Some Topics from the Structure Theory of Banach Spaces with Emphasis on Examples which are related to Set Theory and Logic

Thomas Schlumprecht

Texas A&M University

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Overview

Introduction

Let X be a Banach space with norm $\|\cdot\|$, i.e. X is a vector space over $\mathbb{R}$ (in our case), and

$\|\cdot\| : X \to [0, \infty)$ such that

$$\|x\| = 0 \iff x = 0 \ \& \quad \|x + y\| \leq \|x\| + \|y\|$$

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$$(X, \|\cdot\|) \text{ is complete (Cauchy sequences converge)}$$

A sequence (x_n) in X is called Schauder basis or simply basis of X, if for every $x \in X$ there is a unique sequence $(a_n) \subset \mathbb{R}$ so that

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

Assume (x_n) is a basis of X.
For $n \in \mathbb{N}$ the n-th canonical projection is the map

$$P_n : X \to X, \quad \sum_{i \in \mathbb{N}} a_i x_i \mapsto \sum_{i=1}^{n} a_i x_i.$$

Note:

- $\dim(P_n(X)) = n$,
- $P_m \circ P_n = P_{\min(m,n)}$
- $P_n(x) \to x$, as $n \to \infty$, $x \in X$.

Uniform boundedness of the P_n's: For fixed $x \in X$, say

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We can therefore use the Uniform Boundedness Principle to get

$$C = \sup_{n\in\mathbb{N}} \|P_n\| = \sup_{n\in\mathbb{N}} \sup_{x\in X, x\in \text{Ball}_X} \|P_n(x)\| < \infty.$$

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C is called Basis constant of (x_n) and $|||\cdot|||$ is an equivalent norm on X:

$$\|x\| \le |||x||| \le C\|x\|.$$

Conversely: Assume that X is a separable Banach space and for $n \in \mathbb{N}$ there are projections $P_n : X \to X$ so that

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If we choose $x_n \in P_n(X) \cap \mathrm{Ker}(P_{n-1})$, $x_n \neq 0$, (with $P_0 = 0$). Then (x_n) is a basis of X.

Coordinate functionals:

Let X be a space with basis (x_n) the n-th coordinate functional is the map

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Using the coordinate functionals we can write every $x \in X$ as

$$x = \sum_{i\in\mathbb{N}} x_i^*(x) x_i.$$

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Identify x_n with $\|x_n\| \cdot e_n$. X is then the space of sequences $(\xi_i) = (a_i \|x_i\|)$

$$\text{so that} \quad \sum_{i=1}^{n} \xi_i e_i \text{ converges in } X.$$

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Thus X is the completion of c_{00} with respect to $\|\cdot\|$ with $\|e_i\| = 1$, $i = 1, 2 \ldots$.

Often we define a separable Banach space, by defining a norm $\|\cdot\|$ on c_{00}, so that there is a constant C (usually $C = 1$) so that for all $n \in \mathbb{N}$ and $(a_i)_{i=1}^n$ and all $m \leq n$.

$$\Big\|\sum_{i=1}^{m} a_i e_i\Big\| \leq C\Big\|\sum_{i=1}^{n} a_i e_i\Big\|.$$

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In this case the completion of c_{00} with respect to $\|\cdot\|$, i.e. the vector space X of all sequences (a_i) for which

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is Cauchy, is a Banach space and c_{00} is dense in X. Moreover, if $\|e_i\| = 1$ (or more generally $0 < a < \|e_i\| < b$)), X is a subset of the space of all sequences converging to 0.

Examples

- For $1 \leq p < \infty$

$$\ell_p = \Big\{ (\xi_i) : \sum_{i=1}^{\infty} |\xi_i|^p < \infty \Big\} \text{ with } \|(\xi_i)\|_p = \Big(\sum_{i=1}^{\infty} |\xi_i|^p \Big)^{1/p}.$$

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- Similarly

$$c_0 = \big\{ (\xi_i) : \lim_{i \to \infty} \xi_i = 0 \big\} \text{ with } \|(\xi_i)\|_\infty = \max_{i \in \mathbb{N}} |\xi_i|.$$

- For a separable probability space $(\Omega, \mathcal{M}, \mu)$ and $1 \leq p < \infty$

$$L_p(\mu) = \left\{ f : \Omega \to \mathbb{R} : \|f\|_p := \left(\int_\Omega |f|^p d\mu \right)^{1/p} < \infty \right\}$$

