Graph coding trees and level orders

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Abstract

Recursive methods are developed for counting the number of structures consisting of a generalized Joyce subtree of the complete binary tree of sequences of length at most $2m$ with $m+1$ leaves paired with a vip ordering of the nodes. The approach is also applied to counting the number of such structures in which the set of leaves of the generalized Joyce tree codes an independent set or anticlique. Actual values are computed for $m < 5$.

These numbers are the critical values for Ramsey theoretic results of Džamonja, Larson and Mitchell for certain very large random (Rado) graphs and certain very large dense linear orders respectively.

1 Introduction

Džamonja, Larson and Mitchell [3] have generalized Ramsey results for $\mathbb{RG} = (\omega, E_{\mathbb{RG}})$, the infinite Rado (random) graph, and for $\mathbb{Q}$, the dense linear order without first and last elements. They prove, for a weakly compact cardinal

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κ with additional nice properties, that the κ-Rado (random) graph, $\mathbb{RG}_\kappa = (\kappa, E_{\mathbb{RG}_\kappa})$ satisfies the following partition relations for all finite positive n:

$$\mathbb{RG}_\kappa \rightarrow [\mathbb{RG}_\kappa]^n_{<\omega, r_n^+} \text{ and } \mathbb{RG}_\kappa \not\rightarrow [\mathbb{RG}_\kappa]^n_{r_n^+}.$$

The κ-Rado graph is the ultrahomogeneous graph on κ universal for all graphs of cardinality less than κ. The arrow notation above says that for any coloring of the n-tuples of the κ-Rado graph with finitely many colors, there is an induced subgraph isomorphic to the entire graph whose n-tuples are colored by at most r_n^+ colors and there is a coloring with r_n^+ many colors none of which can be omitted in any induced subgraph isomorphic to the entire graph.

Let ${}^{2m\geq}2$ denote the complete binary tree of sequences of length at most $2m$. Džamonja, Larson and Mitchell [3] also show that number r_{m+1}^+ is the number of pairs $(T, \lessdot)$ where $T \subseteq {}^{2m\geq}2$ is a graph coding generalized Joyce tree and $\lessdot$ is a vip level order of T. They call such pairs *vip* $(m+1)$*-types* for short.

Furthermore, they generalize a Ramsey result of D. Devlin [2] for the order type of the rationals by showing that the (strongly) κ-dense linear order $\mathbb{Q}_\kappa = ({}^{\kappa>}2, <_{\mathrm{Q}})$ satisfies the following partition relations for all finite positive n:

$$\mathbb{Q}_\kappa \rightarrow [\mathbb{Q}_\kappa]^n_{<\omega, t_n^+} \text{ and } \mathbb{Q}_\kappa \not\rightarrow [\mathbb{Q}_\kappa]^n_{t_n^+}.$$

They show that number t_{m+1}^+ is the number of pairs $(T, \lessdot)$ where $T \subseteq {}^{2m\geq}2$ is an anticlique coding generalized Joyce tree and $\lessdot$ is a vip level order of T. They call such pairs *anticlique vip* $(m+1)$*-types* for short.

The main goal of this paper is to provide recursive methods for computing t_{m+1}^+ and r_{m+1}^+. Values for $m < 5$ are computed explicitly. For $m > 1$, the values of t_{m+1}^+ and r_{m+1}^+ are larger than the corresponding values for the countable dense linear order and the countable Rado graph.

In Section 2, the number of anticlique coding generalized Joyce trees with $m+1$ leaves is computed to be the $m+1$st tangent number: $t_{m+1} = \tan^{(2m+1)}(0)$. The main tool is a bijection between anticlique coding generalized Joyce trees and alternating permutations. In Section 3, vip orders are introduced and a bijection is defined between the collection of vip $(m+1)$-types $\tau = (T, \lessdot)$ and well-chosen set of quadruples $(\tau_0, \tau_1, \langle \prec_i : 1 \leq i \leq 2m \rangle, \langle p_i : 1 \leq i \leq 2m \rangle)$. In Section 4, the bijection is used prove recursive formulas for computing the number of vip $(m+1)$-types and anticlique vip $(m+1)$-types. In Section 5, actual values for $m < 5$ are computed by implementing the methods of Section 4.

After a brief review of some notational conventions and basic definitions, the remainder of this section is devoted to reviewing and or reporting on some results for countable structures and their relationship to the work of Džamonja, Larson and Mitchell. It also introduces the concepts of graph coding tree, generalized Joyce tree and level ordering.

By an abuse of notation, identify a sequence $\langle a_0, a_1, \ldots, a_{m-1}\rangle$ with the function $a : n \to \{ a_0, a_1, \ldots, a_{m-1} \}$ where $a(j) = a_j$, and with the set of order pairs $\{ (j, a_j) : j < m \}$. Then one may write $a \subseteq b$ to indicate that $\langle a_0, a_1, \ldots, a_{i-1}\rangle$ is an initial segment of $\langle b_0, b_1, \ldots, b_{j-1}\rangle$, write $|a| = m$ to indicate that the length of the sequence a is m, and write $a(|b|) = a_j$ if $|a| > j = |b|$. Also, one may denote the empty sequence by $\emptyset$. Let ${}^{\mathbb{N}>}2$ denote the collection of all finite sequences of 0's and 1's. For notational convenience, write $a \wedge b$ for the *meet* of a and b, namely the longest initial segment common to both.

Erdős, Hajnal and Pósa [4] used an embedding of the vertices of infinite random graph $\mathbb{RG} = (V, E_{\mathbb{RG}})$ into the complete binary tree of finite sequences of 0's and 1's in their proof that there is a coloring of the edges of the infinite random graph with two colors so that every induced subgraph isomorphic to the random graph has both colors. Specifically they assign to the nth vertex v_n a sequence $\vec{u}_n = \langle u_n(0), u_n(1), \ldots, u_n(n-1)\rangle$ where $\vec{u}_n(i) = 1$ if v_i and v_n are joined by an edge and $\vec{u}_n(i) = 0$ otherwise.

Call an edge between v_i and v_n for $i < n$ an "up" edge if $\vec{u}_i$ is an initial segment of $\vec{u}_n$ or they are incomparable and $\vec{u}_i <_{\text{lex}} \vec{u}_n$, and otherwise call it a "down" edge. Then every induced subgraph isomorphic to the random graph has both "up" and "down" edges. This result is optimal in that one can show that for any partition of the edges into finitely many classes there is an induced subgraph isomorphic to the random graph which uses only two colors.

Using arrow notation one writes

$$\mathbb{RG} \to [\mathbb{RG}]^2_{<\omega,2} \text{ and } \mathbb{RG} \nrightarrow [\mathbb{RG}]^2_2.$$

For any incomparable pair, $\vec{u}_i$ and $\vec{u}_n$ with $i < n$, one may associate the tree obtained from $\{\vec{u}_i, \vec{u}_n\}$ by adding the initial segment of $\vec{u}_n$ on the same level as $\vec{u}_i$, namely $\vec{u}_n{\restriction}i$, and the meet, $\vec{u}_i \wedge \vec{u}_n$. Note that if $\vec{u}_i(|\vec{u}_i \wedge \vec{u}_n|) = \delta$ and $\{\vec{u}_i, \vec{u}_n\}$ is an edge, then the tree so formed is isomorphic to

$\{\,\emptyset, \langle\delta\rangle, \langle 1-\delta\rangle, \langle 1-\delta, 1\rangle\,\}$. The two choices for δ represent the two critical types, since one can show that there is an induced subgraph of the random graph isomorphic to the entire graph whose nodes are pairwise incomparable. This result has been generalized from colorings of pairs to colorings of arbitrary n-tuples. See Sauer [10], Laflamme, Sauer and Vuksanovic [8] for the identification of the critical types, and Vuksanovic [15] and Larson [9] for information on the number of such critical types. Sauer and Laflamme, Sauer and Vuksanovic identify equivalence classes of $m+1$ element sets under similarity each unavoidable similarity class of which includes a strongly diagonal set. Such a set is an antichain A whose meet closure has $2m+1$ elements, all of different lengths. The unique subset $L \subseteq {}^{2m\geq}2$ in such a similarity class has the property that when L is closed under initial segments, the result is a graph coding generalized Joyce tree. Moreover, different unavoidable similarity classes lead to different generalized Joyce trees.

The (dyadic) rationals can be identified with the nodes of the complete binary tree $2^{<\mathbb{N}}$ of finite sequences of zeros and ones. Moreover, the order on the rationals corresponds to the following tree order: $s <_Q t$ if and only if one of the following holds: (a) $t^\frown\langle 0\rangle$ is an initial segment of s; (b) $s^\frown\langle 1\rangle$ is an initial segment of t; or (c) s and t are incomparable and s precedes t in the lexicographic order, i.e. for the least i where they differ, $s(i) < t(i)$. Thus proofs about the order isomorphism type of the rationals can be couched in terms of the structure $(2^{<\mathbb{N}}, <_{\mathrm{Q}})$.

In his Ph.D. thesis, Denis Devlin [2] proved that

$$\mathbb{Q} \to (\mathbb{Q})^n_{<\omega,\, t_n} \text{ and } \mathbb{Q} \not\to (\mathbb{Q})^n_{<\omega,\, t_n-1}$$

where the value of $t_n = \tan^{(2n-1)}(0)$ is the n-th tangent number. Devlin used the language of category theory as in the original proof of the Halpern-Läuchli Theorem. His work extended that of Galvin (unpublished) who showed that $\mathbb{Q} \to (\mathbb{Q})^2_{<\omega,2}$ and asked about generalizations. See [5] for a citation of Galvin's work. For a sketch of a proof of Devlin's Theorem using the tree approach, see Todorcevic and Farah [14]. For a complete proof using this approach, see Vuksanovic [16]. The essential types in the Vuksanovic proof are strongly diagonal and more, so the process above for collapsing them collapse down leads to anticlique coding generalized Joyce trees.

Recall that a set theoretic tree is a partial order with the property that predecessors of any node are well-ordered by the partial order. For this paper, the trees of interest are (rooted) subtrees of ${}^{2m\geq}2$ and the partial order is $\subseteq$. By abuse of notation the tree $(T, \subseteq)$ is referred to by its underlying set T.

Definition 1.1. Suppose $T \subseteq {}^{2m\geq}2$ is a tree. If different leaves of T have different lengths, then call T a *graph coding tree*, where the *graph coded by* T is (L, E) for L the set of leaves of T and $E := \{\, \{s, t\} : |s| < |t| \wedge t(|s|) = 1 \,\}$.

For any set $A \subseteq {}^{2m\geq}2$, let the *meet closure* of A be the set $A^\wedge$ of all $a \wedge b$ for $a, b \in A$. Note that $A \subseteq A^\wedge$ since $a = a \wedge a$.

By definition, a *Joyce tree* is a rooted binary tree which can be drawn so that no two nodes have the same level and all levels up to that of the top leaf have a node. They were named by Ross Street [13] after the physicist William P. Joyce [7] who was using them in his calculations. Ryan Crompton and Tam Pham were undergraduates under the supervision of Street, who reported on his students' work showing that the number of Joyce trees with n nodes is the nth Taylor coefficient of $y = \tan x$. Crompton and Pham use what they call *tremolo permutations* and are also called *alternating permutations* (see Stanley [12]).

Definition 1.2. Call $T \subseteq {}^{2m\geq}2$ a *generalized Joyce tree* if it is a rooted binary tree closed under initial segments such that

1. the set L of leaves of T has $m + 1$ elements;
2. the meet closure of L, $L^\wedge$, has exactly one node on each of the $2m + 1$ levels of ${}^{2m\geq}2$; and
3. for all $s \in L^\wedge$ and $t \in L$, if $|s| < |t|$ and $t(|s|) = 1$, then either s is an initial segment of t or s is a leaf of T.

Note that since $L \subseteq L^\wedge$, every generalized Joyce tree is a graph coding tree. Moreover, the meet closure of the set of leaves of a generalized Joyce tree is a rooted binary tree which may be drawn so that no two nodes appear on the same level and every level from 0 to $2m$ has a node.

The reason for closing the trees under initial segments is that the proofs for the κ-Rado graph and κ-dense linear order use *level orders.*

Definition 1.3. Say that $\prec$ is a *level order* on $T \subseteq {}^{2m\geq}2$ or alternatively that $\prec$ is *an order on the levels* of ${}^{\kappa>}2$, if $\prec$ is a linear order on the nodes of T which extends the length order, i.e. $|s| < |t|$ implies $s \prec t$. If, in addition, T is a generalized Joyce tree, then $\prec$ is a *vip level order* if for each $i \leq 2m$, the unique element d_i of the meet closure of the set of leaves of T is the $\prec$ least element of the ith level of T, and for all u, v on the ith level different from d_i, the following conditions hold:

1. if $d_i \wedge u \subsetneq d_i \wedge v$, then $u \prec v$;
2. if $d_i \wedge u = d_i \wedge v$ and $u(i) < v(i)$, then $u \prec v$.

Let $\mathrm{Vip}(m)$ denote the set of all pairs $(T, \lessdot)$ where T is a generalized Joyce tree with m leaves and $\lessdot$ is a vip level order.

2 Meet indicator sequences

In this section, *meet indicator sequences* are introduced and used to compute the sizes of levels of generalized Joyce trees.

Definition 2.1. Suppose T is a generalized Joyce tree and L is the set of $m+1$ leaves of T. Enumerate $L^\wedge$ in increasing order of length as $d_0, d_1, \ldots, d_{2m-2}$. Then the *meet indicator sequence of* T is $\mu^T : 2m+1 \to \{+1, -1\}$ by $\mu^T(i) = +1$ if $d_i \in L$ and $\mu^T(i) = -1$ if $d_i \in L^\wedge \setminus L$.

Lemma 2.2. *Suppose T is a generalized Joyce tree and L is the set of the $m+1$ leaves of T. Then for all $i \leq 2m$, the cardinality of level i of T is given by the following equation*

$$*(i) : |\{\, x{\restriction}i : x \in L \,\}| = \sum_{i \leq j \leq 2m} \mu^T(j)$$

Proof. By Definition 1.2, $T = \{\, y{\restriction}i : y \in L \wedge i \leq 2m \,\}$. Let $x_0, x_1, \ldots, x_{2m}$ enumerate $L^\wedge$ in increasing order of length.

Use induction on $i \leq 2m$ in reverse order to prove $*(i)$. For $i = 2m$, there is only one leaf of length at least i, namely x_{2m}, the longest leaf of L. Thus $*(2m)$ holds. To continue the induction, suppose $k < 2m$ and $*(i)$ holds for all i with $k < i \leq 2m$.

