BIRANKS FOR PARTITIONS INTO TWO COLORS

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Dedicated to the Memory of
My Friend Richard Lewis (1962-2007)

PARITITIONS

Let $p(n) = \#$ of partitions of n

Let $E(q) = \prod_{n=1}^{\infty} (1-q^n)$ $\qquad (|q|<1)$.

Then
$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{E(q)}$$

MULTIPARTITION

Let $r \geq 1$. Let $\mathcal{P}$ = set of partitions.
A multipartition with r components (or an r-colored partition) of n is an r-tuple

$$\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_r) \in \mathcal{P} \times \mathcal{P} \times \cdots \times \mathcal{P} = \mathcal{P}^r$$

where
$$\sum_{k=1}^{r} |\pi_k| = n.$$

Let $p_{-r}(n)$ denote the number of r-colored partitions of n

Then
$$\sum_{n \geq 0} p_{-r}(n) q^n = E(q)^{-r} = \frac{1}{(E(q))^r}$$

BIPARTITIONS ($r=2$)

Bipartitions of $n=3$

3	$(3, -)$
$2+1$	$(2+1, -)$
$1+1+1$	$(1+1+1, -)$
$2+1$	$(2, 1)$
$1+1+1$	$(1+1, 1)$
$2+1$	$(1, 2)$
$1+1+1$	$(1, 1+1)$
3	$(-, 3)$
$2+1$	$(-, 2+1)$
$1+1+1$	$(-, 1+1+1)$

n	$p_{-2}(n)$
0	1
1	2
2	5
3	10
4	20
5	36
6	65
7	110
8	185
⋮	⋮
99	15560315348120

CONGRUENCES

RAMANUJAN

$$p(5n+4) \equiv 0 \pmod 5$$

$$p(7n+5) \equiv 0 \pmod 7$$

$$p(11n+6) \equiv 0 \pmod{11}$$

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod 5$$

Some Elementary Congruences (Gandhi?)

Let $t > 3$ be prime.

(i) If $\left(\frac{24n+1}{t}\right) = -1$, then $p_{1-t}(n) \equiv 0 \pmod t$

(ii) If $\left(\frac{8n+1}{t}\right) \neq 1$ then $p_{3-t}(n) \equiv 0 \pmod t$

PROOF of BIPARTITION CONGRUENCES MOD 5

$$\sum_{n\geq 0} p_{-2}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^n)^5}$$

$$\equiv \frac{\prod_{n=1}^{\infty} (1-q^n)^3}{\prod_{n=1}^{\infty} (1-q^{5n})} \pmod 5$$

$$= \frac{\sum_{n\geq 0} (-1)^n (2n+1) q^{n(n+1)/2}}{\prod_{n=1}^{\infty} (1-q^{5n})}$$

n	$n(n+1)/2 \pmod 5$
0	0
1	1
2	3
3	1
4	0

THE HAMMOND–LEWIS BIRANK (2004)

Define

$$\text{birank}(\pi_1, \pi_2) = \#(\pi_1) - \#(\pi_2)$$

THEOREM (Hammond & Lewis)
The residue of the birank mod 5 divides the partitions of $5n+4$ into 5 equal classes

Bipartitions of 3	birank (mod 5)
$(3, -)$	$1-0 \equiv 1$
$(2+1, -)$	$2-0 \equiv 2$
$(1+1+1, -)$	$3-0 \equiv 3$
$(2, 1)$	$1-1 \equiv 0$
$(1+1, 1)$	$2-1 \equiv 1$
$(1, 2)$	$1-1 \equiv 0$
$(1, 1+1)$	$1-2 \equiv 4$
$(-, 3)$	$0-1 \equiv 4$
$(-, 2+1)$	$0-2 \equiv 3$
$(-, 1+1+1)$	$0-3 \equiv 2$

Let $N_{HL}(r,t,n) = \#$ of bipartitions of n with birank $\equiv r \pmod t$.

$$N_{HL}(0,5,3) = N_{HL}(1,5,3) = N_{HL}(2,5,3) = N_{HL}(3,5,3) = N_{HL}(4,5,3) = 2$$

PROOF of HAMMOND-LEWIS THEOREM

<u>Notation:</u> $$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots = \prod_{n=1}^{\infty}(1-aq^{n-1})$$

$$(z;q)_\infty (qz^{-1};q)_\infty (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2}$$

$$\sum_{\lambda \in (\pi_1,\pi_2)} z^{\mathrm{birank}(\lambda)} q^{|\lambda|} = \frac{1}{(zq;q)_\infty (z^{-1}q;q)_\infty}$$

Let ζ be a primitive 5^{th} root of unity.