Choose sequence (A_n) of measurable sets with positive measure so that

1. $A_1 = \Omega$,
2. for $n > 0$ and all $m < n$, either $A_n \subset A_m$ and $A_m \setminus A_n$ also has positive measure, or $A_n \cap A_m = \emptyset$.
3. $\{A_n : n \in \mathbb{N}\}$ generates $\mathcal{M}$

Let $\mathcal{M}_n = \sigma(A_1, \ldots, A_n)$, and $P_n(f) = \mathbb{E}(f|\mathcal{M}_n)$.
Choose $h_n \in P_n(L_p(\mu) \cap \mathrm{Ker}(P_{n-1}), \|h_n\| = 1$.
(h_n) is called Haar basis of L_p.

Existence of Bases

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Ovsepian & Pełczyński 1975: Every separable Banach space X admits a bounded, total, biorthogonal system $(e_i, e_i^*) \subset X \times X^*$,

- $\overline{\text{span}(e_i : i \in \mathbb{N})} = X$,
- $e_i^*(e_j) = \delta_{(i,j)}$,
- $\forall x \in X \big(\forall i \in \mathbb{N}\ e_i^*(x) = 0 \Rightarrow x = 0\big)$,
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Every Banach space contains basic sequences (sequences which are a basis for their closed linear span).
Every normalized weakly null sequence has a subsequence which is basic.

Block bases

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If the property passes to subspaces and is stable "under small perturbation" the question becomes a question on block bases:
If (x_n) is a basis a sequence $(y_n) \subset \text{span}(x_n : n \in \mathbb{N}) \setminus \{0\}$ is called block basis of (x_n) if

$$y_n = \sum_{j=k_{n-1}+1}^{k_n} a_j x_j, \text{ with } (a_i) \subset \mathbb{R} \text{ and } 0 = k_0 < k_1 < k_2 \ldots$$

- Every infinite dimensional Banach space contains a basic sequence,
- and to every infinite dimensional subspace Y of a space X with basis (x_n) there is a block basis (y_i) of (x_i) whose span is "almost in Y", i.e. for given $\varepsilon_n \searrow 0$

$$\sup\left\{\mathrm{dist}(z, Y) : z \in \mathrm{span}(y_i : i \geq n), \|z\| = 1\right\} \leq \varepsilon_n.$$

Thus aforementioned question takes the following form:

- Does every/certain basic sequences have a block basis with property (P)?

Equivalence Relations for Bases

Definition

Let (y_n) and (x_n) be two normalized basic sequences, we say (x_n) *C- dominates* (y_n), if

$$\Big\|\sum a_i y_i\Big\| \le C\Big\|\sum a_i x_i\Big\|, \ (a_i) \in c_{00},$$

and write $(y_n) \preceq_C (x_n)$.
We say (y_n) and (x_n) are C-equivalent, and write $(x_n) \sim_C (y_n)$, if $(x_n) \preceq_C (y_n)$ and $(y_n) \preceq_C (x_n)$.

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Theorem (Zippin 1966)

If (x_n) is equivalent to all of its normalized blocks, then (x_n) is equivalent to the ℓ_p or c_0 unit vector basis.

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Problem

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Theorem (Komorowski and Tomczak-Jaegermann (1995) and Gowers (1996))

An infinite dimensional space which is isomorphic to all its infinite dimensional subspaces is isomorphic to Hilbert space.

Shrinking and Boundedly complete bases

Let X be a Banach space and X^* the continuous linear functionals on X. Canonical embedding (By Hahn Banach an isometric embedding):

$$j : X \to X^{**}, \quad x \mapsto j(x), \text{ with } j(x)(x^*) = x^*(x).$$

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X is called reflexive if $j(X) = X^{**}$.

Proposition

X reflexive $\iff$ X^* reflexive $\iff$ $(Ball_X, w)$ compact. $\iff$ all closed subspaces are reflexive.

Assume X has a basis (x_n). What properties of (x_n) ensure that X is reflexive ?

Let X be reflexive with a normalized basis (x_n).

- (x_n) is weakly null. Since every block basis is a basis of a subspace, every normalized block basis is weakly null. (x_n) is shrinking.

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  Limit point y has to be of the form $y = \sum_{i=1}^\infty b_i x_i$.
  Applying coordinate functionals: $y = \sum_{i=1}^\infty a_i x_i$.
  Thus the partial sums converge. (boundedly complete)

These two properties characterize the basis of a reflexive space:

Proposition

Let X be a Banach space with a basis (x_n). Then X is reflexive if and only if (x_n) is boundedly complete and shrinking.