For the first case, suppose $\mu^T(k) = -1$. Then x_k is a meet, so $x_k{}^\frown\langle 0\rangle$ and $x_k{}^\frown\langle 1\rangle$ are both elements of $\{\, y{\restriction}(k+1) : y \in L \,\}$ and for all $y \in L$ with $x_k \not\subseteq y$, $y(k) = 0$ or $y(k) = 1$ but not both, since the meet closure of L has at most one element of length k. Thus the cardinality of $setx{\restriction}k : x \in L$ is one less than the cardinality of $\{\, x{\restriction}(k+1) : x \in L \,\}$ and $*(k)$ holds by the induction hypothesis.

For the second case, suppose $\mu^T(k) = +1$. Then x_k is a leaf, so x_k has no extension in $\{\, y{\restriction}(k+1) : y \in L \,\}$ and for all $y \in L$ with $x_k \not\subseteq y$, $y(k) = 0$ or $y(k) = 1$ but not both, since the meet closure of L has at most one element of length k. Thus the cardinality of $sety{\restriction}k : y \in L$ is one more than the cardinality of $\{\, y{\restriction}(k+1) : y \in L \,\}$ and $*(k)$ holds by the induction hypothesis. □

Corollary 2.3. *Suppose T is a generalized Joyce tree with $m+1$ leaves. Then for all $i \leq 2m$, $\sum_{i \leq j \leq 2m} \mu^T(j) \geq 1$ and $\sum_{j \leq 2m} \mu^T(j) = 1$.*

Proof. The indicated sums are the sizes of the levels of T, which are all nonempty. Since level 0 has size one, the sum of the entire sequence is 1. □

These sequences have a natural definition in terms of the generalized Joyce trees, and their reverses have already been examined.

Definition 2.4. Call μ a *reverse Raney sequence* if $\operatorname{dom}\mu = 2m+1$ for some $m < \omega$, $\operatorname{ran}\mu \subseteq \{-1,+1\}$, $\sum_{i \leq j \leq 2m} \mu(j) \geq 1$ and $\sum_{j \leq 2m} \mu(j) = 1$. Let $\mathcal{R}$ denote the set of all reverse Raney sequences and let $\mathcal{R}(m)$ denote those with domain $2m+1$.

Corollary 2.3 says that the meet indicator sequence of a generalized Joyce tree with $m+1$ leaves is a reverse Raney sequence. Next we show that every reverse Raney sequence is the meet indicator sequence of some generalized Joyce tree with $m+1$ leaves.

Lemma 2.5. *Suppose μ is a reverse Raney sequence and* $\operatorname{dom}\mu = 2m+1$. *Then there is a generalized Joyce tree $T \in \mathrm{AJ}(m+1)$ such that $\mu^T = \mu$.*

Proof. Let $a_0, a_2, \dots, a_{2m}$ list in decreasing order those indices ℓ with $\mu(\ell) = +1$. There are $m+1$ of them, since $\sum_{i \leq 2m} \mu(i) = +1$. Let $a_1, a_3, \dots, a_{2m-1}$ list in decreasing order those ℓ with $\mu(\ell) = -1$. Since every tail of μ has a positive sum, it follows that $a_{2i+1} < a_{2i+2}$ for $i < m$. Thus $\vec{a} := \langle a_0, a_1, \dots, a_{2m} \rangle$ is an alternating sequence on $\{\, 0, 1, \dots, 2m \,\}$. By Lemma 3.3, $\vec{a}$ is in the range of π_0, so there is some $T \in \mathrm{AJ}(m+1)$ with $\pi_0(T) = \vec{a}$. By the construction of $\vec{a}$ and the definition of π_0, it follows that $\mu^T = \mu$. □

3 Alternating permutations

In this section a bijection π_0 is defined between anticlique coding generalized Joyce trees and alternating permutations. This bijection is then used to show that the number of anticlique coding generalized Joyce trees with $m+1$ leaves is the tangent number $t_{m+1} = \tan^{2m+1}(0)$. In the next section, π_0 is used in the definition of other bijections which are used for the recursive counts.

Definition 3.1. A permutation $a = \langle a_0, a_1, \ldots, a_k \rangle$ of the natural numbers $0, 1, \ldots, k$ is an *alternating permutation* [12] or *down-up permutation* [11] if $a_0 > a_1 < a_2 > a_3 < \ldots$.

The study of alternating permutations and the alternating group dates back to André [1] in 1881.

Notation 3.2. Let $\mathrm{AltPerm}(0) = \{ \emptyset \}$ and for $m > 0$, let $\mathrm{AltPerm}(m)$ be the collection of alternating permutations on $\{ 0, 1, \ldots, m-1 \}$.

Let $\mathrm{AJ}(m)$ denote the set of anticlique coding generalized Joyce trees with m leaves. Let π_0 be mapping $T \mapsto \langle |x_0|, |x_0 \wedge x_1|, |x_1|, \ldots, |x_{m-1} \wedge x_m|, |x_m| \rangle$ for $T \in \mathrm{AJ}(m+1)$ with set of leaves $\{ x_0, x_1, \ldots, x_m \}_{<_{\mathrm{lex}}}$.

The next lemma corresponds to one on tremolo permutations and Joyce trees in the analogous counting by Ross Street [13] and his students.

Lemma 3.3. *For each m, the restriction of the mapping π_0 to $\mathrm{AJ}(m+1)$ is a bijection between $\mathrm{AJ}(m+1)$ and $\mathrm{AltPerm}(m+1)$*

Proof. First note that for every $T \in \mathrm{AJ}(m+1)$, $\pi_0(T)$ is an alternating permutation on $\{ 0, 1, \ldots, 2m \}$, since the meet closure of the set L of leaves of T, has one element of each length from 0 to $2m$ and the meet of any two incomparable elements is shorter than either of them.

Claim 3.3.a. *π_0 is one-to-one.*

Proof. Suppose T and U in $\mathrm{AJ}(m+1)$ are such that $\pi_0(T) = \pi_0(U) = \langle n_0, n_1, \ldots, n_{2m} \rangle$. Let $L = \{ x_0, x_1, \ldots, x_m \}_{<_{\mathrm{lex}}}$ and $M = \{ y_0, y_1, \ldots, y_m \}_{<_{\mathrm{lex}}}$ be their respective sets of leaves. Assume toward a contradiction that $T \neq U$. Since T and U are the closure under initial segments of their sets of leaves, it follows that $L \neq M$. Let i be the least index j with $x_j \neq y_j$. Since $\pi_0(T) = \pi_0(U)$, it follows that $|x_i| = |y_i|$. Note that x_0 and y_0 are both sequences of all zeros, since T and U are anticlique coding graphs and x_0 and y_0 are the lexicographically least among L and M respectively. Thus $i > 0$. Let j be the least k such that $x_i(k) \neq y_i(k)$ and assume without loss of generality that $x_i <_{\mathrm{lex}} y_i$. Then $x_i(j) = 0 < 1 = y_i(j)$. Since $x_{i-1} = y_{i-1}$ and $\pi_0(T) = \pi_0(U)$, it follows that $|x_{i-1} \wedge x_i| = |y_{i-1} \wedge y_i| < j$. Thus $x_{i-1} \wedge x_i = y_{i-1} \wedge y_i$. Let $s \in L^{\wedge}$ be such that $|s| = j$. By the definition of generalized Joyce tree, either $s \subseteq y_i$ or $s \in L$. Since $y_i(|s|) = 1$ and U is an anticlique coding tree, s is not in L so it must be an initial segment of y_i which is the meet of two elements of L, $s = y_k \wedge y_\ell$, where $s^\frown\langle 0 \rangle \subseteq y_k$ and

$s^\frown\langle 1\rangle \subseteq y_\ell$. Then $k \leq i-1$ and without loss of generality, $\ell = i$. If $k = i-1$ we have a contradiction that $|y_{i-1} \wedge y_i| < j$ and $|y_{i-1} \wedge y_i| = j$. If $k < i-1$, we have a contradiction that that $y_k <_{\text{lex}} y_i$ and $y_i <_{\text{lex}} y_k$. ☐

Claim 3.3.b. *π_0 maps* $\mathrm{AJ}(m+1)$ *onto* $\mathrm{AltPerm}(m+1)$.

Proof. Let $a = \langle a_0, a_1, \ldots, a_{2m-1}, a_{2m}\rangle$ be an arbitrary alternating permutations in $\mathrm{AltPerm}(m+1)$. Define $x_0, x_1, \ldots, x_m$ by recursion. To start the recursion, let x_0 be a sequence of zeros of length a_0.

Suppose $k < m$ and $x_0, \ldots, x_k$ have been defined so that for all $i \leq k$, $|x_i| = a_{2k}$ and if $i < k$, then $|x_i \wedge x_{i+1}| = a_{2i+1}$ and $x_{i+1}(a_{2i+1}) = 1$ and $x_{i+1}(j) = 0$ for all $j > a_{2i+1}$. Let x_{k+1} be the sequence of length a_{2k+2} which is an extension by zeros of $(x_k \restriction a_{2k+1})^\frown\langle 1\rangle$.

By construction, $L = \{\, x_0, x_1, \ldots, x_m \,\}$ is listed in increasing lexicographic order, and $\langle |x_0|, |x_0 \wedge x_1|, |x_1|, \ldots, |x_{m-1} \wedge x_m|, |x_m|\rangle = a$. Thus $L^\wedge$ has elements of every length from 0 to $2m$. Moreover, by construction, if $x_j(k) = 1$ for some $j, k \leq 2m$, then there is $i < j$ with $x_i \wedge x_{i+1} = x_j \restriction k$. Let T be the closure of L under taking initial segments. Then T is a generalized Joyce tree. Moreover, there are no edges in the graph it codes, so $T \in \mathrm{AJ}(m+1)$. Moreover $\pi_0(T) = a$. Since a was arbitrary, the function π_0 maps $\mathrm{AJ}(m+1)$ onto $\mathrm{AltPerm}(m+1)$. ☐

From the claims, the restriction of π_0 to $\mathrm{AJ}(m+1)$ is a bijection between $\mathrm{AJ}(m+1)$ and $\mathrm{AltPerm}(m+1)$ for each m. ☐

Lemma 3.4. *[See [12]] The number of alternating permutations on the integers $0, 1, \ldots, 2m-2$ is t_m, where t_m is the m-th tangent number, defined recursively by $t_1 = 1$ and*

$$t_n = \sum_{i=1}^{n-1} \binom{2n-2}{2i-1} t_i t_{n-i}.$$

Corollary 3.5. *The number of anticlique coding generalized Joyce trees with $m+1$ leaves is $|\,\mathrm{AJ}(m+1)| = t_{m+1}$.*

Remark 3.5.1. The proof that the number of alternating sequences on the set $\{\, 0, 1, \ldots, 2m \,\}$ satisfies the same recurrence as the tangent numbers may be based on the fact that there is a bijection between such alternating sequences and triples (A, b, c), where $A \subseteq \{\, 1, 2, \ldots, 2m \,\}$ is a set of size $2n+1$ for some $n < m$, b is an alternating sequence on $\{\, 0, 1, \ldots, 2n \,\}$ and c is an alternating sequence on $\{\, 0, 1, \dots, 2(m-n-1) \,\}$. The bijection is simple to state since if $f : \{\, 0, 1, \dots, 2n \,\} \to A$ and $g : \{\, 0, 1, \dots, 2(m-n-1) \,\} \to \{\, 1, 2, \dots, 2m \,\} \setminus A$ are the increasing bijections, then

$$\langle f(b(0)), f(b(1)), \dots, f(b(2n)), 0, g(c(0)), g(c(1)), \dots, g(c(2(m-n-1)))\rangle$$

is the alternating permutation that corresponds to (A, b, c).

4 The basic bijection

In this section a one-to-one map π_1 is defined on the set Vip of generalized Joyce trees with vip level orders which is the basis of the recursive counting methods. In order to describe the map, some additional notation is needed.

Notation 4.1. Suppose T is a generalized Joyce tree with set L of $m+1$ leaves for some positive m. Introduce the following notation:

1. write $L(T)$ for the set of leaves of T;
2. for $\delta < 2$, let $L_\delta(T)$ denote the leaves of T extending $\langle \delta \rangle$;
3. write $L^\wedge_\delta(T)$ for the meet closure of L_δ;
4. set $A^T := \{\, |t| : t \in L^\wedge_0(T) \,\}$;
5. for all i with $1 \le i \le 2m$, let $T(i) := \{\, t \in T : |t| = i \,\}$ be the ith *level* of T;
6. for all i with $1 \le i \le 2m$, let $B^T_i := \{\, t \in T : |t| = i \wedge \langle \delta_i \rangle \subseteq t \,\}$, where $\delta_i = 1$ if $i \in A$ and $\delta_i = 0$ otherwise; and
7. let $p^T = \langle p_1, p_2, \dots, p_{2m} \rangle$ be defined by $p_i = \bigl| \{\, t \in B^T_i : t^\frown\langle 0 \rangle \in T \,\} \bigr|$.

Strong embeddings are the maps used in the proof that generalized Joyce trees with vip level orders are the critical types.

Definition 4.2. For any subsets S_0 and S_1 of ${}^{\mathbb{N}>}2$, a function $e : S_0 \to S_1$ is a *strong embedding* if it is an injection with the following preservation properties:

1. (extension) $s \subseteq t$ if and only if $e(s) \subseteq e(t)$;

2. (level order) $|s| < |t|$ if and only if $|e(s)| < |e(t)|$ and $|s| = |t|$ if and only if $|e(s)| = |e(t)|$;
3. (passing number) if $|s| < |t|$, then $e(t)(|e(s)|) = t(|s|)$.

Definition 4.3. For any finite set $x \subseteq {}^{\mathbb{N}>}2$, define $\mathrm{clp}(x)$ to be the subtree y of ${}^{\mathbb{N}>}2$ that includes the root and is of minimal possible height such that there is a strong embedding from $x^{\wedge}$ onto the meet closure $L(y)^{\wedge}$. If x is presented as a subset of the leaves of a generalized Joyce tree T with level order $\prec$, then $\prec_x$ is the order on $\mathrm{clp}(x)$ induced by the the strong embedding from $x^{\wedge}$ and $\prec$.