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{4} N_{HL}(k,5,n)\,\zeta^k\right) q^n = \frac{1}{(\zeta q;q)_\infty (\zeta^{-1}q;q)_\infty}$$

$$= \frac{(\zeta^2 q;q)_\infty (\zeta^{-2}q;q)_\infty (q;q)_\infty}{(q;q)_\infty (\zeta q;q)_\infty (\zeta^2 q;q)_\infty (\zeta^{-2}q;q)_\infty (\zeta^{-1}q;q)_\infty}$$

$$= \frac{1}{1-\zeta^2}\,\frac{\sum_{n} (-1)^n \zeta^{2n} q^{n(n-1)/2}}{(q^5;q^5)_\infty}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{4} N_{HL}(k,5,n)\,\zeta^{k}\right) q^{n}$$

$$= \frac{1}{1-\zeta^{2}} \frac{\sum_{n\geq 0} (-1)^{n} \zeta^{2n}\left(1-\zeta^{4(2n+1)}\right) q^{n(n+1)/2}}{\prod_{n=1}^{\infty}\left(1-q^{5n}\right)}$$

Coeff $q^{n} = 0$ when $n \equiv 2, 3, 4 \pmod 5$

$$\sum_{k=0}^{4} N_{HL}(k,5,n)\,\zeta^{k} = 0 \quad \text{when } n \equiv 2,3,4 \pmod 5$$

$$N_{HL}(0,5,n) = N_{NH}(1,5,n) = \cdots = N_{NH}(4,5,n)$$

for $n \equiv 2, 3, 4 \pmod 5$

$$\Downarrow$$

HAMMOND LEWIS BIRANK THEOREM

FIRST ANALOGUE – THE DYSON BIRANK

The DYSON RANK of a partition is the largest part minus the number of parts.

THEOREM (Atkin & SND 1953)

* The residue of the Dyson rank mod 5 divides the partitions of $5n+4$ into 5 equal classes
* The residue of the Dyson rank mod 7 divides the partitions of $7n+5$ into 7 equal classes.

The DYSON-BIRANK of a bipartition is defined by

$$\text{Dyson-Birank}(\pi_1, \pi_2) = \text{rank}(\pi_1) + 2\,\text{rank}(\pi_2)$$

THEOREM (G.) (Conj. by Hammond)

The DYSON-BIRANK mod 5 divides the bipartitions of n into FIVE equal classes when $n \equiv 2$ or $4 \pmod 5$.

EXAMPLE ($n=2$)

BIPTNS of 2	DYSON-BIRANK mod 5
$(2, -)$	$1+0 \equiv 1 \pmod 5$
$(1+1, -)$	$-1+0 \equiv 4$
$(1, 1)$	$0+0 \equiv 0$
$(-, 2)$	$0+2 \equiv 2$
$(-, 1+1)$	$0+(-2) \equiv 3$

$$N_D(0,5,2) = N_D(1,5,2) = N_D(2,5,2)$$
$$= N_D(3,5,2) = N_D(4,5,2) = 1$$

Gen-Func-RANK

$$f(z,q) = \sum_{\pi} z^{\text{rank}} q^{|\pi|}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n}$$

RAMANUJAN:

$$(*)\ f(\zeta,q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\zeta q;q)_n (\zeta^{-1} q;q)_n}$$

$$= \left(A(q^5) - (\zeta + \zeta^2 + \zeta^3)\,\phi(q^5)\right)$$
$$+\ q B(q^5) \qquad + q^2 (\zeta + \zeta^4)\, C(q^5)$$
$$+\ q^3 \big((1 + \zeta^2 + \zeta^3)\, D(q^5) + (1 + 2\zeta^2 + 2\zeta^3)\, \phi(q^5)$$

where $A(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n})}{(1-q^{5n-1})^2 (1-q^{5n-4})^2}$

$B(q) = \text{. . . . .}$

DYSON BIRANK GEN FUNC

$$\sum_{\pi=(\pi_1,\pi_2)} z^{\text{DYSON-BIRANK}(\pi)} q^{|\pi|}$$

$$= f(z,q)\, f(z^2,q)$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{4} N_D(k,5,n)\, \zeta^k\right) q^n$$

$$= f(\zeta,q)\, f(\zeta^2,q)$$

$\boxed{\text{DYSON-BIRANK THM}} \Longleftrightarrow$ Coeff of $q^n = 0$ for $n \equiv 2, 4 \pmod 5$