Unconditional Bases

The basis (x_n) of a Banach space X is called unconditional if for every

$$x = \sum_{i=1}^{\infty} a_i x_i \in X$$

the convergence is unconditional.

$$\iff \exists c_u \ forall (a_i) \in c_{00} \ \forall (\varepsilon_i) \in \{-1, +1\}^{\mathbb{N}}$$

$$\left\| \sum_{i=1}^{\infty} \varepsilon_i a_i x_i \right\| \leq c_u \left\| \sum_{i=1}^{\infty} a_i x_i \right\|$$

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$$\left\| \sum_{i=1}^{\infty} \varepsilon_i a_i x_i \right\| \leq c_u \left\| \sum_{i=1}^{\infty} a_i x_i \right\|$$

James 1950: Let X be a Banach space with unconditional basis. Then X is reflexive, or contains c_0 or ℓ_1.
Two steps:

- If (x_i) is not boundedly complete, X contains c_0.
- If (x_i) is not shrinking, X contains ℓ_1.

Example

The unit vector bases of c_0 and ℓ_p, $1 \leq p < \infty$ are unconditional. The summing basis (s_n) of c_0 is not unconditional.

$$s_n = (\underbrace{1, 1, \ldots 1}_{n \text{times}}, 0, 0, \ldots).$$

$(\|s_1 + s_2 \ldots s_n\| = n$, but $\|s_1 - s_2 + s_3, \ldots \pm s_n\| = 1)$
The Haar basis of L_p, $1 < p < \infty$, is unconditional. For $p = 1$ this **is not true**. Actually L_1 has no unconditional basis.

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Ramsey's Theorem

Ramsey's Theorem is usually used in Banach space theory for the following type of problem:

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Often a result of this type involves **Ramsey's theorem.**

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Note that for a net $\{N_i : i \in I\}$ in $[\mathbb{N}]^\omega$,

$$\lim_{i\in I} N_i = N \iff \forall n\in\mathbb{N}\,\exists i_o\in I\,\forall i\geq i_0 \quad \{1,2,\ldots n\}\cap N_i = \{1,2,\ldots n\}\cap N$$

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$$\lim_{i\in I} N_i = N \iff \forall n\in\mathbb{N}\,\exists i_o\in I\,\forall i\geq i_0$$
$$\{1,2,\ldots n\}\cap N_i = \{1,2,\ldots n\}\cap N$$

Theorem (Ramsey 1930, Silver 1970, Galvin&Pikry 1973, Ellentuck 1974)

Assume $\mathcal{A} \subset [\mathbb{N}]^\omega$ is analytic (or closed).
Then there is an infinite $N \subset \mathbb{N}$ so that

$$\textit{Either } [N]^\omega \subset \mathcal{A} \textit{ or } [N]^\omega \cap \mathcal{A} = \emptyset.$$

Corollary

If $(\mathcal{A}_i : i = 1, 2 \ldots n)$ *is a finite partition of* $[\mathbb{N}]^\omega$ *into Borel sets, then there is an* $i_0 \in \{1, 2 \ldots n\}$ *and an* $N \in [\mathbb{N}]^\omega$ *so that* $[N]^\omega \subset \mathcal{A}_{i_0}$.

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If $(\mathcal{A}_i : i = 1, 2 \ldots n)$ is a finite partition of $[\mathbb{N}]^\omega$ into Borel sets, then there is an $i_0 \in \{1, 2 \ldots n\}$ and an $N \in [\mathbb{N}]^\omega$ so that $[N]^\omega \subset \mathcal{A}_{i_0}$.

Corollary ("Ramsey for analysts")

Let (K, d) be compact and metric and let $f : [\mathbb{N}]^\omega \to K$ be Borel. Let $\varepsilon_n \searrow 0$.
Then there exists an $x_0 \in K$ and an $N = \{n_1 < n_2, \ldots\} \in [\mathbb{N}]^\omega$ so that

$$d(f(M), x_0) < \varepsilon_k \text{ whenever } M \in [\{n_k, n_{k+1} \ldots\}]^\omega.$$

Spreading Models

Definition

A basic sequence (e_i) is called spreading if for some constant $C \geq 1$ so that for all $(a_i) \in c_{00}$

$$\frac{1}{C}\Big\|\sum a_i x_i\Big\| \leq \Big\|\sum a_i x_{n_i}\Big\| \leq C\Big\|\sum a_i x_i\Big\|,$$

it is called subsymmetric if it is unconditional and spreading.