Lemma 4.4. *Suppose* $(T, \lessdot) \in \mathrm{Vip}(m+1)$ *for some positive* m. *Let* $L_0 = L_0(T)$ *and* $L_1 = L_1(T)$. *Then* $\mathrm{clp}(L_0)$ *and* $\mathrm{clp}(L_1)$ *are generalized Joyce trees with vip level orders* $\lessdot_{L_0}$ *and* $\lessdot_{L_1}$ *respectively.*

Proof. Let z_δ be the set of leaves of $\mathrm{clp}(L_\delta)$ and $e_\delta : L_\delta^{\wedge} \to z_\delta^{\wedge}$ be the strong embeddings witnessing that z_δ is the set of leaves of $\mathrm{clp}(L_\delta)$. Let $n_\delta + 1 = |L_\delta|$, and let $\{\, a_{\delta,i} : i \leq 2n_\delta \,\}$ enumerate $L_\delta^{\wedge}$ in increasing order of length. For $i \leq 2n$, set $f_\delta(i) = |a_{\delta,i}|$. Use induction and the fact that $\mathrm{clp}(L_\delta)$ has minimal height to show that $|e_\delta(a_{\delta,i})| = i$ for all $i \leq 2n_\delta$. More specifically, $e_\delta(a_{\delta,0}) = \emptyset$, and for $0 < i \leq 2n$, the sequence $e_\delta(a_{\delta,i})$ satisfies $e_\delta(a_{\delta,i})(j) = a_{\delta,i}(f_\delta(j))$ for all $j < i$. Thus by the definitions of generalized Joyce tree and strong embedding, $\mathrm{clp}(L_\delta)$ is a generalized Joyce tree. Moreover, if $\lessdot$ is a vip level order, then since e preserves extension, passing numbers and level order, and carries leaves to leaves, it follows that $\lessdot_{L_\delta}$ is also a vip level order. ☐

Definition 4.5. Suppose $(T, \lessdot) \in \mathrm{Vip}(m+1)$ for some positive m. For all i with $1 \leq i \leq 2m$, if $|B_i^T| \leq 1$, let $\prec_i^T = \emptyset$ and otherwise, for B_i^T listed in increasing lexicographic order, let $\prec_i^T$ be the order induced on the indices by the order $\lessdot$ restricted to B_i^T. For notational simplicity, let $L_0 = L_0(T)$ and $L_1 = L_1(T)$. Define

$$\pi_1(T, \lessdot) := (\mu^T, A^T, (\mathrm{clp}(L_0), \lessdot_{L_0}), (\mathrm{clp}(L_1), \lessdot_{L_1}), \left\langle \prec_i^T : 1 \leq i \leq 2m \right\rangle, p^T).$$

Lemma 4.6. *The mapping* π_1 *is one-to-one.*

Proof. Suppose $(T, \lessdot)$ and $(T', \lessdot')$ are elements of Vip with $\pi_1(T, \lessdot) = \pi_1(T', \lessdot')$, and let $(\mu, A, (U, <_U), (V, <_V), \langle \prec_i : 1 \leq i \leq 2m \rangle, p)$ be the common value. Then for some m, $|L(T)| = |L(T')| = m + 1$, since $\mu^T = \mu = \mu^{T'}$ and m is positive, since π_1 is only defined on non-trivial members of Vip.

Since $A = A^T$, $|A|$ is the cardinality of the meet closure of the set of leaves of $L(T)$ that extend, $\langle 0\rangle$, there is some n such that $|A| = 2n + 1$.

Let $L(T) = \{ x_0, x_1, \ldots, x_m \}_{<_{\text{lex}}}$ and $L(T') = \{ y_0, y_1, \ldots, y_m \}_{<_{\text{lex}}}$ be listed in increasing lexicographic order. Then

- $L_0(T) = \{ x_0, x_1, \ldots, x_n \}$;
- $L_1(T) = \{ x_{n+1}, \ldots, x_{2m} \}$;
- $L_1(T') = \{ y_0, y_1, \ldots, y_n \}$;
- $L_1(T') = \{ y_{n+1}, \ldots, y_{2m} \}$.

Since $A^T = A = A^{T'}$, the sets $\{ |x_0|, |x_0 \wedge x_1|, \ldots, |x_{n-1} \wedge x_n|, |x_n| \}$ and $\{ |y_0|, |y_0 \wedge y_1|, \ldots, |y_{n-1} \wedge y_n|, |y_n| \}$ are equal. In addition, $\operatorname{clp}(L_0(T)) = U = \operatorname{clp}(L_0(T'))$, $\operatorname{clp}(L_1(T)) = V = \operatorname{clp}(L_1(T'))$, so by Remark 3.5.1,

$$\langle |x_0|, |x_0 \wedge x_1|, \ldots, |x_{2n-1} \wedge x_{2n}|, |x_{2n}|\rangle = \langle |y_0|, |y_0 \wedge y_1|, \ldots, |y_{2n-1} \wedge y_{2n}|, |y_{2n}|\rangle.$$

By the definition of clp, there are strong embeddings e_0, e_1, e'_0, e'_1 from $L_0^{\wedge}(T)$, $L_1^{\wedge}(T)$, $L_0^{\wedge}(T')$, $L_1^{\wedge}(T')$ respectively to the meet closures of the sets of leaves of $\operatorname{clp}(L_0(T))$, $\operatorname{clp}(L_1(T))$, $\operatorname{clp}(L_0(T'))$, $\operatorname{clp}(L_1(T'))$ respectively. It follows that the mapping e defined by $e(x_i) = y_i$ and $e(x_i \wedge x_{i+1}) = y_i \wedge y_{i+1}$ is also a strong embedding.

By Lemma 2.2, for each $i \leq 2m$, the ith levels, $T(i)$ and $T'(i)$, of T and T' respectively have the same size. Since T and T' are generalized Joyce trees, they have the same root, namely $\emptyset$.

Use induction on $k < 2m$ to show that the following statement is true:

$$** (k) : (T(k+1), \lessdot) = (T'(k+1), \lessdot').$$

This statement uses $\lessdot$ and $\lessdot'$ where more properly they should be replaced by their restrictions to the given level.

For the basis case, $T(1) = \{ \langle 0\rangle, \langle 1\rangle \} = T(2)$ and $\langle 0\rangle \lessdot \langle 1\rangle$ if and only if $1 \in A$ if and only if $\langle 0\rangle \lessdot' \langle 1\rangle$. Thus $(T(1), \lessdot) = (T'(1), \lessdot')$.

For the induction step, suppose $0 < k < 2m$ and $** (i)$ is true for all $i < k$. Then $(T(k), \lessdot) = (T'(k), \lessdot')$. First show the unique element of $T(k)$ in the meet closure of L is the same as the unique element of $T'(k)$ in the meet closure of L'. Since the alternating permutations derived from L and L' are the same, these unique elements are either $x_j \wedge x_{j+1}$ and $y_j \wedge y_{j+1}$ or x_j and y_j for some suitably chosen j. Since they have the same relative position in the lexicographic order on $T(k) = T'(k)$, they must be the same. By a similar argument, one can show that the unique element of $T(k+1)$ in the meet closure of L and the unique element of $T'(k+1)$ in the meet closure of L' are one point extensions of the same element of $T(k) = T'(k)$.

Let δ and ε be such that the unique element of $T(k) \cap L^\wedge$ extends $\langle \delta \rangle$ and the unique element of $T(k+1) \cap L^\wedge$ extends $\langle \varepsilon \rangle$.

Every element $s \in T(k+1)$ that extends $\langle \varepsilon \rangle$ satisfies $s \lessdot t$ for every element $t \in T(k+1)$ that extends $\langle 1-\varepsilon \rangle$, and a similar statement holds true for elements of $T'(k+1)$. Note that $\lessdot_{L_\varepsilon}$ determines the order on the elements of $T(k+1)$ that extend $\langle \varepsilon \rangle$, $\lessdot'_{L'_\varepsilon}$ determines the order on the elements of $T'(k+1)$ that extend $\langle \varepsilon \rangle$, and $\prec_{k+1}$ determines the order on both the elements of $T(k+1)$ that extend $\langle 1-\varepsilon \rangle$ and the elements of $T'(k+1)$ that extend $\langle 1-\varepsilon \rangle$.

Thus to prove $**(k)$ holds, it suffices to show that $T(k+1) = T'(k+1)$. If $\mu(k) = -1$, then the unique element of $T(k) \cap L^\wedge$ is $x_j \wedge x_{j+1}$ for some $j < 2m$. Since T and T' are both generalized Joyce trees, $T(k+1) = T'(k+1)$ may be obtained from $T(k) = T(k')$ by extending each sequence by $\langle 0 \rangle$ and extending $x_j \wedge x_{j+1} = y_j \wedge y_{j+1}$ by $\langle 1 \rangle$ as well.

Next suppose $\mu(k) = +1$. Then the unique element of $T(k) \cap L^\wedge$ is $x_j = y_j$ for some suitably chose j. Since e_δ and e'_δ are both passing number preserving maps into the same tree, the elements of $T(k+1)$ and $T'(k+1)$ that extend $\langle \varepsilon \rangle$ are the same. Notice that the lexicographic order on $T(k+1)$ and $T'(k+1)$ is determined by that on $T(k)$ and $T'(k)$. Let $u_0, u_1, \ldots, u_{\ell-1}$ enumerate the elements of $T(k)$ that extend $\langle 1-\varepsilon \rangle$ in increasing lexicographic order. Then $u_i{}^\frown\langle 0 \rangle \in T(k+1)$ if and only if i occurs among the first p_{i+1} elements in the $\prec_{i+1}$-increasing enumeration of ℓ. Since an analogous statement is true for $T'(k+1)$, it follows that the sets of elements of $T(k+1)$ and of $T'(k+1)$ that extend $\langle 1-\varepsilon \rangle$ are equal, so $T(k+1) = T'(k+1)$. This equality completes the proof of the induction step.

Since $(T(0), \lessdot) = (T'(0), \lessdot')$ and for all $k < 2m$, $(T(k+1), \lessdot) = (T'(k+1), \lessdot')$, it follows that $(T, \lessdot) = (T', \lessdot')$. Therefore π_1 is one-to-one. $\square$

An interesting statistic for generalized Joyce trees with a set L of $m+1$ leaves is the number of nodes on level i that extend $\langle 1-\delta_i \rangle$ where the unique node on that level from $L^\wedge$ extends $\langle \delta_i \rangle$. The function defined below can be used to compute this statistic for generalized Joyce trees whose meet indicator sequence is μ, whose set of leaves is L, and the set of lengths of elements of the meet closure of L extending $\langle 0 \rangle$ is $A = \{\, |t| : \langle 0 \rangle \subseteq t \in L^\wedge \,\}$.

Definition 4.7. Suppose $\mu \in \mathcal{R}(m)$ and $A \subseteq \{1, 2, \ldots, 2m\}$. Define

$$q_i(\mu, A) := \begin{cases} 1 & \text{if } i = 0, \\ 0 & A \subseteq i \text{ or } |\mu| \setminus i \subseteq A, \\ \sum_{i<j\notin A} \mu(j) & \text{if } i \in A \text{ and } |\mu| \setminus i \not\subseteq A, \\ \sum_{i<j\in A} \mu(j) & \text{if } i \notin A \text{ and } A \not\subseteq i. \end{cases}$$

Lemma 4.8. *Suppose T is a generalized Joyce tree with a set L of $m+1$ leaves for some positive L and $A = A^T$. Define χ_A for $i \leq 2m$ by $\chi_A(i) = 1$ if $i \in A$ and $\chi_A(i) = 0$ otherwise. Then for all i with $1 \leq i \leq 2m$,*

$$q_i(\mu^T, A) = |\{ t \in T(i) : \langle \chi_A(i) \rangle \subseteq t \}|.$$

Proof. Notice that if $A \subseteq i$, then $\chi_A(i) = 0 = q_i(\mu^T, A)$ and set in question, $\{ t \in T(i) : \langle \chi_A(i) \rangle \subseteq t \}$, is empty. Similarly, if $2m \setminus i \subseteq A$, then $\chi_A(i) = 1 = q_i(\mu^T, A)$ and the $\{ t \in T(i) : \langle \chi_A(i) \rangle \subseteq t \}$ is empty.

Next suppose i is such that $i \in A$ and $2m \setminus i \not\subseteq A$. The sum $\sum_{i<j\notin A} \mu(j)$ is the difference between the number of leaves of $L_1(T)$ of length greater than i and the number of meets of different leaves of $L_1(T)$ of length greater than i. Thus it is the size of the set $\{ t \in T(i) : \langle 1 \rangle \subseteq t \}$. That is, $q_i(\mu^T, A) = \sum_{i<j\notin A} \mu(j) = |\{ t \in T(i) : \langle 1 \rangle \subseteq t \}|$.

Finally suppose $0 < i$ is such that $i \notin A$ and $A \not\subseteq i$. The sum $\sum_{i<j\in A} \mu(j)$ is the difference between the number of leaves of $L_0(T)$ of length greater than i and the number of meets of different leaves of $L_0(T)$ of length greater than i. Thus it is the size of the set $\{ t \in T(i) : \langle 0 \rangle \subseteq t \}$. That is, $q_i(\mu^T, A) = \sum_{i<j\in A} \mu(j) = |\{ t \in T(i) : \langle 0 \rangle \subseteq t \}|$. ☐

Definition 4.9. Suppose $\mu \in \mathcal{R}(m)$ for some positive m. Further suppose that $A = \{ a_0, a_1, \ldots, a_{k-1} \}_<$ is a non-empty proper subset of $\{1, 2, \ldots, 2m\}$ and $B = \{ b_0, b_1, \ldots, b_{\ell-1} \}_<$ its complement. Let μ_A denote the sequence of length k such that $\mu_A(i) = \mu(a_i)$ for all $i < k$, and $\mu'_A = \mu_B$ denote the sequence of length ℓ such that $\mu'_A(j) = \mu(b_j)$ for all $j < \ell$.

The careful reader may check that the following lemma is a consequence of the definitions of μ_A, μ'_A and of clp.

Lemma 4.10. *Suppose T is a generalized Joyce tree with $|L(T)| = m+1$ for some positive m. Then μ_A is the meet indicator sequence of* $\mathrm{clp}(L_0(T))$ *and μ'_A is the meet indicator sequence of* $\mathrm{clp}(L_1(T))$.

