_{n+1}(d)$ the following properties:*

1. *the degree of $\alpha_{n+1}(d)$ is $n(n+1)/2$;*
2. *the leading coefficient of $\alpha_{n+1}(d)$ is $n!(n+1)!$;*
3. *the lowest degree term of $\alpha_{n+1}(d)$ is $2^n d^n$; and*
4. *$n!(n+1)!\, d^{n(n+1)/2} \le \alpha_{n+1}(d) < (2n)!\, d^{n(n+1)/2}$.*

Proof. Each 2-Raney sequence $\sigma \in \mathcal{R}(n)$ contributes to the $d^{Q(\sigma)}$ term of $\alpha_{n+1}(d)$ the quantity $\prod_{j\leq 2n \wedge \sigma(j)<0} \tau_\sigma(j)(\tau_\sigma(j)-1)$.

By the definitions of τ_σ and 2-Raney sequences, the value of $\tau_\sigma(j)$ is positive for all positive $j < 2n$ and $\tau_\sigma(0) = 0$. Thus $Q(\sigma) \geq n$, since there are $n+1$ indices j for which $\sigma(j) > 0$. This value n is achieved by the 2-Raney sequence $\sigma_* := \langle 1\rangle^\frown\langle 1, -1\rangle^n$ and by no other 2-Raney sequence. Item 3 follows since $\tau_* := \tau_{\mu_*} = \langle 0\rangle^\frown\langle 1, 2\rangle^n$, and $\prod_{\substack{j<2n \\ \sigma_*(j)<0}} \tau_*(j)(\tau_*(j)-1) = 2^n$.

For a 2-Raney sequence $\sigma \in \mathcal{R}(n)$, the function $\tau_\sigma(j)$ counts the difference between the number of $k < j$ with $\sigma(k) = +1$ and the number of $k < j$ with $\sigma(k) = -1$. Thus the largest possible value for $Q(\sigma)$ is $0 + 1 + \cdots + n = n(n+1)/2$. This value is achieved uniquely for $\sigma^* := \langle +1\rangle^{n+1\frown}\langle -1\rangle^n$. Note that $\tau^* := \tau_{\sigma^*} = \langle 0, 1, 2 \ldots, n, n+1, n, n-1, n-2, \ldots, 2\rangle$. Thus the degree of $\alpha_{n+1}(d)$ is $Q(\sigma^*) = n(n+1)/2$, and the leading coefficient is

$$\prod_{\substack{j\leq 2n \\ \sigma(j)<0}} \tau^*(j)(\tau^*(j)-1) = \prod_{2\leq \ell\leq n+1} \ell(\ell-1) = n!(n+1)!\,.$$

To get the lower bound of item 4, simply truncate the polynomial to its leading term. Note that the computation of the leading coefficient gives the largest value that can be contributed toward the polynomial by any 2-Raney sequence. Since $|\mathcal{R}(n)| = C(n) = (2n)!/[(n!)((n+1)!)]$, the upper bound is obtained by using $n!(n+1)!d^{n(n+1)/2}$ as an estimate for every Raney sequence. □

Theorem 9.10. *For $\sigma \in \mathcal{R}(n)$, set $R(\sigma) := \sum_{j<2n} [\tau_\sigma(j) + \sigma(j) - 1]$. For n, d with $2 \leq n, d < \omega$, let $\rho_n(d) := |\Delta(n, d)|$ be the number of diagonal subsets of ${}^{(2n-2)\geq}d$. Then*

$$\rho_{n+1}(d) := \sum_{\sigma\in\mathcal{R}(n)} d^{R(\sigma)} \left[\frac{d(d-1)}{2}\right]^n \prod_{\substack{j\leq 2n \\ \sigma(j)>0}} \tau_A(j)\,(\tau_A(j)-1)\,.$$