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it is called subsymmetric if it is unconditional and spreading.
Let X be a Banach space and (x_i) be a sequence. A spreading sequence (e_i) is called the spreading model of (x_n) if for all $k \in \mathbb{N}$ and $(a_i)_{i=1}^n$

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\ldots\lim_{n_k\to\infty}\Big\|\sum_{i=1}^k a_i x_{n_i}\Big\| = \Big\|\sum_{i=1}^k a_i e_i\Big\|.$$

Proposition (Brunel and Sucheston, 1973)

Every non compact seminormalized sequence (x_n) *in a Banach spaces* X *has a subsequence with a spreading model* (e_i).

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$$\left| \Big\| \sum_{i=1}^{k} a_i x_{n_{\ell_i}} \Big\| - \Big\| \sum_{i=1}^{k} a_i e_i \Big\| \right| < \varepsilon_k,$$

whenever $k \in \mathbb{N}$, $k \leq \ell_1 < \ell_2 < \ldots < \ell_k$, and $(a_i)_{i=1}^{k} \subset \mathbb{R}$.

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whenever $k \in \mathbb{N}$, $k \leq \ell_1 < \ell_2 < \ldots < \ell_k$, *and* $(a_i)_{i=1}^{k} \subset \mathbb{R}$.

Proof: Apply "Ramsey for Analysts" to the functions

$$f_k : [\mathbb{N}]^k \to \mathcal{M}_k = \{s : \mathbb{R}^k \to [0,\infty) \text{ seminorm}\}$$

$$(n_1, n_2, \ldots, n_k) \mapsto \left[(a_i)_{i=1}^{k} \mapsto \| \sum_{i=1}^{k} a_i x_{n_i} \| \right].$$

Proposition

If (x_n) *is a semi normalized weakly null sequence, it has a subsequence which has an unconditional spreading model.*

Rosenthal's ℓ_1 Theorem

Theorem

Let (x_n) be a bounded sequence in a Banach space X. Then (x_n) has a subsequence (y_n) so that,
either (y_n) is w-Cauchy
or (y_n) is equivalent to the ℓ_1 unit vector basis.

Consider $X \subset C(Ball(X^*), w^*) = S$. The Crucial Lemma

Lemma

Let (A_n, B_n) with $A_n, B_n \subset S$ disjoint for $n \in \mathbb{N}$. Then there is a subsequence (A_{n_i}, B_{n_i}) which is
Either Boolean independent, i.e.

$$\bigcap_{i \in I_1} A_i \cap \bigcap_{i \in I_2} B_i \neq \emptyset \text{ whenever } I_1, I_2 \subset \mathbb{N} \text{ are disjoint}$$

Or convergent

$$\forall s \in S \quad \{i \in \mathbb{N} : s \in A_{n_i}\} \text{ or } \{i \in \mathbb{N} : s \in B_{n_i}\} \text{ finite.}$$

Partial Unconditionality

Example (Maurey Rosenthal 1977)

There is a normalized weakly sequence which has no unconditional basic subsequence.

(proof uses idea of coding)

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Problem

Given a basic sequence (x_i). Can we pass to a subsequence, so that there is a constant C so that

$$\Big\| \sum_{i \in E} a_i y_i \Big\| \leq C \Big\| \sum a_i y_i \Big\|$$

for certain E's and/or certain coefficients ?

Theorem (Elton 1980)

For each $1 > \delta > 0$ *there exists* $K(\delta) < \infty$ *(actually* $K(\delta) \sim_C \log(1/\delta)$*) with the following property.*

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Let (x_n) *be a normalized weakly null sequence.*
Then there exists a subsequence (y_n) *of* (x_n) *such that if* $(a_i)_{i=1}^{\infty} \subseteq [-1, 1]$ *and* $I \subseteq \{i : |a_i| \geq \delta\}$ *then*

$$\|\sum_I a_i y_i\| \leq K(\delta)\|\sum a_i y_i\|.$$

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Problem: Is $K(\delta)$ independent of δ?
Dilworth, Kalton and Kutzarova 2003: Yes, if (x_n) does not have a subsequence whose spreading model is equivalent to the c_0-unit vector bases.