Definition 4.11. For $\mu \in \mathcal{R}(m)$ and $A \subseteq \{1, 2, \ldots, 2m\}$, define $\mathrm{Vip}(\mu)$, $\mathrm{Vip}(\mu, A)$, $\mathbb{L}(\mu, A)$ and $\mathbb{P}(\mu, A)$ as follows:

1. $\mathrm{Vip}(\mu) := \{ (T, \lessdot) \in \mathrm{Vip} : \mu^T = \mu \}$;
2. $\mathrm{Vip}(\mu, A) := \{ (T, \lessdot) \in \mathrm{Vip} : \mu^T = \mu \wedge A^T = A \}$;
3. $\mathbb{L}(\mu, A)$ is the set of all sequences $\langle \prec_i : 1 \leq i < 2m \rangle$ such that for all positive $i < 2m$, if $q_i(\mu, A) > 1$, then $\prec_i$ is a linear order on $\{0, 1, \ldots, q_i(\mu, A) - 1\}$ and $\prec_i = \emptyset$ otherwise;
4. $\mathbb{P}(\mu, A)$ is the set of all sequences $p = \langle p_1, p_2, \ldots, p_{2m-1} \rangle$ such that for all positive $i < 2m$, the entry p_i satisfies the inequality $0 \leq p_i \leq q_i(\mu, A)$ and if $\mu(i) = -1$, then $p_i = q_i(\mu, A)$.

Lemma 4.12. *Suppose $\mu \in \mathcal{R}(m)$ for some $m > 0$ and $A \subseteq \{1, 2, \ldots, 2m\}$. If $\mathrm{Vip}(\mu, A) \neq \emptyset$, then $\mu_A \in \mathcal{R}$, $\mu'_A \in \mathcal{R}$, and $\pi_1[\mathrm{Vip}(\mu, A)] = \mathcal{V}(\mu, A)$, where*

$$\mathcal{V}(\mu, A) := \{ (\mu, A) \} \times \mathrm{Vip}(\mu_A) \times \mathrm{Vip}(\mu'_A) \times \mathbb{L}(\mu, A) \times \mathbb{P}(\mu, A).$$

Proof. The proof proceeds by a series of claims.

Claim 4.12.a. *If $(S, \lhd) \in \mathrm{Vip}(\mu, A)$, then $(\mathrm{clp}(L_0(S)), \lhd_{L_0(S)}) \in \mathrm{Vip}(\mu_A)$, $(\mathrm{clp}(L_1(S)), \lhd_{L_1(S)}) \in \mathrm{Vip}(\mu'_A)$, and $\{ \mu_A, \mu'_A \} \subseteq \mathcal{R}$.*

Proof. Suppose $(S, \lhd) \in \mathrm{Vip}(\mu, A)$ is arbitrary. Apply Lemma 4.4 to see that both $(\mathrm{clp}(L_0(S)), \lhd_{L_0(S)})$ and $(\mathrm{clp}(L_1(S)), \lhd_{L_1(S)})$ are generalized Joyce trees with vip level orders. By Lemma 4.10, μ_A is the meet indicator sequence of $\mathrm{clp}(L_0(S))$ and μ'_A is the meet indicator sequence of $\mathrm{clp}(L_1(S))$, so both μ_A and μ'_A are in $\mathcal{R}$ by Corollary 2.3. Consequently, $(\mathrm{clp}(L_0(S)), \lhd_{L_0(S)}) \in \mathrm{Vip}(\mu_A)$ and $(\mathrm{clp}(L_1(S)), \lhd_{L_1(S)}) \in \mathrm{Vip}(\mu'_A)$. ☐

Claim 4.12.b. *The range of π_1 is a subset of $\mathcal{V}(\mu, A)$.*

Proof. Let $(S, \lhd) \in \mathrm{Vip}(\mu, A)$ be an arbitrary element of $\mathrm{Vip}(\mu, A)$. By Lemma 4.8, for all i with $1 \leq i \leq 2m$, the set B_i^S has cardinality $|B_i^S| = q_i(\mu, A)$. From the definitions of p^S and $\prec_i^S$ for all i with $1 \leq i < 2m$, it follows that $p^S \in \mathbb{P}$ and $\langle \prec_i^S : 1 \leq i < 2m \in \mathbb{L}$. Thus by the previous claim, $\pi_1(S, \lhd) \in \mathcal{V}(\mu, A)$. Since $(S, \lhd)$ was arbitrary, it follows that $\pi[\mathrm{Vip}(\mu, A)] \subseteq \mathcal{V}(\mu, A)$. ☐

Claim 4.12.c. *The set $\mathcal{V}(\mu, A)$ is a subset of the range of π_1.*

Proof. Let $\sigma := (\mu, A, (U_0, <_0), (U_1, <_1), \langle \prec_i \colon 1 \leq i \leq 2m \rangle, p)$ be an arbitrary element of $\mathcal{V}(\mu, A)$. For notational convenience, let $A_0 = A$ and $A_1 = \{ 1, 2, \ldots, 2m \} \setminus A$. Then $\mu^{U_0} = \mu_{A_0}$ and $\mu^{U_1} = \mu_{A_1} = \mu'_A$. For $1 \leq i \leq 2m$, let $\chi(i)$ be the unique δ with $i \in A_\delta$.

Use recursion on i with $1 \leq i \leq 2m$ to define sets $T_0(i)$ and $T_1(i)$ and mappings $g_{0,i} : U_0(|A_0 \cap i|) \to T_0(i)$ and $g_{1,i} : U_1(|A_1 \cap i|) \to T_1(i)$ as follows.

To start the recursion, set $T_0(1) = \{ \langle 0 \rangle \}$, $T_1(1) = \{ \langle 1 \rangle \}$, and for $\delta < 2$, let $g_{\delta,i}$ be the unique map from $\{\emptyset\}$ to $T_\delta(1)$.

Next suppose that $T_0(k)$, $T_1(k)$, $g_{0,k}$ and $g_{1,k}$ have all been defined for some $k < 2m$. Let $\varepsilon < 2$ be such that $k \in A_\varepsilon$, and let $j = |A_\varepsilon \cap k|$. Then $|A_\varepsilon \cap (k+1)| = j+1$. If $U_\varepsilon(j+1) \neq \emptyset$, define $g_{\varepsilon,k+1}(u^\frown\langle d \rangle) = g_{\varepsilon,k}(u)^\frown\langle d \rangle$ for all $u^\frown\langle d \rangle \in U_\varepsilon(j+1)$ and let $T_\varepsilon(k+1)$ be the range of $g_{\varepsilon,k+1}$. If $U_\varepsilon(j+1) = \emptyset$, then $T_\varepsilon(k+1) = \emptyset = g_{\varepsilon,k+1}$.

If $T_{1-\varepsilon}(k) = \emptyset$, then $T_{1-\varepsilon}(k+1) = \emptyset = g_{1-\varepsilon,k+1}$. So suppose $T_{1-\varepsilon}(k) \neq \emptyset$. Let $\ell := |A_{1-\varepsilon} \cap k| = |A_{1-\varepsilon} \cap (k+1)|$. If $k+1 \in A_{1-\varepsilon}$, then let $\lhd$ be the order defined on $T_{1-\varepsilon}(k)$ by $g(u) \lhd g(v)$ if and only if $u <_{1-\varepsilon} v$; otherwise, let $\lhd$ be the order on $T_{1-\varepsilon}(k)$ with the property that when the set is listed in increasing lexicographic order, the order $\lhd$ induces on the indices matches $\prec_i$. Define $g_{1-\varepsilon,k+1}(u) = u^\frown\langle 0 \rangle$ for all $u \in U_{1-\varepsilon}(j)$ for which $g_{1-\varepsilon,k}(u)$ has fewer than p_k many predecessors in the $\lhd$ order on $T_{1-\varepsilon}(k)$, and set $g_{1-\varepsilon,k+1}(u) = u^\frown\langle 1 \rangle$ otherwise. Let $T_\varepsilon(k+1)$ be the range of $g_{\varepsilon,k+1}$.

Let $T = \{\emptyset\} \cup \bigcup \{ T_0(i) \cup T_1(i) : 1 \leq i \leq 2m \}$. By induction one can show that T is a subset of ${}^{2m\geq}2$ which is a tree closed under initial segments. In particular, $\emptyset$ is its root.

For $u \in U_\varepsilon(j)$ with $j = |A_\varepsilon \cap k|$ and $k \in \varepsilon$, if u is a leaf, then it has no extensions in $U_\varepsilon(j)$ and if it is a meet, then it has two extensions in $U_\varepsilon(j)$. Thus by the recursive construction, $g_{\varepsilon,k}(u)$ is in the meet closure of the leaves of T and is a meet or a leaf if u is. Also any $s \in T_{1-\varepsilon}(k)$ has exactly one extension. So T has exactly $m+1$ leaves and it has exactly one node of the meet closure of its leaves on each of the $2m+1$ levels of ${}^{2m\geq}2$. Since $p \in \mathbb{P}$, T has the property that for all $s \in L^\wedge(T)$ and $t \in L(T)$, if $|s| < |t|$ and $t(|s|) = 1$, then either s is an initial segment of t or $s \in L(T)$. Thus T is a generalized Joyce tree. Note that $\mu^T = \mu_A$ and $A^T = A$.

Furthermore, for $\delta < 2$, the set $g_\delta = \bigcup \{ g_{\delta,i} : i \in A_\delta \}$ is a strong embedding from U_δ to $\bigcup \{ T_\delta(i) : i \in A_\delta \}$, hence $\operatorname{clp}(L_\delta(T)) = U_\delta$.

Let $\lessdot$ be the unique level order on T such that for all i and ε with $1 \leq i < 2m$ and $i \in A_\varepsilon$, for all $s, t \in T(i)$, $s \lessdot t$ if and only if one of the following conditions holds:

1. $s \in T_\varepsilon(i)$ and $t \in T_{1-\varepsilon}(i)$;

2. $s, t \in T_\varepsilon(i)$, $s = g_{\varepsilon,i}(u)$, $t = g_{\varepsilon,i}(v)$ and $u <_\varepsilon v$;

3. $s, t \in T_{1-\varepsilon}(i)$, $s = g_{1-\varepsilon,i}(u)$, $t = g_{1-\varepsilon,i}(v)$ and $u \prec_i v$.

Since the meet closure of the leaves of $L_0(T)$ are in the range of g_0 and the meet closure of the leaves of $L_1(T)$ are in the range of g_1, by condition 2, $\lessdot_{L_0(T)} = <_0$ and $\lessdot_{L_1(T)} = <_1$. Since these are both vip orders, it follows from conditions 1 and 2 that $\lessdot$ is a vip order. Thus $(T, \lessdot) \in \mathrm{Vip}(\mu, A)$.

Condition 3 guarantees that $\langle \prec_i^T \colon 1 \leq i < 2m \rangle = \langle \prec_i \colon 1 \leq i < 2m \rangle$. Moreover, by construction, $p^T = p$. Therefore $\pi_1(T, \lessdot) = \sigma$, as required. □

Now the lemma follows from the claims. □

Definition 4.13. For $\mu \in \mathcal{R}(m)$ and $A \subseteq \{ 1, 2, \ldots, 2m \}$, let Vip^- be the set of all $(T, \lessdot)$ in Vip such that T codes an anticlique. Let $\mathrm{Vip}^-(m)$ be the set of anticlique vip m-types, set $\mathrm{Vip}^-(\mu) := \mathrm{Vip}(\mu) \cap \mathrm{Vip}^-$ and $\mathrm{Vip}^-(\mu, A) := \mathrm{Vip}(\mu, A) \cap \mathrm{Vip}^-$.

Corollary 4.14. *Suppose $\mu \in \mathcal{R}(m)$ for $m > 0$ and $A \subseteq \{ 1, 2, \ldots, 2m \}$. If $\mathrm{Vip}(\mu, A) \neq \emptyset$, then $\mu_A \in \mathcal{R}$, $\mu'_A \in \mathcal{R}$, and $\pi_1 \left[\mathrm{Vip}^-(\mu, A) \right] = \mathcal{V}^-(\mu, A)$, where*

$$\mathcal{V}^-(\mu, A) := \{ (\mu, A) \} \times \mathrm{Vip}^-(\mu_A) \times \mathrm{Vip}^-(\mu'_A) \times \mathbb{L}(\mu, A) \times \{ \langle q_i(\mu, A) \rangle \}$$

Proof. Note that if $(T, \lessdot)$ is an anticlique vip type, then for $L_0 = L_0(T)$ and $L_1 = L_1(T)$, both $(\mathrm{clp}(L_0), \lessdot_{L_0})$ and $(\mathrm{clp}(L_1), \lessdot_{L_1})$ are anticlique vip types. Furthermore, if $\mu = \mu^T$ and $A = A^T$, then $p^T = \langle q_i(\mu, A) : 1 \leq i < 2m \rangle$. Now apply Lemma 4.12. □

5 Counting methods

This section contains methods for computing the number of vip $(m+1)$-types with a given meet indicator sequence.

Lemma 5.1. *Let $\mu \in \mathcal{R}(m)$ for some positive m, and let $A \subseteq \{ 1, 2, \ldots, 2m \}$ be such that μ_A and μ'_A are both in $\mathcal{R}$. Then*

$$|\mathbb{L}(\mu, A)| = Q(\mu, A) := \prod_{1 \leq i < 2m} q_i(\mu, A)! \text{ and}$$

$$|\mathbb{P}(\mu, A)| = P(\mu, A) := \prod_{\substack{i<2m \\ \mu(i)>0}} (q_i(\mu, A) + 1).$$

Proof. For $0 < i < 2m$, if $q_i(\mu, A) = 0$ or $q_i(\mu, A) = 1$, then the ith entry of every sequence in $\mathbb{L}(\mu, A)$ is $\prec_i = \emptyset$; otherwise the ith entry is one of $q_i(\mu, A)!$ possible linear orders on $q_i(\mu, A)$. Since $0! = 1! = 1$, the above computation of $|\mathbb{L}(\mu, A)$ is correct.

For $0 < i < 2m$, if $\mu(i) = -1$, then the ith entry of every sequence in $\mathbb{P}(\mu, A)$ is $q_i(\mu, A)$, so these entries are determined. For $0 < i < 2m$, if $\mu(i) = +1$, then the possibilities for the ith entry p_i of a sequence in $\mathbb{P}(\mu, A)$ satisfies the inequality $0 \le p_i \le q_i(\mu, A)$ so there are $q_i(\mu, A) + 1$ choices available. Thus the computation above counts all sequences in $\mathbb{P}(\mu, A)$. ☐

Lemma 5.2. *Suppose* $m > 0$, $\mu \in \mathcal{R}(m)$ *and* $0 \notin A \subset 2m - 1$. *If* $\mu_A \in \mathcal{R}$ *and* $\mu'_A \in \mathcal{R}$, *then* $|\operatorname{Vip}^-(\mu, A)| = |\operatorname{Vip}^-(\mu_A)| \cdot |\operatorname{Vip}^-(\mu'_A)| \cdot |\mathbb{L}(\mu, A)|$ *and* $|\operatorname{Vip}(\mu, A)| = |\operatorname{Vip}(\mu_A)| \cdot |\operatorname{Vip}(\mu'_A)| \cdot |\mathbb{L}(\mu, A)| \cdot |\mathbb{P}(\mu, A)|$.