Proof. Sum the results of Lemma 8.1 □

We translate this theorem into an algorithm via a Maple procedure using the techniques applied above to Theorem 9.4.

```
q := proc(k,m,x) option remember;
```

```
  if (k < 1 or m < 0 or 2*m >= k) then
    0
  elif (m = 0) then
    x^((k)*(k-1)/2)
  else
    q(k-1,m-1,x)*(k-1-2*(m-1))*(k-2-2*(m-1))*x^(k-1-2*(m-1)-2)
      # ends in -1
    +q(k-1,m,x)*(x^(k-1-2*(m-1)))
      # ends in +1
  end if end proc:
Q := n -> q(2*n-1,n-1,x)*((x^2-x)/2)^(n-1);

for n from 1 to 4 do
  sort(expand(Q(n)));
end do;
```

Examples 9.11. Here are the values of $\rho_n(2)$ for $n = 1, \dots, 10$:

1 :	1
2 :	4
3 :	208
4 :	84544
5 :	225285376
6 :	3562673554432
7 :	313228604408713216
8 :	146151093077541238226944
9 :	349492125813998287750324092928
10 :	416817372663146443348345786611033 7024

Examples 9.12. The following are a few examples of the polynomials $\rho_n(d)$:

- $\rho_2(d) = d^3 - d^2$;
- $\rho_3(d) = 3d^8 - 6d^7 + 4d^6 - 2d^5 + d^4$;
- $\rho_4(d) = 18d^{15} - 54d^{14} + 63d^{13} - 45d^{12} + 33d^{10} + 19d^9 - 9d^8 + 3d^7 - d^6$;

10 Applications

The first application is to weak embedding types of certain trees. The definition below is based on one introduced by Milliken [5] and made more specific by Vuksanovic [14] by use of the lexicographic order.

Definition 10.1. Two rooted subtrees $A, B \subseteq {}^{(2n)\geq}d$ *have the same embedding type*, in symbols $A \sim_{Em} B$, if and only if there is a bijection $f : A \to B$ which preserves $\subseteq$, length order, and passing numbers.

Lemma 10.2. *The number of weak embedding types of* $(2n-1)$*-element binary transversal subtrees of* ${}^{(2n-2)\geq}d$ *equals the number of* n*-element diagonal subsets of* ${}^{(2n-2)\geq}d$.

Proof. A binary transversal subtree of ${}^{(2n-2)\geq}d$ with $(2n-1)$-elements has n leaves and is the meet closure of the set of its leaves. Also, the set of its leaves is a diagonal set. Since it has $2n-1$ elements all on different levels, it is its own collapse.

Since having the same embedding type is an equivalence relation, to prove the lemma, it suffices to show that if $A, B \subseteq {}^{(2n-2)\geq}d$ are $(2n-1)$-element subtrees and $A \sim_{Em} B$, then $A = B$. However, transversal subsets of ${}^{(2n-2)\geq}d$ must have one element of each of the lengths $0, 1, \ldots, 2n-2$, so every value of every sequence in A and B is a passing number, and by definition these are preserved. □

Theorem 10.3. *For* n, d *with* $2 \leq n, d < \omega$, $\rho_n(d)$ *is the cardinality of the set of (weak) embedding types of binary transversal subtrees of* ${}^{(2n-2)\geq}d$ *with* $2n-1$ *nodes.*

Proof. Use Theorem 9.10 together with Lemma 10.2. □

The second application is in the Ramsey theory of homogeneous relational structures. A *purely binary relational structure* is a structure with finitely many binary relations and a single unary type. That is, each of the binary relations is either reflexive or irreflexive, so none of the binary relations codes a non-trivial unary relation. A *universal* purely binary relational structure $\mathbb{U}$ is one completely determined by its two element substructures in the following sense: it embeds every finite purely binary relational structure $\mathbb{A}$ in the same language having the same unary type and having the property that each two element substructure of $\mathbb{A}$ is isomorphic to a two element substructure of $\mathbb{U}$. See [9] for more information about universal countable binary relational structures.

We define the notion of canonical partition below after introducing some preliminary concepts, and then motivate this application by listing examples of universal purely binary relational structures.

Suppose $\mathbb{U}$ is a universal countable purely binary relational structure with universe U. A set $Q \subseteq [U]^n$ is *persistent* if every induced substructure of $\mathbb{U}$ isomorphic to $\mathbb{U}$ has an element of Q. It is *indivisible* if for every partition of Q into finitely many class, there is an induced substructure of $\mathbb{U}$ isomorphic to $\mathbb{U}$ with the property that all elements of Q in the substructure are in the same class.