Theorem (Odell 1993 "Schreier Unconditionality")

If (x_n) is a normalized weakly null sequence and $\varepsilon > 0$ then there is a subsequence (y_n) which is $(2+\varepsilon)$-Schreier unconditional : for each

$$E \in \{F \subset \mathbb{N} : \min F \geq |F|\} =: \mathcal{S}_1$$

so that

$$\Big\| \sum_{i \in E} a_i y_i \Big\| \leq (2+\varepsilon) \Big\| \sum a_i y_i \Big\|$$

Remark: Statement becomes wrong if one replaces $\mathcal{S}_1$ by

$$\mathcal{S}_2 = \Big\{ \bigcup_{i=1}^{n} E_i : n \leq \min E_i \ \& \ E_i \in \mathcal{S}_1 \Big\}.$$

Theorem (Argyros, Mercourakis and Tsarpalias 1998)

Every normalized weakly null sequence (x_n) has a convexly unconditional subsequence (y_n), meaning that for all $\delta > 0$ there is a c_δ so that:
If $F \subset \mathbb{N}$ finite and $(\lambda_i)_{i\in\mathbb{N}}$ such that

$$\delta < \Big\| \sum_{i\in F} \lambda_i y_i \Big\| \le \sum_{i\in F} |\lambda_i| = 1.$$

Then

$$\Big\| \sum_{i=1}^{\infty} \lambda_i y_i \Big\| \ge c_\delta.$$

Problem

Does Ramsey's Theorem extend to Banach spaces? Let $\mathcal{A} \subset S_X$, $\dim(X) = \infty$.
Is there an infinite dimensional subspace Y *so that* $S_Y \subset \mathcal{A}$ *or* $\mathcal{A} \cap S_Y = \emptyset$*?*

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Answer: of course not!

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Answer: of course not!

Example

(x_i) basis of X. For $x = \sum_{i=1}^{\infty} x_i^*(x) x_i \in S_X$ let $i_0(x) \in \mathbb{N}$ be the minimum of all i's in $\mathbb{N}$ for which $|x_i^*(x)| = \max_{j\in\mathbb{N}} x_i^*(x)|$. Then let

$$A = \{x \in S_X : x^*_{i_0(x)}(x) > 0\} \text{ and } B = \{x \in S_X : x^*_{i_0(x)}(x) < 0\}.$$

More sensible question:

Problem

Assume $A \subset S_X$ and $\varepsilon > 0$.
Is there a infinite dimensional subspace $Y \subset X$, so that

$$S_Y \subset A_\varepsilon \text{ or } S_Y \subset (S_X \setminus A)_\varepsilon?$$

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Equivalently, let $f : S_X \to \mathbb{R}$ be Lipschitz/uniformly continuous, and $\varepsilon > 0$, is there an infinite dimensional $Y \subset X$ so that

$$osc(f, S_Y) = \sup_{y,z \in S_Y} |f(z) - f(y)| \leq \varepsilon?$$

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How about, if f *is an equivalent norm?*

Definition

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We say that $|||\cdot|||$ is a λ-distortion of $\|\cdot\|$ if for all infinite dimensional subspaces $Y \subset X$,

$$\sup\left\{\frac{|||y_1|||}{|||y_2|||} : \|y_1\| = \|y_2\| = 1\right\} \geq \lambda.$$

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We say that X is λ-distortable if such an equivalent norm exists and
that X is distortable if there is a $\lambda > 1$ for which X is λ-distortable,
and we say that X is arbitrary distortable if for all $\lambda > 1$ X is λ-distortable.

Theorem (James 1964)

c_0 and ℓ_1 are not distortable.

Idea of proof (for ℓ_1)
Let $|||\cdot|||$ equivalent norm on ℓ_1

$$r = \lim_{n\to\infty} \inf\{|||x||| : x \in \text{span}(e_i : i \geq n), \|x\| = 1\} > 0,$$

choose block basis (y_i) with $|||y_i||| \sim r$. Then

$$\Big|\Big|\Big| \sum a_i y_i \Big|\Big|\Big| \sim r \text{ whenever } \sum_{i=1}^{\infty} |a_i| = 1.$$

Tsirelson's space

Theorem (Tsirelson 1974)

There is a separable space T which does not contain any copy of ℓ_p, $1 \leq p < \infty$, or c_0.

Remark

This space was the first space known to be distortable. We will present the construction of its dual which was considered by Figiel and Johnson, 1974.