Proof. Use Lemmas 4.12, 5.1 and Corollary 4.14. ☐

Corollary 5.3. *There is a single reverse Raney sequence* $\langle +1 \rangle$ *in* $\mathcal{R}(0)$, *and* $\operatorname{Vip}(1) = \operatorname{Vip}^-(1) = \{\, (\{\emptyset\}, \emptyset)\,\}$ *has only one member. For all* $\mu \in \mathcal{R}(m)$ *with* $m > 0$,

$$|\operatorname{Vip}^-(\mu)| = \sum_{\substack{0 \notin A \subseteq 2m-1 \\ \mu_A, \mu'_A \in \mathcal{R}}} |\operatorname{Vip}^-(\mu_A)| \cdot |\operatorname{Vip}^-(\mu'_A)| \cdot |\mathbb{L}(\mu, A)|, \text{ and}$$

$$|\operatorname{Vip}(\mu)| = \sum_{\substack{0 \notin A \subseteq 2m-1 \\ \mu_A, \mu'_A \in \mathcal{R}}} |\operatorname{Vip}(\mu_A)| \cdot |\operatorname{Vip}(\mu'_A)| \cdot |\mathbb{L}(\mu, A)| \cdot |\mathbb{P}(\mu, A)|.$$

The next lemma follows from the fact that the meet indicator sequence of a generalized Joyce tree is a reverse Raney sequence (see Corollary 2.3).

Lemma 5.4. *For all* m,

$$|\operatorname{Vip}^-(m+1)| = \sum_{\mu \in \mathcal{R}(m)} |\operatorname{Vip}^-(\mu)| \quad \text{and} \quad |\operatorname{Vip}^-(m+1)| = \sum_{\mu \in \mathcal{R}(m)} |\operatorname{Vip}^-(\mu)|$$

For hand computation, the following lemma allows one to cut the work in half.

Lemma 5.5 (Duality Lemma). *Suppose* $m > 0$*,* $\mu \in \mathcal{R}(m)$ *and* $A \subseteq \{1, 2, \ldots, 2m\}$*. If* $\mu_A \in \mathcal{R}$*,* $\mu'_A \in \mathcal{R}$ *and* $B = \{1, 2, \ldots, 2m\} \setminus A$*, then* $\mathrm{Vip}^-(\mu, A) = \mathrm{Vip}^-(\mu, B)$ *and* $\mathrm{Vip}(\mu, A) = \mathrm{Vip}(\mu, B)$*.*

Proof. First notice that $\mu_B = \mu'_A$ and $\mu'_B = \mu_A$. Next check that for all $i \leq 2m$, $q_i(\mu, A) = q_i(\mu, B)$. By Lemma 5.1, $|\mathbb{L}(\mu, A)| = |\mathbb{L}(\mu, B)|$ and $|\mathbb{P}(\mu, A)| = |\mathbb{P}(\mu, B)|$. Finally, apply Lemma 5.2. ☐

An additional definition simplifies use of the Duality Lemma.

Definition 5.6. For all $\mu \in \mathcal{R}(m)$, let $I(\mu)$ be the following set:

$$I(\mu) := \left\{ A \in [2m+1]^{\leq m} : 0 \notin A \wedge \mu_A, \mu'_A \in \mathcal{R} \wedge (|A| < m \vee 1 \in A) \right\}$$

Theorem 5.7. *There is a single reverse Raney sequence* $\langle +1 \rangle$ *in* $\mathcal{R}(0)$*, and* $\mathrm{Vip}(1) = \mathrm{Vip}^-(1) = \{ (\{\emptyset\}, \emptyset) \}$ *has only one member. For all* $\mu \in \mathcal{R}(m)$ *with* $m > 0$*,*

$$|\mathrm{Vip}^-(\mu)| = \sum_{A \in I(\mu)} 2Q(\mu, A) |\mathrm{Vip}^-(\mu_A)| \cdot |\mathrm{Vip}^-(\mu'_A)| \text{ and}$$

$$|\mathrm{Vip}(\mu)| = \sum_{A \in I(\mu)} 2Q(\mu, A) P(\mu, A) |\mathrm{Vip}(\mu_A)| \cdot |\mathrm{Vip}(\mu'_A)|.$$

Proof. The set $I(\mu)$ has the property that for all $B \subseteq \{1, 2, \ldots, 2m\}$ with $\mu_B \in \mathcal{R}$ and $\mu'_B \in \mathcal{R}$, either B is in $I(\mu)$ or $\{1, 2, \ldots 2m\} \setminus B$ is in $I(\mu)$ but not both.

Thus the theorem follows from Corollary 5.3, using Lemma 5.1 to compute $Q(\mu, A) = |\mathcal{L}(\mu, A)|$ and $P(\mu, A) = |\mathcal{P}(\mu, A)|$ and the Duality Lemma 5.5 to cut the index set to half its original size compensating by doubling the factor for each set $A \in I(\mu)$. ☐

The next lemma, which has a bijective proof, is the simplest way to compute $|\mathrm{Vip}^-(\mu)|$ and $|\mathrm{Vip}(\mu)|$ from smaller values, but only applies to special μ.

Lemma 5.8 (Product Lemma). *Suppose* $\mu \in \mathcal{R}$ *is a reverse Raney sequence and* $\mu = \nu^\frown \nu'$*, where* ν' *is a non-trivial Raney sequence and* $\nu \neq \emptyset$*. Then* $\nu^+ := \nu^\frown \langle +1 \rangle$ *is a reverse Raney sequence,*

$$|\mathrm{Vip}^-(\mu)| = |\mathrm{Vip}^-(\nu^+)| \cdot |\mathrm{Vip}^-(\nu')| \text{ and}$$

$$|\mathrm{Vip}(\mu)| = |\mathrm{Vip}(\nu^+)| \cdot |\mathrm{Vip}(\nu')|.$$

Proof. Let n be such that $\nu' \in \mathcal{R}(n)$. Then ν' has length $2n+1$. Thus ν has length $(2m+1)-(2n+1) = 2(m-n)$. Since ν' is a reverse Raney sequences, $\sum_{2(m-n)\leq j \leq 2m} \mu(j) = 1$. for all $i \leq 2(m-n)$,

$$\sum_{i\leq j\leq 2(m-n)} \nu^+(j) = 1 + \sum_{i\leq j<2(m-n)} \mu(j) = \sum_{i\leq j\leq 2m} \mu(j).$$

Since μ is a reverse Raney sequence, it follows that ν^+ is a reverse Raney sequence.

For $(U, <_U) \in \mathrm{Vip}(\nu^+)$ with longest leaf w and and $(V, <_V) \in \mathrm{Vip}(\nu')$, let $h((U, <_U), (V, <_V)) = (T, \lessdot)$, where $T := U \cup \{\, w^\frown v : v \in V \,\}$ and for $s, t \in T$, $s \lessdot t$ if and only if one of the following conditions holds:

1. $|s| \leq |w|$, $|t| \leq |w|$ and $s <_U t$;
2. $s = w^\frown p$, $t = w^\frown q$ and $p <_V q$; or
3. $|s| < |w|$ and $|w| < |t|$.

Note that h maps $\mathrm{Vip}(\nu^+) \times \mathrm{Vip}(\nu')$ into $\mathrm{Vip}(\mu)$, and $\mathrm{Vip}^-(\nu^+) \times \mathrm{Vip}^-(\nu')$ into $\mathrm{Vip}^-(\mu)$. The reader may check that h and its restriction to $\mathrm{Vip}^-(\nu^+) \times \mathrm{Vip}^-(\nu')$ are bijections required to complete the proof. □

6 Small values

In this section, Lemma 5.4 is used to compute the sizes of $\mathrm{Vip}^-(m+)$, $\mathrm{Vip}(m+1)$ for $m < 4$. To compute these values, Theorem 5.7 and the Product Lemma 5.8 are used to compute the sizes of $\mathrm{Vip}^-(\mu)$ and $\mathrm{Vip}(\mu)$ for $\mu \in \mathcal{R}(m)$. To save space, especially in tables, the alternate notation $\mathcal{V}$ is used in place of Vip, and frequently the angle brackets, commas and ones are omitted from the reverse Raney sequences, so $\langle -1, +1, +1\rangle$ becomes $- + +$, for example.

Proposition 6.1. *The sizes of* $\mathrm{Vip}^-(m+1)$, $\mathrm{Vip}(m+1)$ *and* $\mathrm{Vip}^-(\mu)$, $\mathrm{Vip}(\mu)$ *for* $\mu \in \mathcal{R}(m)$ *for* $m \leq 1$ *are given below:*

m	*name*	μ	$\lvert\mathcal{V}^-(\mu)\rvert$	$\lvert\mathcal{V}(\mu)\rvert$
0	$\mu_{0,0}$	$+$	1	1
1	$\mu_{1,0}$	$-++$	2	4

Proof. For $m = 0$, there is a single reverse Raney sequence $\langle +1\rangle$ in $\mathcal{R}(0)$. As reported in Theorem 5.3, there is a single vip 1-type and it trivially codes an anticlique, so $|\operatorname{Vip}^-(1)| = |\operatorname{Vip}(1)| = 1$.

For $m = 1$, there is only one reverse Raney sequence in $\mathcal{R}(1)$, namely $\mu = \langle -1, +1, +1\rangle$. The set $I(\mu)$ is a singleton whose only member is $A = \{1\}$. Since $\vec{q}(\mu, A) = \langle 1, 1, 0\rangle$, it follows that $Q(\mu, A) = 1$, $P(\mu, A) = 2$, $\mu_A = \langle +1\rangle = \mu'_A$. Thus for $\mu = \langle -1, +1, +1\rangle$,

$$\begin{aligned}
|\operatorname{Vip}^-(2)| &= |\operatorname{Vip}^-(\mu)| \\
&= 2Q(\mu, \{1\})|\operatorname{Vip}^-(\langle 1\rangle)| \cdot |\operatorname{Vip}^-(\langle 1\rangle)| \\
&= 2(1)|\operatorname{Vip}^-(\langle 1\rangle)| \cdot |\operatorname{Vip}^-(\langle 1\rangle)| \\
&= 2, \text{ and}
\end{aligned}$$

$$\begin{aligned}
|\operatorname{Vip}(2)| &= |\operatorname{Vip}(\mu)| \\
&= 2Q(\mu, \{1\})P(\mu, \{1\})|\operatorname{Vip}(\langle 1\rangle)| \cdot |\operatorname{Vip}(\langle 1\rangle)| \\
&= 2(1)(2)|\operatorname{Vip}(\langle 1\rangle)| \cdot |\operatorname{Vip}(\langle 1\rangle)| \\
&= 4.
\end{aligned}$$

□

Proposition 6.2. *The sizes of* $\operatorname{Vip}^-(m)$, $\operatorname{Vip}(m)$ *and* $\operatorname{Vip}^-(\mu)$, $\operatorname{Vip}(\mu)$ *for* $\mu \in \mathcal{R}(m)$ *for* $m = 2$ *are given below, and the totals are* $|\operatorname{Vip}^-(3)|$ *and* $|\operatorname{Vip}(3)|$*:*

name	μ	$\lvert\mathcal{V}^-(\mu)\rvert$	$\lvert\mathcal{V}(\mu)\rvert$
$\mu_{2,0}$	$--+++$	16	112
$\mu_{2,1}$	$-+-++$	4	16
Totals		20	128

Proof. For $m = 2$, there are two reverse Raney sequences in $\mathcal{R}(3)$, namely $\mu_{2,0} = \langle -1, -1, +1, +1, +1\rangle$ and $\mu_{2,1} = \langle -1, +1, -1, +1, +1\rangle$. Since $\mu_{2,1} = \langle -1, +1\rangle ^\frown \langle -1, +1, +1\rangle$, by the Product Lemma 5.8,

$$|\operatorname{Vip}^-(\mu_{2,1})| = |\operatorname{Vip}^-(\langle -1, +1, +1\rangle)| \cdot |\operatorname{Vip}^-(\langle -1, +1, +1\rangle)| = (2)(2) = 4,$$

and

$$|\operatorname{Vip}(\mu_{2,1})| = |\operatorname{Vip}(\langle -1, +1, +1\rangle)| \cdot |\operatorname{Vip}(\langle -1, +1, +1\rangle)| = (4)(4) = 16.$$

The set $I(\mu_{2,0})$ has three elements: $\{\ell\}$ for $2 \le \ell \le 4$. In Figure 1, the sizes of $\mathcal{V}^-(\mu_A)$, $\mathcal{V}^-(\mu'_A)$ and $\mathcal{V}^-(\mu, A)$ are computed for $A \in I(\mu_{2,0}$ in the three columns under the heading $\mathcal{V}^-$, and the corresponding values for

$\mathcal{V}$ are computed, again using Lemma 5.2, under columns the headed by $\mathcal{V}$. Note that $Q(\mu, A) = |\mathbb{L}(\mu, A)|$ is computed under the column headed by Q and $P(\mu, A) = |\mathbb{P}(\mu, A)|$ is computed under the column headed by P using Lemma 5.1. Finally, the sizes of $\operatorname{Vip}^-(\mu_{2,0})$ and $\operatorname{Vip}(\mu_{2,0})$ are computed using Theorem 5.7.

								$\mathcal{V}^-$			$\mathcal{V}$		
A	$q_i(\mu, A)$					Q	P	μ_A	μ'_A	μ, A	μ_A	μ'_A	μ, A
$\mu_{2,0}$	$-$	$-$	$+$	$+$	$+$								
$\{2\}$	1	1	2	0	0	2!	3	1	2	4	1	4	24
$\{3\}$	1	1	1	1	0	1	2	1	2	2	1	4	16
$\{4\}$	1	1	1	1	0	1	2	1	2	2	1	4	16
Totals										8			56
Doubles										$\lvert\mathcal{V}^-(\mu_{2,0})\rvert = 16$			$\lvert\mathcal{V}(\mu_{2,0})\rvert = 112$

Figure 1: Sizes of $\mathcal{V}^-(\mu, A)$ and $\mathcal{V}(\mu, A)$ for $\mu = \mu_{2,0}$ and $|A| < 2$.

Thus $|\operatorname{Vip}^-(3)| = |\operatorname{Vip}^-(\mu_{2,0})| + |\operatorname{Vip}^-(\mu_{2,1})| = 16 + 4 = 20$, and $|\operatorname{Vip}(3)| = |\operatorname{Vip}(\mu_{2,0})| + |\operatorname{Vip}(\mu_{2,1})| = 112 + 16 = 128$, Lemma 5.4. $\square$

Figures 2, 3, and 4 are used in the proof of the next proposition.