Definition 10.4 (See Definition 2.1 [6]). Suppose $\mathbb{U}$ is a universal countable purely binary relational structure with universe U. A *canonical equivalence relation* on $[U]^n$ is an equivalence relation with finitely many equivalence classes so that each of the classes is persistent and indivisible. A *canonical partition* is the set of equivalence classes of a canonical equivalence relation.

All canonical partitions of a structure have the same number of cells. The number of cells of a canonical partition of the n-tuples of a universal countable purely binary relational structure depends on the *degree* of the structure. For readers familiar with logic, the *degree* of the structure is the number of quantifier-free types of ordered pairs of the structure. Suppose $\mathbb{U} = (U, R_0, R_1, \ldots, R_{m-1})$ is a binary relational structure and all of its relations are either reflexive or irreflexive. For each $i < m$, there are four possibilities for truth about the relation R_i for an given order pair of points (u, v) from U: R_i holds for neither (u, v) nor (v, u); R_i holds for (u, v) but not (v, u); R_i holds for (v, u) but not (u, v); or R_i holds for both (u, v) and (v, u). Since there are m binary relations, there are 4^m possibilities for the truth of all the relations for a given ordered pair. The *degree* of $\mathbb{U}$ is the number of possibilities that are realized by ordered pairs from U.

For readers unfamiliar with universal purely binary relational structures, we describe a typical example. Start with a complete graph $G = (\omega, [\omega]^2, c)$ on a countable set of vertices with a coloring $c : [\omega]^2 \to \{\, 0, 1, \ldots, d-1 \,\}$ of the edges which has the property that every finite complete graph colored with d colors can be embedded as an induced colored subgraph. For $\ell < d$, let R_ℓ be the binary relation $i \, R_\ell j$ if and only if $c(\{i, j\}) = \ell$. Then R_ℓ is an irreflexive symmetric binary relation on ω, and $(\omega, R_0, R_1, \ldots, R_{d-1})$ is an example of a universal countable purely binary relational structure of degree d. Note that one and only one of the relations R_ℓ holds for any given pair $\{i, j\}$ with $i < j$. There is a natural way to embed this structure into the d-ary sequences. Define $\pi(0) = \langle\rangle$, and for positive j, let $\pi(j)$ be be the sequence whose ith entry is the color of the edge $\{i, j\}$.

Sauer, in private conversation, pointed out that every universal countable purely binary relational structure of degree d is isomorphic to the structure above defined from some coloring of the complete graph.

In their paper, Laflamme, Sauer and Vuksanovic [6] show how to code an arbitrary universal countable purely binary relational structure of degree d as a cofinal transversal subset of ${}^{\omega>}d$. Below is the definition of the partition of n-element subsets of ${}^{\omega>}d$ that Laflamme, Sauer and Vuksanovic pull back to define a canonical equivalence relation for universal countable purely binary relational structures of finite degree d (see Definition 7.2 of [6]).

Definition 10.5. For $2 \leq n, d < \omega$, fix an enumeration $P_0, \ldots, P_{r-1}$ of the similarity classes of n-element diagonal sets of ${}^{\omega>}d$ for some $r = r_n$, and let $\mathrm{nd}_n({}^{\omega>}d)$ denote the collection of non-diagonal n-element subsets ${}^{\omega>}d$. Set

$$\mathcal{C}_n({}^{\omega>}d) := \{\, P_0 \cup \mathrm{nd}_r({}^{\omega>}d), P_1, \ldots, P_{r-1} \,\}\,.$$

Theorem 10.6 (Theorem 7.1 [6]). *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable purely binary relational structure of degree d, and let $\pi : U \to {}^{\omega>}d$ be an embedding of U into a cofinal transversal set. For $i < r$, define $Q_i := \{\, A \in [U]^n : \pi[A] \in P_i \,\}$, and let $\mathcal{C}_n(\mathbb{U}) := \{\, Q_0, \ldots, Q_{r-1} \,\}$ be the partition obtained as the inverse image of $\mathcal{C}_n({}^{\omega>}d)$ under the embedding of $\mathbb{U}$ in ${}^{\omega>}d$. Then $\mathcal{C}_n(\mathbb{U})$ is a canonical partition of the n-element subsets of U.*