There is a norm on c_{00} which satisfies the implicit equation

$$||x|| = \|x\|_{c_0} \vee \sup_{n \leq E_1 < E_2 < \ldots E_n} \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\|.$$

Where:
For $E \subset \mathbb{N}$ finite and $x = \sum a_i e_i \in c_{00}$: $E(x) = \sum_{i \in E} a_i e_i$,
$n \leq E_1 < E_2 \ldots E_n$ means $n \leq \min E_1 \leq \max E_2 < \min E_2 \ldots$

Indeed define by induction: $\|x\|_0 = \max |x_i|$ text and

$$\|x\|_{n+1} = \|x\|_n \vee \sup_{n \leq E_1 < E_2 < \ldots E_n} \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\|_n.$$

Then take $\|x\| := \max \|x\|_n$.

$$||x|| = \|x\|_{c_0} \vee \sup_{n \le E_1 < E_2 < \ldots E_n} \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\|.$$

Sketch

$$||x|| = \|x\|_{c_0} \vee \sup_{n \le E_1 < E_2 < \dots E_n} \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\|.$$

Sketch

1. ℓ_1 is finitely block represented in any block space Y of T, i.e.

$$\forall \varepsilon > 0, n \in \mathbb{N}\ \exists \text{block} (y_i)_{i=1}^n \quad (y_i)_{i=1}^n \sim_{(1+\varepsilon)} \ell_1^n \text{un.vect.basis.}$$

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2. For $\ell \in \mathbb{N}$

$$|||x|||_\ell = \max_{E_1 < E_2 < \ldots E_\ell} \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\|.$$

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$$|||x|||_\ell = \max_{E_1 < E_2 < \ldots E_\ell} \frac{1}{2} \sum_{i=1}^{n} \|E_i(x)\|.$$

3. If $n >> \ell$ and $y = \frac{1}{n} \sum_{i=1}^n x_i$, (x_i) block then $|||y|||_\ell \leq 1/2$

4. If $x = \frac{1}{n}\sum_{i=1}^{n} x_i$, $(x_i) \sim_{1+\varepsilon} \ell_1^n$un.vect.basis,then

$$||x|| \sim 1 \text{ and } |||x|||_2 \sim \frac{1}{2}$$

4. If $x = \frac{1}{n}\sum_{i=1}^{n} x_i$, $(x_i) \sim_{1+\varepsilon} \ell_1^n$un.vect.basis,then

$$||x|| \sim 1 \text{ and } |||x|||_2 \sim \frac{1}{2}$$

5. If $x = \frac{1}{n_1}\sum_{i=1}^{n} x_i$, and $y = \frac{1}{n_2}\sum_{i=1}^{n_2} x_i$, with $n_2 >> \max \mathrm{supp} x$, then

$$||x+y|| \sim \frac{3}{2} \text{ and } |||x+y||| \sim 1.$$

Example (Sch, 1991)

Consider the norm on c_{00} given by the following implicit norm

$$\|x\| = \|x\|_{c_0} \vee \sup_{\ell \geq 2, E_1 < E_2 \ldots E_\ell} \frac{1}{\log_2(\ell+1)} \sum \|E_i(x)\|.$$

Then the completion of c_{00} under that norm is arbitrary distortable.

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Problem

Are there spaces which are distortable but not arbitrarily distortable? Is Tsirelson's space such a space?

Example (Argyros and Deliyanni 1997)

There is an arbitrary distortable Banach space, the spreading models all of its block bases are (uniformly) equivalent to ℓ_1.

Remark

The fact that ℓ_p , $1 < p < \infty$, is distortable, implies that there is also a Lipschitz function on ℓ_1 which does not stabilize on an infinite dimensional subspace.

But

Theorem (Gowers 1992)

Every Lipschitz function on S_{c_0} stabilizes on infinite dimensional subspaces.

Proof uses a "vector space" version of Glaser's theorem that compact semi groups have idempotent elements.

Gowers' dichotomy Theorem

Restatement of Ramsey's Theorem:
Consider game between two players: Let $\mathcal{A} \subset [\mathbb{N}]^\omega$ closed
Player I chooses $N_1 \in [\mathbb{N}]^\omega$, Player II: $n_1 \in \mathbb{N}$
Player I chooses $N_2 \in [\mathbb{N}]^\omega$, Player II: $n_2 \in \mathbb{N}$
$\vdots$
Player I wins if $\{n_i : i \in \mathbb{N}\} \in \mathcal{A}$.

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One way to interpret Galvin & Pikry's proof:
Either $\exists N \in [\mathbb{N}]^\omega \forall M \in [N]^\omega$ Player I has winning strategy on M
In that case there is $\exists N' \in [\mathbb{N}]^\omega \quad [N']^\omega \subset \mathcal{A}$.