										$\mathcal{V}^-$			$\mathcal{V}$		
A	$q_i(\mu, A)$							Q	P	μ_A	μ'_A	μ, A	μ_A	μ'_A	μ, A
$\mu_{3,0}$	$-$	$-$	$-$	$+$	$+$	$+$	$+$								
$\{3\}$	1	1	1	3	0	0	0	6	4	1	16	96	1	112	2688
$\{4\}$	1	1	1	1	2	0	0	2	6	1	16	32	1	112	1344
$\{5\}$	1	1	1	1	1	1	0	1	8	1	16	16	1	112	896
$\{6\}$	1	1	1	1	1	1	0	1	8	1	16	16	1	112	896
$\{1, 3, 4\}$	1	1	2	2	2	0	0	8	9	2	2	32	4	4	1152
$\{1, 3, 5\}$	1	1	2	2	1	1	0	4	12	2	2	16	4	4	768
$\{1, 3, 6\}$	1	1	2	2	1	0	0	4	12	2	2	16	4	4	768
$\{1, 4, 5\}$	1	1	2	2	1	1	0	4	12	2	2	16	4	4	768
$\{1, 4, 6\}$	1	1	2	2	1	1	0	4	12	2	2	16	4	4	768
$\{1, 5, 6\}$	1	1	2	2	2	0	0	8	9	2	2	32	4	4	1152
Totals												288			11200
Doubles												$\lvert\mathcal{V}^-(\mu)\rvert = 576$			$\lvert\mathcal{V}(\mu)\rvert = 22400$

Figure 2: Sizes of $\mathcal{V}^-(\mu_{3,0}, A)$ and $\mathcal{V}(\mu_{3,0}, A)$.

Proposition 6.3. *The sizes of* $\mathrm{Vip}^-(m)$*,* $\mathrm{Vip}(m)$ *and* $\mathrm{Vip}^-(\mu)$*,* $\mathrm{Vip}(\mu)$ *for* $\mu \in \mathcal{R}(3)$ *are given below, and the totals are* $|\,\mathrm{Vip}^-(4)|$ *and* $|\,\mathrm{Vip}(4)|$*:*

name	μ	$\lvert\mathcal{V}^-(\mu)\rvert$	$\lvert\mathcal{V}(\mu)\rvert$
$\mu_{3,0}$	$---++++$	576	22400
$\mu_{3,1}$	$--+-+++$	128	3008
$\mu_{3,2}$	$--++-++$	32	448
$\mu_{3,3}$	$-+--+++$	32	448
$\mu_{3,4}$	$-+-+-++$	8	64
Totals		776	$26,368$

Proof. For $m = 3$, there are five reverse Raney sequences in $\mathcal{R}(3)$, of which three are splittable in the sense that Lemma 5.8 applies.

In Figure 2, the values of $|\mathcal{V}^-(\mu_{3,0})|$ and $|\mathcal{V}(\mu_{3,0})|$ are computed, using the fact that the set $I(\mu_{3,0})$ is the set of singletons $\{\,\ell\,\}$ for $3 \le \ell \le 6$ and the collection of triples $\{\,1, k, \ell\,\}$ where $3 \le k < \ell \le 6$.

In Figure 3, the values of $|\mathcal{V}^-(\mu_{3,1})|$ and $|\mathcal{V}(\mu_{3,1})|$ are computed, using the fact that the set $I(\mu_{3,1})$ is the set of singletons $\{\,\ell\,\}$ for $3 \le \ell \le 6$ and together with the collection of triples $\{\,1, 2, \ell\,\}$ where $3 \le \ell \le 6$, since if 1 is in A, then so must 2 in order for μ'_A to be in $\mathcal{R}(1)$.

										$\mathcal{V}^-$			$\mathcal{V}$		
A	$q_i(\mu, A)$							Q	P	μ_A	μ'_A	μ, A	μ_A	μ'_A	μ, A
$\mu_{3,1}$	$-$	$-$	$+$	$-$	$+$	$+$	$+$								
$\{3\}$	1	1	2	0	0	0	0	2	3	1	16	32	1	112	672
$\{4\}$	1	1	1	1	2	0	0	2	6	1	4	8	1	16	192
$\{5\}$	1	1	1	1	1	1	0	1	8	1	4	4	1	16	128
$\{6\}$	1	1	1	1	1	1	0	1	8	1	4	4	1	16	128
$\{1, 2, 4\}$	1	1	1	1	2	0	0	2	6	2	2	8	4	4	192
$\{1, 2, 5\}$	1	1	1	1	1	1	0	1	8	1	4	4	1	16	128
$\{1, 2, 6\}$	1	1	1	1	1	0	0	1	4	1	4	4	1	16	64
Totals												64			1504
Doubles												$\lvert\mathcal{V}^-(\mu)\rvert = 128$			$\lvert\mathcal{V}(\mu)\rvert = 3008$

Figure 3: Sizes of $\mathcal{V}^-(\mu_{3,1}, A)$ and $\mathcal{V}(\mu_{3,1}, A)$.

In Figure 4, the splittable reverse Raney sequences are displayed with a box marking the place where the sequence may be split. In the figure, the sizes of $\mathrm{Vip}^-(\mu)$ and $\mathrm{Vip}(\mu)$ are computed using the Product Lemma 5.5.

	$\mu = \nu^\frown \nu'$	$\mathcal{V}^-$			$\mathcal{V}$		
name	$\nu^\frown\Box^\frown\nu'$	ν^+	ν'	μ	ν^+	ν'	μ
$\mu_{3,2}$	$--++\ \Box-++$	16	2	32	112	4	448
$\mu_{3,3}$	$-+\ \Box--+++$	2	16	32	4	112	448
$\mu_{3,4}$	$-+-+\ \Box-++$	4	2	8	16	4	64

Figure 4: Sizes of $\mathcal{V}^-(\mu, A)$ and $\mathcal{V}(\mu, A)$ for splittable $\mu \in \mathcal{R}(3)$.

$\Box$

Proposition 6.4. *The sizes of* $\mathrm{Vip}^-(\mu)$ *and* $\mathrm{Vip}(\mu)$ *for* $\mu \in \mathcal{R}(4)$ *are given below, and the totals are* $|\mathrm{Vip}^-(5)|$ *and* $|\mathrm{Vip}(5)|$*:*

name	μ	$\lvert\mathcal{V}^-(\mu)\rvert$	$\lvert\mathcal{V}(\mu)\rvert$
$\mu_{4,0}$	$----+++++$	$121,600$	$35,926,016$
$\mu_{4,1}$	$---+-++++$	$20,736$	$4,453,376$
$\mu_{4,2}$	$--+--++++$	$4,608$	$627,200$
$\mu_{4,3}$	$---++-+++$	$4,608$	$617,984$
$\mu_{4,4}$	$--+-+-+++$	$1,024$	$86,272$
$\mu_{4,5}$	$---+++-++$	$1,152$	$89,600$
$\mu_{4,6}$	$-+---++++$	$1,152$	$89,600$
$\mu_{4,7}$	$--++--+++$	256	$12,544$
$\mu_{4,8}$	$--+-++-++$	256	$12,032$
$\mu_{4,9}$	$-+--+-+++$	256	$12,032$
$\mu_{4,10}$	$--++-+-++$	64	$1,792$
$\mu_{4,11}$	$-+--++-++$	64	$1,792$
$\mu_{4,12}$	$-+-+--+++$	64	$1,792$
$\mu_{4,13}$	$-+-+-+-++$	16	256
Totals		$155,856$	$41,932,288$

Proof. Start with the application of the Product Lemma 5.8 to compute the easy values of $|\mathrm{Vip}^-(\mu)|$ and $|\mathrm{Vip}(\mu)|$.

For $\mu = \mu_{4,i}$ with $i < 5$, compute $\mathcal{V}^-\mu, A$ and $\mathcal{V}\mu, A$ using Theorem 5.7 for $A \in I(\mu)$ as before using information from Figures 6, 7, 8, 9 and 10. $\Box$

	$\mu = \nu^\frown\nu'$	$\mathcal{V}^-$			$\mathcal{V}$		
name	$\nu^\frown\square^\frown\nu'$	ν^+	ν'	μ	ν^+	ν'	μ
$\mu_{4,5}$	$---+++\ \square-++$	576	2	$1,152$	$22,400$	4	$89,600$
$\mu_{4,6}$	$-+\ \square---++++$	2	576	$1,152$	4	$22,400$	$89,600$
$\mu_{4,7}$	$--++\ \square--+++$	16	16	256	112	112	$12,544$
$\mu_{4,8}$	$--+-++\ \square-++$	128	2	256	$3,008$	4	$12,032$
$\mu_{4,9}$	$-+\ \square--+-+++$	2	128	256	4	$3,008$	$12,032$
$\mu_{4,10}$	$--++-+\ \square-++$	32	2	64	448	4	$1,792$
$\mu_{4,11}$	$-+\ \square--++-++$	2	32	64	4	448	$1,792$
$\mu_{4,12}$	$-+-+\ \square--+++$	4	16	64	16	112	$1,792$
$\mu_{4,13}$	$-+-+-+\ \square-++$	8	2	16	64	4	256

Figure 5: Sizes of $\mathcal{V}^-(\mu, A)$ and $\mathcal{V}(\mu, A)$ for splittable $\mu \in \mathcal{R}(4)$.

A	$q_i(\mu, A)$									Q	P	$\mathcal{V}^-$	$\mathcal{V}$
$\{4\}$	1	1	1	1	4	0	0	0	0	24	5	13824	2688000
$\{5\}$	1	1	1	1	1	3	0	0	0	6	8	3456	1075200
$\{6\}$	1	1	1	1	1	1	2	0	0	2	12	1152	537600
$\{7\}$	1	1	1	1	1	1	1	1	0	1	16	576	358400
$\{8\}$	1	1	1	1	1	1	1	1	0	1	16	576	358400
$\{1,4,5\}$	1	1	2	2	3	3	0	0	0	144	16	4608	1032192
$\{1,4,6\}$	1	1	2	2	3	1	2	0	0	48	24	1536	516096
$\{1,4,7\}$	1	1	2	2	3	1	1	0	0	24	16	768	172032
$\{1,4,8\}$	1	1	2	2	3	1	1	0	0	24	16	768	172032
$\{2,4,5\}$	1	1	2	2	3	3	0	0	0	144	16	4608	1032192
$\{2,4,6\}$	1	1	2	2	3	1	2	0	0	48	24	1536	516096
$\{2,4,7\}$	1	1	2	2	3	1	1	0	0	24	16	768	172032
$\{2,4,8\}$	1	1	2	2	3	1	1	0	0	24	16	768	172032
$\{3,4,5\}$	1	1	1	3	3	3	0	0	0	216	16	6912	1548288
$\{3,4,6\}$	1	1	1	3	3	1	2	0	0	72	24	2304	774144
$\{3,4,7\}$	1	1	1	3	3	1	1	0	0	36	16	1152	258048
$\{3,4,8\}$	1	1	1	3	3	1	1	0	0	36	16	1152	258048
$\{1,5,6\}$	1	1	2	2	2	2	2	0	0	32	27	1024	387072
$\{1,5,7\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{1,5,8\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{2,5,6\}$	1	1	2	2	2	2	2	0	0	32	27	1024	387072
$\{2,5,7\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{2,5,8\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{3,5,6\}$	1	1	1	3	2	2	2	0	0	48	27	1536	580608
$\{3,5,7\}$	1	1	1	3	2	2	1	1	0	24	36	768	387072
$\{3,5,8\}$	1	1	1	3	2	2	1	1	0	24	36	768	387072
$\{1,6,7\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{1,6,8\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{2,6,7\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{2,6,8\}$	1	1	2	2	2	2	1	1	0	16	36	512	258048
$\{3,6,7\}$	1	1	1	3	2	2	1	1	0	24	36	768	387072
$\{3,6,8\}$	1	1	1	3	2	2	1	1	0	24	36	768	387072
$\{1,7,8\}$	1	1	2	2	2	2	2	0	0	32	27	1024	387072
$\{2,7,8\}$	1	1	2	2	2	2	2	0	0	32	27	1024	387072
$\{3,7,8\}$	1	1	1	3	2	2	2	0	0	48	27	1536	580608
Totals												60800	17963008
Doubles												$121,600$	$35,926,016$

Figure 6: Sizes of $\mathcal{V}^-(\mu_{4,0}, A)$ and $\mathcal{V}(\mu_{4,0}, A)$ for $A \in I(\mu_{4,0})$.

A	$q_i(\mu, A)$									Q	P	μ'_A	$\mathcal{V}^-$	$\mathcal{V}$
$\{3\}$	1	1	1	3	0	0	0	0	0	6	4	$\mu_{3,0}$	3456	537600
$\{5\}$	1	1	1	1	1	3	0	0	0	6	8	$\mu_{3,1}$	768	144384
$\{6\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{3,1}$	256	72192
$\{7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,1}$	128	48128
$\{8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,1}$	128	48128
$\{1,3,5\}$	1	1	2	2	1	3	0	0	0	24	12	$\mu_{2,0}$	768	129024
$\{1,3,6\}$	1	1	2	2	1	1	2	0	0	8	18	$\mu_{2,0}$	256	64512
$\{1,3,7\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{1,3,8\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{2,3,5\}$	1	1	2	2	1	3	0	0	0	24	12	$\mu_{2,0}$	768	129024
$\{2,3,6\}$	1	1	2	2	1	1	2	0	0	8	18	$\mu_{2,0}$	256	64512
$\{2,3,7\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{2,3,8\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{4,5,6\}$	1	1	1	1	2	2	2	0	0	8	18	$\mu_{2,0}$	256	64512
$\{4,5,7\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{4,5,8\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{4,6,7\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{4,6,8\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,0}$	128	43008
$\{4,7,8\}$	1	1	1	1	2	2	2	0	0	8	18	$\mu_{2,0}$	256	64512
$\{1,5,6\}$	1	1	2	2	2	2	2	0	0	32	27	$\mu_{2,1}$	256	55296
$\{1,5,7\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{1,5,8\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{1,6,7\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{1,6,8\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{1,7,8\}$	1	1	2	2	2	2	2	0	0	32	27	$\mu_{2,1}$	256	55296
$\{2,5,6\}$	1	1	2	2	2	2	2	0	0	32	27	$\mu_{2,1}$	256	55296
$\{2,5,7\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{2,5,8\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{2,6,7\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{2,6,8\}$	1	1	2	2	2	2	1	1	0	16	36	$\mu_{2,1}$	128	36864
$\{2,7,8\}$	1	1	2	2	2	2	2	0	0	32	27	$\mu_{2,1}$	256	55296
Totals													10368	2226688
Doubles													20736	4453376

Figure 7: Sizes of $\mathcal{V}^-(\mu_{4,1}, A)$ and $\mathcal{V}(\mu_{4,1}, A)$ for $A \in I(\mu_{4,1})$.