RAMANUJAN'S IDENTITY for $f(\zeta, q)$

$+$

$B^2(q) = A(q)\,C(q)$ & $C^2(q) = B(q)\,D(q)$

SECOND ANALOG — THE 5-CORE-BIRANK

t-RESIDUE DIAGRAM ($t=5$) of a PTN

Cell (i,j) is labelled $j-i \pmod t$

$4+3+3+3+1+1$

0	1	2	3
4	0	1	
3	4	0	
2	3	4	
1			
0			

For $0 \leq k \leq t-1$, let

$r_k = r_k(\pi) = \#$ of cells labelled k in the t-residue diagram

$r_0 = 4$
$r_1 = 3$
$r_2 = 2$
$r_3 = ?$
$r_4 = ?$

THEOREM (G-KIM-STANTON (1990))

Define

$$\text{5-CORE-CRANK}(\pi) := r_1 + 2r_2 - 2r_3 - r_4$$

Then the 5-CORE-CRANK (π) (mod 5) divides the partitions of $5n+4$ into FIVE equal classes.

[illegible]

THE 5-CORE-BIRANK.

$$\text{5-core-birank}(\pi_1, \pi_2)$$
$$:= \text{5-core-crank}(\pi_1) + 2 * (\text{5corecrank}(\pi_2))$$

THEOREM (G)

The residue of the 5-core-birank (mod 5) divides the bipartitions of n into FIVE equal classes when $n \equiv 2, 3, \text{ or } 4 \pmod 5$.

$$(1+1,1) = \left(\begin{array}{|c|}\hline 0 \\ \hline 4 \\ \hline \end{array} \, , \; \begin{array}{|c|}\hline 0 \\ \hline \end{array} \right)$$

$$5\text{-core-crank}\left(\begin{array}{|c|}\hline 0 \\ \hline 4 \\ \hline \end{array} \right) = -1$$

$$5\text{-core-crank}\left(\begin{array}{|c|}\hline 0 \\ \hline \end{array} \right) = 0$$

$$5\text{-core-birank}\,(1+1,1) = -1+0 = -1.$$

BIPTNS of 3	5-CORE-BIRANK (mod 5)
$(3,-)$	$3+0 \equiv 3$
$(2+1,-)$	$0+0 \equiv 0$
$(1+1+1,-)$	$-3+0 \equiv 2$
$(2,1)$	$1+0 \equiv 1$
$(1+1,1)$	$-1+0 \equiv 4$
$(1,2)$	$0+2 \equiv 2$
$(1,1+1)$	$0-2 \equiv 3$
$(-,3)$	$0+6 \equiv 1$
$(-,2+1)$	$0+0 \equiv 0$
$(-,1+1+1)$	$0-6 \equiv 4$

$$N_{5c}(0,5,3) = N_{5c}(1,5,3) = \ldots = N_{5c}(4,5,3) = 2$$

5-CORE CRANK GEN. FUNC.

(G- KIM- STANTON, 1990)

$$\Phi(z,q) := \sum_{\pi} z^{\text{5-CORE-CRANK}(\pi)} q^{|\pi|}$$

$$= T(z,q) \times \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})^5}$$

where

$$T(z,q) = \sum_{\pi \text{ is 5-core}} z^{\text{5-CORE-CRANK}(\pi)} q^{|\pi|}$$

$$= \sum_{\substack{\vec{n} \in \mathbb{Z}^5 \\ n_0+n_1+n_2+n_3+n_4=0}} z^{n_1+3n_2+n_3} q^{\frac{5}{2}||\vec{n}||^2 + \vec{b}\cdot\vec{n}}$$

where $\vec{b} = (0,1,2,3,4)$

5-DISSECTION of $\Phi(\zeta, q)$

$$T(\zeta, q) = \sum_{\pi \text{ is a 5-core}} \zeta^{\text{5-CORE-CRANK}(\pi)} q^{|\pi|}$$

$$= W(q^5)\Big(1 + q\,R(q^5) + q^2(\zeta + \zeta^{-1})\,R(q^5)^2 - q^3(\zeta + \zeta^{-1})\,R(q^5)^3\Big)$$

where

$$W(q) := J_{2,5}(q)^3\big(J_{10,25}(q) - q\,(\zeta + \zeta^{-1})\,J_{5,25}(q)\big)$$

$$R(q) = \frac{J_{1,5}(q)}{J_{2,5}(q)}$$

$$J_{a,b}(q) = \prod_{n=1}^{\infty}(1 - q^{bn-a})(1 - q^{bn+a-b})(1 - q^{bn})$$

> **NOTE:** Coefficient of q^{5n+4} in the q-expansion of $T(5, q)$ is 0.