Or $\forall N \in [\mathbb{N}]^\omega \exists M \in [N]^\omega$ Player II has winning strategy.
Then $\forall N \in [\mathbb{N}]^\omega \exists M \in [N]^\omega \qquad \mathcal{A} \cap [M]^\omega = \emptyset$.

Gowers

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Player I chooses $X_1 \subset X$, block subspace Player II: $x_1 \in S_{X_1}$

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Player I wins if $\{x_i : i \in \mathbb{N}\} \in \mathcal{A}$.

For $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1)$, we put

$$\mathcal{A}_{\overline{\varepsilon}} = \{(z_i) \subset S_X : \exists (x_i) \in \mathcal{A} : \quad \|x_i - z_i\| < \varepsilon_i\}.$$

Theorem (Gowers 2002)

If on each infinite dimensional subspace Y and for any $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1)$, Player I has a winning strategy for $\mathcal{A}_{\overline{\varepsilon}}$, then for each $\overline{\varepsilon} = (\varepsilon_i) \subset (0,1)$ there is a block space Y so that every normalized block of Y is in $\mathcal{A}_{\overline{\varepsilon}}$.

Problem (Universality Problem)

Given a class $\mathcal{C}$ of separable Banach spaces (we can think of it as a subset of all the subspaces of $C[0,1]$).
Is there an element X in $\mathcal{C}$, or in some related class $\mathcal{C}'$, so that X is universal for $\mathcal{C}$, i.e. every Y in $\mathcal{C}$ embeds (isomorphically) in X?

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Often this problem is easier to solve if one restricts one self to spaces with bases or, more generally with finite dimensional decompositions.

Definition (Finite Dimensional Decompositions)

A sequence of finite dimensional subspaces E_n of a Banach space X is called Finite Dimensional Decomposition of X (FDD) if for all $x \in X$ there is a unique sequence (x_i), with $x_i \in E_i$, so that $X = \sum x_i$.

Assuming a universality problem was solved within the class of Banach spaces with bases/FDDs, the general universality problem becomes an Embedding Problem.

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Problem (Embedding Problem)

Assume X is a Banach space in some class $\mathcal{C}$, can it be embedded into a space Z in $\mathcal{C}$, or some related class $\mathcal{C}'$, with a basis/FDD?

Examples

The space of continuous functions on $[0, 1]$, $C[0, 1]$, is universal for all separable Banach spaces (Banach).
Pełczyński's: ('69) There is a Banach space having a basis/unconditional basis, which is complementably universal for all Banach spaces having basis/unconditional bases.

Szlenk 1968: There is no reflexive, separable space universal for all reflexive separable spaces.
Sketch of Szlenk's argument: Define an "index" which assigns to each reflexive separable space X an ordinal $\alpha = \mathrm{Sz}(X) < \omega_1$ so that

- $\mathrm{Sz}(X)$ invariant under equivalent renormings,
- $\mathrm{Sz}(X)$ is monotone,
- for all $\alpha < \omega_1$ there is a reflexive X so that $\mathrm{Sz}(X) \geq \alpha$.

Bourgain 1980: actually every separable space which is universal for all reflexive spaces must contain $C[0, 1]$, and, thus, must contain **all** separable Banach spaces.

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Is there a separable reflexive space which is universal for all separable uniform convex (super reflexive) spaces?

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- **Pruss** 1980, yes for spaces with FDDs.
- **Odell & Sch** 2005. yes in general (proving embedding) actually there is a space X universal for all super reflexive spaces so that $\mathrm{Sz}(X) \vee \mathrm{Sz}(X^*) = \omega^2$.

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Is there a separable reflexive space which is universal for all separable uniform convex (super reflexive) spaces?