A	$q_i(\mu, A)$									Q	P	μ'_A	$\mathcal{V}^-$	$\mathcal{V}$
$\{2\}$	1	1	2	0	0	0	0	0	0	2	3	$\mu_{3,0}$	1152	134400
$\{5\}$	1	1	1	1	1	3	0	0	0	6	8	$\mu_{3,3}$	192	21504
$\{6\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{3,3}$	64	10752
$\{7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,3}$	32	7168
$\{8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,3}$	32	7168
$\{1,2,5\}$	1	1	1	1	1	3	0	0	0	6	8	$\mu_{2,0}$	192	21504
$\{1,2,6\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{2,0}$	64	10752
$\{1,2,7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,0}$	32	7168
$\{1,2,8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,0}$	32	7168
$\{3,5,6\}$	1	1	1	1	2	2	2	0	0	8	18	$\mu_{2,1}$	64	9216
$\{3,5,7\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{3,5,8\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{3,6,7\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{3,6,8\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{3,7,8\}$	1	1	1	1	2	2	2	0	0	8	18	$\mu_{2,1}$	64	9216
$\{4,5,6\}$	1	1	1	1	2	2	2	0	0	8	18	$\mu_{2,1}$	64	9216
$\{4,5,7\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{4,5,8\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{4,6,7\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{4,6,8\}$	1	1	1	1	2	2	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{4,7,8\}$	1	1	1	1	2	2	2	0	0	8	18	$\mu_{2,1}$	64	9216
Totals													2304	313600
Doubles													4608	627200

Figure 8: Sizes of $\mathcal{V}^-(\mu_{4,2}, A)$ and $\mathcal{V}(\mu_{4,2}, A)$ for $A \in I(\mu_{4,2})$.

A	$q_i(\mu, A)$									Q	P	μ'_A	$\mathcal{V}^-$	$\mathcal{V}$
$\{3\}$	1	1	1	3	0	0	0	0	0	6	4	$\mu_{3,1}$	768	72192
$\{4\}$	1	1	1	1	2	0	0	0	0	2	6	$\mu_{3,1}$	256	36096
$\{6\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{3,2}$	64	10752
$\{7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,2}$	32	7168
$\{8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,2}$	32	7168
$\{1,3,4\}$	1	1	2	2	2	0	0	0	0	8	9	$\mu_{2,0}$	256	32256
$\{2,3,4\}$	1	1	2	2	2	0	0	0	0	8	9	$\mu_{2,0}$	256	32256
$\{5,6,7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,0}$	32	7168
$\{5,6,8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,0}$	32	7168
$\{5,7,8\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{2,0}$	64	10752
$\{1,3,6\}$	1	1	2	2	1	1	2	0	0	8	18	$\mu_{2,1}$	64	9216
$\{1,3,7\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{1,3,8\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{2,3,6\}$	1	1	2	2	1	1	2	0	0	8	18	$\mu_{2,1}$	64	9216
$\{2,3,7\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{2,3,8\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{1,4,6\}$	1	1	2	2	1	1	2	0	0	8	18	$\mu_{2,1}$	64	9216
$\{1,4,7\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{1,4,8\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{2,4,6\}$	1	1	2	2	1	1	2	0	0	8	18	$\mu_{2,1}$	64	9216
$\{2,4,7\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
$\{2,4,8\}$	1	1	2	2	1	1	1	1	0	4	24	$\mu_{2,1}$	32	6144
Totals													2304	308992
Doubles													4608	617984

Figure 9: Sizes of $\mathcal{V}^-(\mu_{4,3}, A)$ and $\mathcal{V}(\mu_{4,3}, A)$ for $A \in I(\mu_{4,3})$.

A	$q_i(\mu, A)$									Q	P	μ'_A	$\mathcal{V}^-$	$\mathcal{V}$
$\{2\}$	1	1	2	0	0	0	0	0	0	2	3	$\mu_{3,1}$	256	18048
$\{4\}$	1	1	1	1	2	0	0	0	0	2	6	$\mu_{3,3}$	64	5376
$\{6\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{3,4}$	16	1536
$\{7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,4}$	8	1024
$\{8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{3,4}$	8	1024
$\{1,2,4\}$	1	1	1	1	2	0	0	0	0	2	6	$\mu_{2,0}$	64	5376
$\{1,2,6\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{2,1}$	16	1536
$\{1,2,7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,1}$	8	1024
$\{1,2,8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,1}$	8	1024
$\{3,4,6\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{2,1}$	16	1536
$\{3,4,7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,1}$	8	1024
$\{3,4,8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,1}$	8	1024
$\{5,6,7\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,1}$	8	1024
$\{5,6,8\}$	1	1	1	1	1	1	1	1	0	1	16	$\mu_{2,1}$	8	1024
$\{5,7,8\}$	1	1	1	1	1	1	2	0	0	2	12	$\mu_{2,1}$	16	1536
Totals													512	43136
Doubles													1024	86272

Figure 10: Sizes of $\mathcal{V}^-(\mu_{4,4}, A)$ and $\mathcal{V}(\mu_{4,4}, A)$ for $A \in I(\mu_{4,4})$.

6.1 The Summary

This section includes tables summarizing the small values we have computed.

First we list the values of $|\mathcal{V}^-(\mu)|$ and $|\mathcal{V}(\mu)|$ for various values of μ.

m	μ	$\lvert\mathcal{V}^-(\mu)\rvert$	$\lvert\mathcal{V}(\mu)\rvert$
0	$+$	1	1
1	$--+$	2	4
2	$--+++$	16	112
	$-+-++$	4	16
3	$---++++$	576	22,400
	$--+-+++$	128	3,008
	$--++-++$	32	448
	$-+--+++$	32	448
	$-+-+-++$	8	64
4	$----+++++$	121,600	35,926,016
	$---+-++++$	20,736	4,453,376
	$---++-+++$	4,608	617,984
	$--+--++++$	4,608	627,200
	$--+-+-+++$	1,024	86,272
	$---+++-++$	1,152	89,600
	$-+---++++$	1,152	89,600
	$--++--+++$	256	12,544
	$--+-++-++$	256	12,032
	$-+--+-+++$	256	12,032
	$--++-+-++$	64	1,792
	$-+--++-++$	64	1,792
	$-+-+--+++$	64	1,792
	$-+-+-+-++$	16	256

Figure 11: Some small values of $R(\mu)$ and $R^+(\mu)$.

We have included information on the quotient $r_m/2^m$, since there is an equivalence relation on vip $(m+1)$-types whose equivalence classes all have size 2^m. We call this equivalence relation *flip similarity.* For a binary sequence s of length less than $2m$, let F_s be the automorphism of ${}^{\leq 2m}2$ defined

m	t_{m+1}	t^+_{m+1}	$t^+_{m+1}/2^m$	r_{m+1}	r^+_{m+1}	$r^+_{m+1}/2^m$
0	1	1	1	1	1	1
1	2	2	1	4	4	2
2	16	20	5	112	128	32
3	272	776	97	$12,352$	$26,368$	$3,296$
4	$7,936$	$155,856$	$19,482$	$4,437,760$	$41,932,288$	$5,241,536$

Figure 12: Some small values of r_m and r^+_m.

by

$$F_s(t) = \begin{cases} s^\frown\langle 1-\delta\rangle^\frown w & \text{if } t = s^\frown\langle\delta\rangle^\frown w, \\ t & \text{otherwise.} \end{cases}$$

Note that F_s preserves length, extension and meets. Hence it preserves the property of being a leaf. It follows that for any generalized Joyce tree T, its image, $F_s[T]$, is also a generalized Joyce tree.

Suppose that $(T, \lessdot)$ is a vip $(m+1)$-type. If $w \in T$ a meet and z is a node of T with $|z| > |w$, then $z(|w|) = 1$ only if $w \subseteq z$. Thus condition 2 in the definition of vip level order (see Definition 1.3) does not apply to $i = |w|$. Hence if $s \in L^{\wedge}(T) \setminus L(T)$ is a meet of T, then for all u in T, $F_s(u)$ differs from u at most at $|s|$. Thus for any vip $(m+1)$-type $(T, \lessdot)$, $F_s(\lessdot) = (F_s(u), F_s(v)) : u \lessdot v$ is a vip level order on $F_s[T]$, and $(F_s(T), F_s(\lessdot))$ is also a vip $(m+1)$-type.

For a binary sequence s of length less than $2m$, define the *flip G_s about s* on vip $(m+1)$-types by $G_s(T, \lessdot)$ is the identity unless $s \in L^{\wedge}(T) \setminus L(T)$ is a meet of T, in which case, $G_s(T, \lessdot) = (F_s(T), F_s(\lessdot))$.

Since the composition of G_s with itself is the identity, and for any s, $F_s(s) = s$, any sequence of these flips can be reversed. Say vip $(m+1)$-type $(T, \lessdot)$ with set of leaves L is *flip similar* to vip $(m+1)$-type $(T', \lessdot')$ with set of leaves L' if and only if either $(T, \lessdot) = (T', \lessdot')$ or there is a finite sequence of flips, $G_{s_0}, G_{s_1}, \dots, G_{s_\ell}$ such that $G_{s_\ell}(G_{s_{\ell-1}}(...(G_{s_0}(T, \lessdot))\dots)) = (T', \lessdot')$.

One can prove a normal form theorem that if $(T, \lessdot)$ and $(T', \lessdot')$ are flip similar but distinct, then there is a sequence of flips with $\lg(s_0) > \lg(s_1) > \cdots > \lg(s_{\ell-1})$ such that $G_{s_i}(T_i, \lessdot_i) \neq (T_i, \lessdot_i)$ for all $i < \ell$. Such a sequence necessarily consists of flips about elements of the meet closure of the leaves of τ which are not leaves themselves. Hence there are exactly 2^m such sequences.

They all lead to different vip $(m+1)$-types, so the equivalence classes all have the same size, namely 2^m.

7 Long leaves

This section is an investigation of vip m-types whose meet indicator sequence is μ^*_m, the concatenation of the sequence of -1's of length m with the sequence of $+1$'s of length $m+1$.

Lemma 7.1. *Suppose $m \le \ell \le 2m$. Then*

$$q_k(\mu^*_m, \{\ell\}) = \begin{cases} 1 & \text{if } k < \ell, \\ 2m - \ell & \text{if } k = \ell \\ 0 & \text{if } \ell < k. \end{cases}$$

Proof. Use the definition of $q_i(\mu, A)$ and and the fact that μ^*_m is the sequence whose first m entries are all -1 and last $m+1$ entries are all $+1$. ☐

The next result requires a similar computation.

Lemma 7.2. *Suppose $1 \le n < m$ and $A = \{\,\ell_0, \dots, \ell_{2n}\,\}$, where and $0 < \ell_0 < \cdots < \ell_{n-1} < m \le \ell_n < \dots \ell_{2n} \le 2m$. Then*

$$q_k(\mu^*_m, A) = \begin{cases} 1 & \text{if } k < \ell_0, \\ i+2 & \text{if } \ell_{i-1} < k < \ell_i, \\ \ell_i - i & \text{if } k = \ell_i \text{ and } i < n, \\ (2m - \ell_i) - (2n - i) & \text{if } k = \ell_i \text{ and } n \le i \le 2n, \\ 2n - i & \text{if } \ell_i < k < \ell_{i+1} \text{ and } n \le i < 2n, \\ 0 & \text{if } \ell_{2n} < k \le 2m. \end{cases}$$

Use Lemma 7.1 and the definitions of P and Q to obtain the next result.

Lemma 7.3. *For all m, ℓ with $1 \le m \le \ell \le 2m$, $Q(\mu^*_m, \{\ell\}) = (2m - \ell)!$, $P(\mu^*_m, \{\ell\}) = 2^{\ell-m}(2m-\ell+1)$ and $Q(\mu^*_m, \{\ell\})P(\mu^*_m, \{\ell\}) = 2^{\ell-m}(2m-\ell+1)!$.*

Definition 7.4. If m is positive, $n \le \lfloor \frac{m-1}{2} \rfloor$ and $\mu \in \mathcal{R}(m)$, then define $I_n(\mu) := I(\mu) \cap [2m+1]^{2n+1}$. Set $Q_{0,0} = 1$ and for $m > 0$ and $n \le \lfloor \frac{m-1}{2} \rfloor$, set $Q_{m,n} := \sum_{A \in I_n(\mu^*_m)} Q(\mu^*_m, A)$.

Lemma 7.5. *For all* $m \geq 0$, $Q_{m,0} =!m = \sum_{k=0}^{m} k!$ *and this function has a generating function* $(Ei(1) - (Ei(1-x)) * exp(1-x)$, *where* $Ei(x)$ *is the exponential integral.*

Proof. To derive the formula for Q from the sum over ℓ from m to $2m$ of the expression from the previous lemma, make the substitution $k = 2m - \ell$, and reverse the order of summation. See the Online Encyclopedia of Integer Sequences [11] for the definition of the lefthand factorial function and its generating function. ☐

Lemma 7.6. *Set* $M_{0,0} = 1$, $M_{1,0} = 4$, *and for* $m > 1$, *set* $M_{m,0} := 2\sum_{j=0}^{m} Q(\mu_m^*(\{m+j\})P(\mu_m^*(\{m+j\})$.

Then $M_{m,1} = \sum_{k=0}^{m}(k+1)!2^{m-j}$ *and this sequence satisfies the recurrence* $M_{1,0} = 1$, $M_{m+1,0} = 2M_{m,0} + (m+2)!$.