1990 G-K-S

5-CORE BIRANK GEN. FUNC.

$$\sum_{\pi=(\pi_1,\pi_2)} z^{\text{5-CORE-BIRANK}(\pi)} q^{|\pi|} = \Phi(z,q)\,\Phi(z^2,q)$$

$$= T(z,q)\,T(z^2,q)\,\frac{1}{\prod_{n=1}^{\infty}(1-q^{5n})^{10}}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{4} N_{5C}(k,5,n)\,\zeta^k\right) q^n$$

$$= \frac{1}{E^{10}(q^5)}\,T(\zeta,q)\,T(\zeta^2,q)$$

$$= \frac{1}{E^{10}(q^5)}\,W^2(q^5)\Big(1 + 2q^5R^5(q^5) + q\,R(q^5)\big(2 - q^5R^5(q^5)\big)\Big)$$

$\boxed{\text{THE 5-CORE BIRANK THEOREM}}$

A THETA-FUNCTION IDENTITY

$$U(z,q) := \sum_{n\in\mathbb{Z}^5} z^{\vec{n}\cdot\vec{1}}\, \zeta^{n_1+3n_2+n_3}\, q^{5\|\vec{n}\|^2+2\vec{b}\cdot\vec{n}}$$

$$= \Theta(z\zeta^4, q^5)\,\Theta(zq^2\zeta, q^5)\,\Theta(zq^4, q^5)\,\Theta(z\zeta q^6, q^5)\,\Theta(z\zeta^3 q^8, q^5)$$

$$= F_0(q)\,S_0(z,q) + F_1(q)\,S_1(z,q) + F_2(q)\,S_2(z,q)$$

$$+ F_3(q)\,S_3(z,q) + F_4(q)\,S_4(z,q)$$

where

$$F_0(q) = W(q^{10})\left(1 + q^2R(q^{10}) + (\zeta^2+\zeta^3)q^4R(q^{10})^2 - (\zeta^2+\zeta^3)q^6R(q^{10})^3\right)$$

$$F_1(q) = W(q^{10})\left(\zeta^4 + q^2R(q^{10}) + (1+\zeta)q^4R^2(q^{10}) - (\zeta^2+\zeta^3)q^6R^3(q^{10})\right)$$

$$F_2(q) = W(q^{10})\left(1 + \zeta^4q^2R(q^{10}) + (1+\zeta)q^4R^2(q^{10}) - (1+\zeta^4)q^6R^3(q^{10})\right)$$

$$F_3(q) = W(q^{10})\left(1 + q^2R(q^{10}) + (1+\zeta^4)q^4R^2(q^{10}) - (1+\zeta)q^6R^3(q^{10})\right)$$

$$F_4(q) = W(q^{10})\left(\zeta + \zeta^4q^2R(q^{10}) + (1+\zeta^4)q^4R^2(q^{10}) - (\zeta^2+\zeta^3)R^3(q^{10})\right)$$

$$S_j(z,q) = \sum_{n=-\infty}^{\infty} z^{5n+j}\, q^{25n^2+(20+j)n+\cdots} \qquad 0\le j\le 4$$

- BOTH SIDES of IDENTITY satisfy
$$V(zq^{10},q) = -z^{-5}q^{-45}V(z,q)$$

- SUFFICES TO SHOW BOTH SIDES AGREE FOR SIX DISTINCT VALUES of z with
$$|q|^{10} < |z| \leqslant 10$$

- ROUTINE TO SHOW BOTH SIDES $= 0$ for
$$z = -q^5\zeta,\ -q^3\zeta^4,\ -q,\ -\zeta^4 q^9,\ -\zeta q^7$$

- USE THE THEORY OF MODULAR FUNCTIONS TO SHOW IDENTITY HOLDS FOR $z=1$

- REDUCES TO VERIFYING 5 IDENTITIES FOR MODULAR FUNCTIONS on $\Gamma_1(50)$

- THE FIRST CASE INVOLVES VERIFYING A SUM OF 135 EXPLICIT THETA PRODUCTS $= \mathcal{U}_{05}(U(-1,q))$

involves computing order at all cusps
and showing identity holds up to [illegible] of q^{n}

ANDREWS BICRANK

For a partition π the CRANK of a partition is the largest part if the partition has no ones and difference between the number of parts > than the number of ones, and the number of ones.