- **Pruss** 1980, yes for spaces with FDDs.
- **Odell & Sch** 2005. yes in general (proving embedding) actually there is a space X universal for all super reflexive spaces so that $\mathrm{Sz}(X) \vee \mathrm{Sz}(X^*) = \omega^2$.
- **Odell, Sch & Zsak** 2006: For any $\alpha < \omega_1$ there is a space X universal for the class of spaces Y with $\mathrm{Sz}(Y) \vee \mathrm{Sz}(Y^*) \leq \omega^{\alpha\omega}$.
  Moreover $\mathrm{Sz}(X) \vee \mathrm{Sz}(X^*) \leq \omega^{\alpha\omega+1}$ (and this is best possible)

Key observation: Let Z be a Banach space with FDD (E_i) and let

$$\mathcal{A} \subset S_Z^\omega = \{(z_i) : z_i \in S_Z\}$$

Define for $\overline{\varepsilon} = (\varepsilon_n) \subset (0,1)$ the $\overline{\varepsilon}$-fattening of $\mathcal{A}$

$$\mathcal{A}_{\overline{\varepsilon}} = \{(y_i) : \exists (z_i) \in \mathcal{A} \quad \|z_i - y_i\| < \varepsilon_i\}$$

Consider the following infinite asymptotic game :
Player I chooses $n_1 \in \mathbb{N}$, Player II: $z_1 \in S_Z \cap \oplus_{i=n_1}^{\infty} E_i$
Player I chooses $n_2 \in \mathbb{N}$, Player II: $z_12 \in S_Z \cap \oplus_{i=n_2}^{\infty} E_i$
...
Player I has won if $(z_i) \in \mathcal{A}$
Assume $\mathcal{A}$ is closed (Odell & Sch, 2002) or only co-analytic (Rosendal, 2006)
The following are equivalent:

- $\forall \overline{\varepsilon}$ Player 1 has a winning strategy for $\mathcal{A}_{\overline{\varepsilon}}$
- $\forall \overline{\varepsilon}$ there is a blocking (F_i) of (E_i) so that every skipped block basis (x_i) with respect to (F_i) is in $\mathcal{A}_{\overline{\varepsilon}}$.

Set theoretic approach to universality problems

Bossard 1995 considered the Effros Borel space of separable infinite Banach spaces

$$\mathrm{Subs} = \{X \subset C[0,1] : X \text{ is closed linear sub space}\},$$

and if $X \in \mathrm{Subs}$

$$\mathrm{Subs}(Y) = \{Y : Y \text{ is closed linear sub space}\},$$

- $\mathrm{Subs}(X)$ is a Borel subset in $\mathcal{F}(X) = \{F \subset X : F \text{ is closed}\}$ Thus $\mathrm{Subs}(X)$ is a standard Borel space.
- $\mathrm{Subs}_\infty(X) = \{X \in \mathrm{Subs} : \dim(X) = \infty\}$ is a Borel subset in $\mathcal{F}(X)$.
- The relation $\{(Y, X) : \ Y \text{ is a closed subspace of } X\}$ is Borel in $\mathrm{Subs} \times \mathrm{Subs}$.
- For $X \in \mathrm{Subs}$ the set $\{Y \in \mathrm{Subs} : Y \text{ embeds into } X\}$ is analytic.
- The isomorphism relation $\simeq$ is analytic but non Borel, i.e. the set $\{(X, Y) : X, Y \in \mathrm{Subs}, X \simeq Y\}$ is analytic in $\mathrm{Subs} \times \mathrm{Subs}$ but not Borel.

Bossard proved the following generalization of Bourgain's result

Theorem (Bossard, 2002)

If $\mathcal{A}$ is an analytic class in Subs which contains all reflexive spaces, then $\mathcal{A}$ contains a copy of $C[0,1]$.

The Connection of Brossard's approach to universality problems was observed by Argyros and Dodos.

Definition (Argyros & Dodos, 2005)

A class $\mathcal{C}$ of separable Banach spaces (identified with subsets of Subs) is said to be strongly bounded if for every analytic set $\mathcal{A} \subset \mathcal{C}$ (analytic in Subs) there is an element in $\mathcal{C}$ which is universal for $\mathcal{A}$.

Theorem (Argyros & Dodos, 2005)

The class of separable reflexive spaces with basis is strongly bounded.
The class of spaces with separable dual and basis is strongly bounded.

Theorem (Dodos & Ferenczi, 2007)

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The class of separable reflexive spaces are strongly bounded. The class of spaces with separable dual and basis is strongly bounded.

Corollary (Dodos & Ferenczi, 2007)

- *Another proof of Bourgain's conjecture.*
- *For every countable $\xi < \omega_1$ there exists a space X_ξ having countable Szlenk index which is universal for all the spaces with Szlenk index $\leq \xi$.*

Problems

- *How different are the two approaches? Really different or the same proof in two different languages?*
- *Is there a set theoretic approach to get the more quantitative results?*
- *Is there a set theoretical approach to get the universality result for*

$$\mathrm{Sz}(X) \vee \mathrm{Sz}(X^*)?$$