Proof. To derive the formula for M from the sum over ℓ from m to $2m$ of the expressions from Lemma 7.3, make the substitution $k = 2m - \ell$, and reverse the order of summation. Use algebra to show M_m satisfies the given recursion. ☐

Lemma 7.7. *Suppose* $1 \leq n < m$ *and* $A = \{\ell_0, \dots, \ell_{2n}\}$, *where and* $0 < \ell_0 < \cdots < \ell_{n-1} < m \leq \ell_n < \dots \ell_{2n} \leq 2m$. *Then, for* $\ell_{2n+1} = 2m$,

$$Q(\mu_m^*, A) = \prod_{i<n} [(i+2)!]^{\ell_{i+1}-\ell_i-1}(\ell_i - i)! \prod_{n \leq i \leq 2n} [(2n-i)!]^{\ell_{i+1}-\ell_i-1}(2m-\ell_i-2n+i)!$$

Proof. Recall $Q(\mu_m^*, A) = \prod_{k<2m} q_k(\mu_m^*, A)!$, and since $q_{2m}(\mu_m^*, A) = 0$, this product can be extended to include an extra term. Now apply Lemma 7.2. ☐

Example 7.8. Here are the values of $Q_{m,n}$ for $m = 5$ and $n \leq \lfloor \frac{m-1}{2} \rfloor = 2$:

$$\begin{aligned} Q_{5,0} &=!5 = 7,216 \\ Q_{5,1} &= 52,000 \\ Q_{5,2} &= 7728 \\ |\operatorname{Vip}^-(\mu_5^*)| &= 5,540,622,336 \end{aligned}$$

The value of $Q_{5,0}$ is computed using Lemma 7.5.

The set $I_1(\mu_5^*)$ may be described as the set of all $\{i+1, 5+j, 6+j+k\}$ where $i < 4$, $j \leq 4$ and $k \leq 4 - j$. Since $(2 \cdot 1 - 1)! = 1$ and $(2 \cdot 1 - 2)! = 1$, by Lemma 7.7, $Q(\mu_5^*, A)$ for $A = \{\, i+1, 5+j, 6+j+k \,\})$ may be computed as follows:

$$\begin{aligned} Q(\mu_5^*, A) &= 2^{5+j-(i+1)-1}(i+1)!(4-j)!(4-j-k)! \\ &= (2^{3-i}(i+1)!)\, 2^j (4-j)!(4-j-k)! \end{aligned}$$

Now sum up over i, j, k, to compute $Q_{5,1}$:

$$\begin{aligned} Q_{5,1} &= \sum_{i<4} \sum_{j \le 4} \sum_{k \le 4-j} \big(2^{3-i}(i+1)!\big)\, 2^j (4-j)!(4-j-k)! \\ &= \sum_{j \le 4} \sum_{k \le 4-j} \left(\sum_{i<4} 2^{3-i}(i+1)! \right) 2^j (4-j)!(4-j-k)! \\ &= \left(\sum_{i<4} 2^{3-i}(i+1)! \right) \left(\sum_{j \le 4} \sum_{k \le 4-j} 2^j (4-j)!(4-j-k)! \right) \\ &= \left(\sum_{i<4} 2^{3-i}(i+1)! \right) \left(\sum_{j \le 4} 2^j (4-j)! \sum_{k \le 4-j} (4-j-k)! \right) \\ &= \left(\sum_{i<4} 2^{3-i}(i+1)! \right) \left(\sum_{j \le 4} 2^j (4-j)! \sum_{\ell \le 4-j} \ell! \right) \\ &= \left(\sum_{i<4} 2^{3-i}(i+1)! \right) \left(\sum_{j \le 4} 2^j (4-j)!(!(4-j)) \right) \\ &= \left(\sum_{i<4} 2^{3-i}(i+1)! \right) \left(\sum_{p \le 4} 2^{4-p} p!(!p)) \right) \end{aligned}$$

Now $\sum_{i<4} 2^{3-i}(i+1)! = 2^3 1! + 2^2 2! + 2^1 3! + 2^0 4! = 8 + 8 + 12 + 24 = 52$. Moreover, $\sum_{p \le 4} 2^(4-p)p!(!p)) = 2^4 0!(!0) + 2^3 1!(!1) + 2^2 2!(!2) + 2^1 3!(!3) + 2^0 4!(!4) = 16(1) + 8(2) + 8(4) + 12(10) + 24(34) = 1000$. Hence $Q_{5,1} = 52{,}000$.

The set $I_2(\mu_5^*)$ is the set of all $\{\, 1, i+2, 8-j, 9-k, 10-\ell \,\}$ where $i < 3$, $\ell \le k \le j \le 3$. By Lemma 7.7, for $A = \{\, 1, i+2, 8-j, 9-k, 10-j-k \,\})$, $Q(\mu_5^*, A)$ may be computed as follows:

$$\begin{aligned} Q(\mu_5^*, A) &= 2^{(i+2)-1-1}(1-0)!(3!)^{8-j-(i+2)-1}(i+1)! 2^{(9-k)-(8-j)-1} j!k!\ell! \\ &= 2^i (3!)^{5-j-i}(i+1)! 2^{j-k} j!k!\ell! \\ &= 2^{i+5-j-i+j-k} 3^{5-j-i}(i+1)! j!k!\ell! \\ &= 2^{5-k} 3^{5-j-i}(i+1)! j!k!\ell! \\ &= 4\,(3^{2-i}(i+1)!)\, 3^{3-j} 2^{3-k} j!k!\ell! \end{aligned}$$

Next sum up over $i < 3$ and $\ell \leq k \leq j \leq 3$, to compute $Q_{5,2}$:

$$\begin{aligned} Q_{5,2} &= \sum_{\ell \leq k \leq j \leq 3} \sum_{i<3} \left(2^{2-i}(i+1)!\right) 3^{3-j} 2^{3-k} j! k! \ell! \\ &= \left(\sum_{i<3} 2^{3-i}(i+1)! \right) \sum_{\ell \leq k \leq j \leq 3} 3^{3-j} 2^{3-k} j! k! \ell! \end{aligned}$$

Now $\sum_{i<3} 2^{2-i}(i+1)! = 2^2 1! + 2^1 2! + 2^0 3! = 4 + 4 + 6 = 14$.

In Figure 13, the function $(j, k, \ell) \mapsto 3^{3-j} 2^{3-k} j! k! \ell!$ is computed for $\ell \leq k \leq j \leq 3$, and the results are summed to obtain a total of 1104.

j	k	ℓ	$3^{3-j}2^{3-k}j!k!\ell!$	j	k	ℓ	$3^{3-j}2^{3-k}j!k!\ell!$
3	3	3	216	2	2	2	48
3	3	2	72	2	2	1	24
3	3	1	36	2	2	0	24
3	3	0	36	2	1	1	24
3	2	2	48	2	1	0	24
3	2	1	24	2	0	0	48
3	2	0	24	1	1	1	36
3	1	1	24	1	1	0	36
3	1	0	24	1	0	0	72
3	0	0	48	0	0	0	216
Subtotals			552				552
Total							1104

Figure 13: Computation of $\sum_{\ell \leq k \leq j \leq 3} 3^{3-j} 2^{3-k} j! k! \ell!$.

Consequently $Q_{5,2} = 14(1104) = 7728$. By Theorem 5.7,

$$|\operatorname{Vip}^-(\mu_5^*) = 2 \sum_{i<3} Q_{5,i} |\operatorname{Vip}^-(\mu_i^*)| |\operatorname{Vip}^-(\mu_{m-i-1}^*)|.$$

That is, $|\operatorname{Vip}^-(\mu_5^*) = 2[Q_{5,0}(1)(60,800) + Q_{5,1}(2)(22,400) + Q_{5,2}(16)(16)]$. Evaluate the sum to obtain $|\operatorname{Vip}^-(\mu_5^*)| = 5,540,622,336$.

8 Bounds

This section includes some upper and lower bounds.

Proposition 8.1. *For all $m \geq 2$, $t^+_{m+1} \geq (t_{m+1} - 2^m) + 2^m \prod_{i \leq m} i!$.*

Proof. For each binary sequence u of length m, define an anticlique coding generalized Joyce tree $A(u)$ with $m+1$ leaves as follows. For each j with $m \leq j < 2m-1$, let $s(u,j)$ be the sequence of length j that extends $u \restriction k$ for $k = j - m$, takes the opposite value from u at k, $s(u,j)(k) = 1 - u(k)$, and satisfies $s(u,j)(\ell) = 0$ for $\ell > k$; let $s(u, 2m-1)$ be the extension by zeros of $u^\frown\langle 1 \rangle$ to length $2m-1$, and let $s(u, 2m)$ be the extension by zeros of u to length $2m$.

Notice that $L(u) := \{\, s(u,j) : m \leq j \leq 2m \,\}$ is an $(m+1)$-element antichain in ${}^{2m\geq}2$ whose meet closure is $L(u,j)$ together with all initial segments of u, a set of size $m+1+m = 2m+1$. It follows that $A(u)$, the closure of $L(u)$ under initial segments, is a generalized Joyce tree. Since all extensions were by zeros beyond what was necessary for the meets, $A(u)$ codes an anticlique.

Let $\mathcal{A}(m)$ be the family of all vip$^-$ $(m+1)$-types whose underlying tree is $A(u)$ for some binary sequence u of length m. Notice that any two vip orders on $A(u)$ agree on levels 0 to $m-1$, since any two meets of $A(u)$ are comparable. Now $A(u)$ admits $\prod_{i<m}(m-i)! = \prod_{k \leq m} k!$ many vip level orders, since the leaf x_i of length $m+i$ has $m-i$ elements of $A(u)$ of the same length whose meet with x_i is $u \restriction i$. Since there are 2^m ways to choose u, there are 2^m many trees of the form $A(u)$ and $|\mathcal{A}(m)| = 2^m \prod_{i \leq m} i!$. Recall that by Corollary 3.5 the number of anticlique coding $(m+1)$-types is t_{m+1}. Thus the lower bound of the lemma is the sum of $|\mathcal{A}(m)|$ and the number of anticlique coding $(m+1)$-types not of the form $A(u)$, without reference to vip order. $\square$

Proposition 8.2. *For all $m \geq 2$, $r^+_{m+1} \geq (t_{m+1} - 2^m) + 2^m (m+1)! \left(\prod_{i \leq m} i!\right)$.*

Proof. Momentarily fix a binary sequence u of length m, a sequence $\vec{p} = \langle\, p_i : i < m \,\rangle$ with $p_i \leq m-i$, and a sequence $\mathcal{O} = \langle\, \prec_i : i < m \,\rangle$ of linear orders with domain of $\prec_i$ being $\{\, 0, 1, \ldots, m-i-1 \,\}$. Modify the construction of $A(u)$ to create, for $m \leq j \leq 2m$, the sequence $b(u, \vec{p}, \mathcal{O}, j)$ of length j which extends $s(u,j) \restriction m$ such that for all i with $m \leq i < j$, if $s(u,j) \restriction i$ is the kth element in listing of that level of $A(u)$, without $s(u,i)$, in increasing lexicographic order and k has p or more predecessors in the linear order $\prec_i$, then $b(u, \vec{p}, \mathcal{O}, j)(i) = 1$ and otherwise, $b(u, \vec{p}, \mathcal{O}, j)(i) = 0$.

Let $B(u, \vec{v}, \mathcal{O})$ be the downwards closure of $\{\, b(u, \vec{p}, \mathcal{O}, j) : m \leq j \leq 2m \,\}$, and let $\lessdot$ be the unique level order which is a vip order and matches the order on the level j for $m \leq j < 2m$ induced by $\prec_{j-m}$ on the indices of the elements other than $b(u, \vec{p}, \mathcal{O}, i)$ listed in increasing lexicographic order. Since the lexicographic order on such a level is determined by the restrictions of the sequences to length m, it matches that of the corresponding level of $A(u)$.

If $u \neq u'$ or $\vec{p} \neq \vec{p}'$ or $\mathcal{O} \neq \mathcal{O}'$, then $(B(u, \vec{p}, \mathcal{O}), \lessdot) \neq (B(u', \vec{p}', \mathcal{O}'), \lessdot')$. Since all of these pairs are vip m-types, the proposition follows by a count of the appropriate family of triples $(u, \vec{p}, \mathcal{O})$. □

Lemma 8.3. *Suppose T is a generalized Joyce tree and L is the set of the $m+1$ leaves of T. Then for all $i \leq 2m$, the size of level i is*

$$|T(i)| = \sum_{i \leq j \leq 2m} \mu^T(j) = i + 1 - 2|\{\, j < i : \mu^T(i) = +1 \,\}|.$$

Moreover, if $i \leq m$, then $|T(i)| \leq i+1$ and $|T(2m-i)| \leq i+1$.

Proof. By Lemma 2.2, $|T(i)| = \sum_{i \leq j \leq 2m} \mu^T(j)$. Thus for $i < m$, $|T(2m - i)| \leq i+1$, with equality if all elements of $L^\wedge$ of length at least $2m - i$ are leaves.

To prove $|T(i)| = i + 1 - 2|\{\, j < i : \mu^T(i) = +1 \,\}|$, use induction on $i \leq 2m$. The basis case is $i = 0$, where $T(0)$ is a singleton and $i + 1 - 2 \cdot 0 = 1$. For the induction step, note that $|T(i+1)|$ is one less than $|T(i)|$ if $\mu(i) = +1$ (the unique element of $L^\wedge$ of length i is a leaf) and that $|T(i+1)|$ is one more than $|T(i)|$ if $\mu(i) = -1$ (the unique element of $L^\wedge$ of length i is a meet).

Notice that if $i \leq m$, then $|T(i)| \leq i + 1$, with equality if all elements of $L^\wedge$ of length less than i are meets. □

Lemma 8.4. *Suppose T is a generalized Joyce tree and L is the set of the $m+1$ leaves of T. Then the number $V(T)$ of vip orders on T, is bounded above:*

$$V(T) \leq \prod_{0 < i < 2m} \left(-1 + \sum_{i \leq j \leq 2m} \mu^T(j) \right)! \leq m! \prod_{i < m} (i!)^2.$$

Proof. The estimate with the meet indicator sequence gives the number of level orders in which for each level, the unique element of $L^\wedge$ on that level is its least element. For the second inequality, use the estimates of Lemma 8.3 and the fact that $0! = 1$. □

Proposition 8.5. *The number of anticlique coding vip $(m+1)$-types is bounded above by $t^+_{m+1} \leq t_{m+1}(m!) \prod_{i<m} (i!)^2$.*

Proof. By Corollary 3.5 the number of anticlique coding $(m+1)$-types is t_{m+1}, so the lemma follows from Lemma 8.4. $\square$

Proposition 8.6. *The number of vip* $(m+1)$*-types is bounded above by* $r^+_{m+1} \leq r_{m+1}(m!)\prod_{i<m}(i!)^2$.

Proof. Since there is a bijection between strongly diagonal subsets $L \subseteq {}^{2m\geq}2$ of size $m+1$ and generalized Joyce trees (map L to its closure under initial segments), the value r_{m+1} identified by Laflamme, Sauer and Vuksanovic [8] is the number of coding $(m+1)$-types is t_{m+1}, so the lemma follows from Lemma 8.4. $\square$

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