THEOREM (Andrews & G. 1988)

* The residue of the crank mod 5 divides the partitions of $5n+4$ into 5 equal classes

* The residue of the crank mod 7 divides the partitions of $7n+5$ into 7 equal classes

* The residue of the crank mod 11 divides the partitions of $11n+6$ into 11 equal classes.

Let $M(m,n)$ denote the number of partitions of n with crank m.
then

$$\sum_{n\geq 0}\sum_{m} M(m,n) z^m q^n = (1-z)q + \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

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Define $M'(m,n)$ by

$$\sum_{n \geq 0} \sum_{m} M'(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

EXTENDED PARTITIONS

$$\mathcal{E} = \{ (-),\ 1_a,\ 1_b,\ 1,\ 2,\ 1+1,\ 3,\ 2+1,\ 1+1+1, \ldots \}$$

$$|(-)| = 0$$

$$|1_a| = |1_b| = |1| = 1$$

$$\omega(\pi) = \begin{cases} -1 & \text{if } \pi = 1_b \\ |\pi| & \text{if } \pi \neq 1_b \end{cases}$$

Then

$$p(n) = \sum_{\substack{\pi \in \mathcal{E} \\ |\pi| = n}} \omega(\pi)$$

CRANK of AN EXTENDED PARTITION

$$\text{crank}(\pi) = \begin{cases} 0 & \text{if } \pi = (-) \\ 1 & \text{if } \pi = 1_a \\ -1 & \text{if } \pi = 1_b \\ [illegible] & \text{otherwise} \end{cases}$$

Then

$$F(z,q) = \sum_{\pi \in \mathcal{E}} \omega(\pi)\, z^{\mathrm{crank}(\pi)}\, q^{|\pi|}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

<u>BICRANK FOR TWO-COLORED EXTENDED PARTITIONS</u>

For $\pi = (\pi_1, \pi_2) \in \mathcal{E} \times \mathcal{E}$
define

$$|\pi| = |\pi_1| + |\pi_2|$$

$$\omega(\pi) = \omega(\pi_1) \cdot \omega(\pi_2)$$

(Andrews) $\mathrm{BIGRANK\,I}(\pi) := \mathrm{CRANK}(\pi_1) + \mathrm{CRANK}(\pi_2)$

(G.) $\mathrm{BICRANK\,II}(\pi) := \mathrm{CRANK}(\pi_1) + 2 * \mathrm{CRANK}(\pi_2)$

THEOREM (Andrews (2008), G (2010))

* THE RESIDUE of BICRANK I divides the extended bipartitions of n into 5 classes of equal weight if $n \equiv 3 \pmod 5$

* THE RESIDUE of BICRANK II divides the extended bipartitions of n into 5 classes of equal weight if $n \equiv 2$ or $4 \pmod 5$.

EXAMPLE ($n=3$)

There are 18 extended bipartitions of 3 with total weight $p_{-2}(3) = 10$

EXTENDED BIPTNS	BICRANK I (mod 5)	WEIGHT
$(2+1, -)$	$0+0 \equiv 0$	1
$(-, 2+1)$	$0+0 \equiv 0$	1
$(2, 1)$	$2-1 \equiv 1$	1
$(1, 2)$	$-1+2 \equiv 1$	1
$(1+1+1, -)$	$-3+0 \equiv 2$	1
$(2, 1_b)$	$2+0 \equiv 2$	-1
$(1+1, 1)$	$-2-1 \equiv 2$	1
$(1_b, 2)$	$0+2 \equiv 2$	-1
$(1, 1+1)$	$-1-2 \equiv 2$	1
$(-, 1+1+1)$	$0-3 \equiv 2$	1
$(3, -)$	$3+0 \equiv 3$	1
$(2, 1_a)$	$2+1 \equiv 3$	$+1$
$(1+1, 1_a)$	$-2+0 \equiv 3$	-1
$(1_a, 2)$	$1+2 \equiv 3$	1
$(1_b, 1+1)$	$0-2 \equiv 3$	-1
$(-, 3)$	$0+3 \equiv 3$	1
$(1+1, 1_a)$	$-2+1 \equiv 4$	1
$(1_a, 1+1)$	$1-2 \equiv 4$	1

CRANK GEN. FUNC.

$$F(z,q) = \sum_{\pi \in \mathcal{E}} z^{\mathrm{crank}(\pi)} q^{|\pi|}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

RAMANUJAN

$$* \quad F(\text{[illegible]}, q) = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-\text{[illegible]}\,q^n)(1-\text{[illegible]}\,q^n)}$$

$$= A(q^5) - q(\text{[illegible]})^2 B(q^5)$$
$$+ q^2(\text{[illegible]}) C(q^5) - q^3(\text{[illegible]}) D(q^5)$$

BICRANK GEN. FUNC.