Corollary 10.7. *For positive $n < \omega$ and d with $2 \leq d < \omega$, if $\mathbb{U}$ is a universal countable purely binary relational structure, then the the number of cells in a canonical partition $\mathcal{C}_n(\mathbb{U})$ of n-element subsets of $\mathbb{U}$ is $\alpha_n(d)$.*

Proof. Apply Theorem 10.6 and Corollary 9.6. □

The random (Rado) graph, $\mathbb{RG} = (\omega, E_{\mathbb{RG}})$, is a special case of a universal countable purely binary relational structure of degree 2. Vuksanovic, in Lemma 2.1 of [15], gives a characterization of the equivalence classes of a canonical partition for the random graph and includes a table with the first few values for $r_n = \alpha_n(2)$: $r_1 = 1$, $r_2 = 4$, $r_3 = 112$.

By definition, the partition relation $\mathbb{U} \to (\mathbb{U})^n_{<\omega/r}$ holds if for every coloring of the n-element subsets of U with finitely many colors, there is an induced substructure of $\mathbb{U}$ isomorphic to $\mathbb{U}$ on whose n-element subsets the coloring takes at most r values. The number of cells of a canonical partition is a critical number for this partition relation.

Corollary 10.8 (Corollary 7.1 [6]). *Let $\mathbb{U} = (U; \mathfrak{L})$ be a universal countable purely binary relational structure of degree d. If r_n is the number of similarity classes of n-element diagonal subsets of ${}^{\omega>}d$, then*

$$\mathbb{U} \to (\mathbb{U})^n_{<\omega/r_n} \text{ and } \mathbb{U} \nrightarrow (\mathbb{U})^n_{<\omega/r_n-1}.$$

Corollary 10.9. *The partition relations $\mathbb{RG} \to (\mathbb{RG})^n_{<\omega/\alpha_n(2)}$ and $\mathbb{RG} \nrightarrow (\mathbb{RG})^n_{<\omega/\alpha_n(2)-1}$ hold. More generally, if $\mathbb{U}$ is a universal countable purely binary relational structure of degree d, then the partition relations below hold:*

$$\mathbb{U} \to (\mathbb{U})^n_{<\omega/\alpha_n(d)} \text{ and } \mathbb{U} \nrightarrow [\mathbb{U}]^n_{\alpha_n(d)}.$$

Proof. The random graph is an example of a countable universal homogeneous relational structure of degree 2. Apply Corollary 10.8. $\square$

For concreteness, here is a short list of universal countable purely binary relational structures, with their degrees.

1. The Rado (random) graph has degree 2.
2. The random oriented graph has degree 3.
3. The random directed graph has degree 4.
4. The random tournament has degree 2.

Note that the random tournament has Ramsey theory equivalent to that of the Rado graph, since they have the same degree.

For clarity, we point out that $(\mathbb{Q}, <)$, the countable dense linear order without endpoints, and the countable homogeneous triangle-free graph are examples of countable homogeneous purely binary relational structures which are *not* universal in the sense used here, since each forbids a three element substructure, so is not determined by its two element substructures.

Devlin [1], in his thesis, proved a parallel Ramsey Theorem for $(\mathbb{Q}, <)$ in which the critical values are the tangent numbers (see also Vuksanovic [14]). Sauer [8] has proved a Ramsey theorem for colorings of the edges of the countable triangle free homogenous graph, but Ramsey questions for larger size subsets remain open.

For more background information on the Ramsey theory of countable homogeneous relational structures, see the introductory section the paper *Coloring subgraphs of the Rado graph* by Sauer [10]. In particular, this work generalizes that of Erdős and Rado [3], who determined the canonical partitions of n element sequences of natural numbers. It also generalizes work of Erdős, Hajnal and Pósa [2], who showed that any partition of the edges of the random graph must have must have at least two colors.

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