CONGRUENCES

$spt_2(n) \equiv 0 \pmod 5$ if $n \equiv 0, 1, 4 \pmod 5$

$spt_2(n) \equiv 0 \pmod 7$ if $n \equiv 0, 1, 5 \pmod 7$

$spt_2(n) \equiv 0 \pmod{11}$ if $n \equiv 0 \pmod{11}$

$spt_3(n) \equiv 0 \pmod 7$, if $n \not\equiv 3, 6 \pmod 7$

$spt_3(n) \equiv 0 \pmod 2$, if $n \equiv 1 \pmod 4$

$spt_4(n) \equiv 0 \pmod 3$, if $n \equiv 0 \pmod 3$

THEOREM (Bringmann, G., Mahlburg)

For any given prime $\ell > 3$ & any fixed k, j $\exists$ ∞ly arithmetic progressions

$$\eta_{2k}(An+B) \equiv 0 \pmod{\ell^j}$$

COROLLARY

Analog holds for $spt_k(n)$

THEOREM (ATKIN & G.)

There are exact relations between

$N_{2k}(n)$ and $N_2(n)$

and lower order crank moments

for $k =$

EXAMPLE

$$N_4(n) = -\frac{2}{3}(3n+1)M_2(n) + \frac{8}{3}M_4(n) + (1-12n)N_2(n)$$

(35)

$$\boxed{spt_2(11n) \equiv 0 \pmod{11}}$$

PROOF:

$$spt_2(n) = \mu_4(n) - \eta_4(n)$$

$$= \frac{1}{24}\left(M_4(n) - M_2(n) - N_4(n) + N_2(n)\right)$$

and

$$24\, spt_2(n) = \left(2n - \tfrac{1}{3}\right) M_2(n) - \tfrac{5}{3} M_4(n) + 12\, n\, N_2(n)$$

and

$$spt_2(n) \equiv M_4(n) + (n+9)\, M_2(n) + 6n\, N_2(n) \pmod{11}$$

$$(n+5)^3 M_4(n)$$
$$\equiv (5n^4 + 10n^3 + 8n^2 + 8n + 9)\, M_2(n) \pmod{11}$$

and $\quad M_2(n) = 2n\, p(n)$.

Hence

$$M_4(11n) \equiv M_2(11n) \equiv 0 \pmod{11}$$

and

$$spt_2(11n) \equiv 0 \pmod{11}$$

(37)

$$\boxed{spt_3(4n+1) \equiv 0 \pmod{2}}$$

PROOF:

$$spt_3(n) = \frac{-7}{7920} M_6(n)$$

$$+ \frac{1}{1584}(60n+13) M_4(n)$$

$$+ \frac{1}{3780}(7 - 78n - 108n^2) M_2(n)$$

$$- \frac{1}{20} n(1+3n) N_2(n)$$

Luckily $N_2(n) \equiv 0 \pmod{2}$

$$spt_3(4n+1) \equiv s_3(4n+1) \pmod 2$$

where

$$s_3(n) = \frac{-7}{7920} M_6(n) + \frac{1}{1584}(160n+13) M_4(n)$$
$$+ \frac{1}{3960}(7 - 78n - 108n^2) M_2(n)$$

Let $S_3 = \sum_{n=1}^{\infty} s_3(n) q^n$

Then

$$S_3 \in P \cdot \mathcal{W}_3$$

where $P = \frac{1}{(q)_\infty}$ & $\mathcal{W}_3$ is space of quasi-modular forms of wt ≤ 6.

$\dim(P \cdot \mathcal{W}_3) = 6$ &

&

$\delta_q(P)$, $\delta_q^2(P)$, $\delta_q^3(P)$, P_3, $\delta_q(P_3)$, P_5 form a basis where $\delta_q = q \frac{d}{dq}$

& $P_j = P(q) \sum_{n=1}^{\infty} \sigma_j(n) q^n$

we find that

$$s_3(n) = \frac{n}{270}(5 - 12n - 147n^2)p(n)$$
$$+ \frac{1}{12}(6n+1)p_3(n)$$
$$- \frac{7}{540} p_5(n) \quad \&$$

$$s_3(4n+1) \equiv p(4n+1) + p_3(4n+1)$$
$$(\bmod 2)$$

$$S_q(P) = \sum_{n=1}^{\infty} n\,p(n)\,q^n$$

$$= P \cdot \sum_{n=1}^{\infty} \sigma(n)\,q^n$$

$$\equiv P \cdot \sum_{n=1}^{\infty} \sigma_3(n)\,q^n \pmod 2$$

$$\equiv \sum_{n=1}^{\infty} p_3(n)\,q^n \pmod 2$$

$$n\,p(n) \equiv p_3(n) \pmod 2$$

$$p(4n+1) \equiv p_3(4n+1) \pmod 2$$

& $s_3(4n+1) \equiv p(4n+1) + p_3(4n+1)$

$$\equiv 0 \pmod 2. \quad \square$$

$$\boxed{spt_4(3n) \equiv 0 \pmod 3}$$ (4i)

PROOF:

Let

$$S_k(b,t) := \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n q^{n(kn+1)/2 + bn}}{(1-q^{tn})}$$

Then

$$\sum_{n=1}^{\infty} spt_4(n)\, q^n$$

$$\equiv \frac{1}{(q)_\infty}\Big(-S_1(4,9) + S_1(5,9) + S_3(4,9) - S_3(5,9)\Big)$$

$$\pmod 3$$

Let $N(r,t,n)$ = # of partitions of n with rank $\equiv r \pmod{t}$

Let $M(r,t,n)$ = # of partitions of n with crank $\equiv r \pmod{t}$.

Then

$$\sum_{n=0}^{\infty} N(r,t,n)\,q^n = \frac{1}{(q)_\infty}\left(S_3(r,t)+S_3(t-r,t)\right)$$

&

$$\sum_{n=0}^{\infty} M(r,t,n)\,q^n = \frac{1}{(q)_\infty}\left(S_1(r,t)+S_1(t-r,t)\right)$$

$$S_k(b,t) = -S_k(t-1-b,t)$$

Hence

$$S_1(4,9) = S_3(4,9) = 0.$$

and

$$spt_4(n) \equiv M(4,9,n) - N(4,9,n) \pmod 3$$

But

$$\boxed{M(4,9,3n) = N(4,9,3n)}$$

LEWIS

and

$$spt_4(3n) \equiv 0 \pmod 3. \quad \square$$

COROLLARY

$$
\begin{aligned}
\tau(n) \equiv\ & (588 + 297n + 258n^2 + 9n^3 + 108n^4 + 456n^5)\,\sigma(n) \\
& + (60 + 255n + 189n^2 + 612n^3 + 162n^4)\,\sigma_3(n) \\
& + (306 + 297n + 540n^2 + 180n^3)\,\sigma_5(n) \\
& + (177 + 576n + 454n^2)\,\sigma_7(n) \\
& + (201 + 690n)\,\sigma_9(n) \\
& + 117\,\sigma_{11}(n) \pmod{3^6}.
\end{aligned}
$$

Similar ASHWORTH (1966)