Let $k = 1$ or 2

$$\sum_{\pi \in \mathcal{E}\times\mathcal{E}} z^{\mathrm{BICRANK}(k)(\pi)} q^{|\pi|} \omega(\pi)$$

$$= F(z, q)\, F(z^k, q)$$

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{4} M_k(m, 5, n)\, \zeta^m \right) q^n$$

$$= F(\zeta, q)\, F(\zeta^k, q)$$

$\boxed{\text{BICRANK THM}}$ $\Longleftrightarrow$ $\boxed{\begin{array}{l} \text{coeff of } q^n = 0 \\ \text{for } n \equiv 3 \pmod 5 \quad k = 1 \\ \& \; n \equiv 2, 4 \pmod 5 \quad k = 2 \end{array}}$

$\boxed{\begin{array}{c} \text{RAMANUJAN'S IDENTITY for } F(\zeta, q) \\ + \\ B^2(q) = A(q)\,C(q) \;\&\; C^2(q) = B(q)\,D(q) \end{array}}$

MULTIRANKS & MULTICRANKS

Let $\mathcal{P}$ = set of partitions

<u>generalised Hammond-Lewis multirank:</u>

$$\text{gHL multirank}(\pi) := \sum_{k=1}^{r/2} k\left(\#(\pi_k) - \#(\pi_{r+1-k})\right)$$

where r is even &

$$\pi \in \mathcal{P}^r = \mathcal{P} \times \mathcal{P} \times \cdots \times \mathcal{P}$$

<u>THEOREM</u>: Let $t > 3$ be prime.

* The residue of the gHL multirank mod t divides the multipartitions of n with $r = t-1$ components into t equal classes provided $24n+1$ is a q.n.r. mod t

* . . . . $r = t-3$
. . . .
. . . .
. . . $8n+1$ is NOT a q.r. (mod t)

<u>EXAMPLE</u> ($t=7$, $r=4$, $n=2$) 27

multiptns of 2 with 4 comp.	gtl2multirank (mod 7)
(–, –, –, 1+1)	$-2 \equiv 5$
(–, –, –, 2)	$-1 \equiv 6$
(–, –, 1, 1)	$-3 \equiv 4$
(–, –, 1+1, –)	$-4 \equiv 3$
(–, –, 2, –)	$-2 \equiv 5$
(–, 1, –, 1)	$1 \equiv 1$
(–, 1, 1, –)	$0 \equiv 0$
(–, 1+1, –, –)	$4 \equiv 4$
(–, 2, –, –)	$2 \equiv 2$
(1, –, –, 1)	$0 \equiv 0$
(1, 1, –, –)	$3 \equiv 3$
(1+1, –, –, –)	$2 \equiv 2$
(2, –, –, –)	$1 \equiv 1$
(1, –, 1, –)	$-1 \equiv 6$

gtl2multirank$(\pi_1, \pi_2, \pi_3, \pi_4)$

$$= \#(\pi_1) - \#(\pi_4)$$

$$+ 2 * (\#(\pi_2) - \#(\pi_3)).$$

MULTICRANKS

MULTICRANK I

Let $\pi \in \mathcal{E}^{r/2} \times \mathcal{P}^{r/2}$ (r even)

$$\mathrm{multiCRANKI}(\pi) := \sum_{k=1}^{r/2} k * \mathrm{crank}(\pi_k)$$

MULTICRANK II

For $\pi \in \mathcal{E} \times \mathcal{E} \times \mathcal{P}^{r-2}$

define

$$\mathrm{multiCRANKII}(\pi) := \mathrm{crank}(\pi_1) + 2 * \mathrm{crank}(\pi_2) + \sum_{k=3}^{r} k * (\#(\pi_k) - \#(\pi_{r-k+3}))$$

$$|\pi| := \sum |\pi_k|$$

$$\omega(\pi) := \prod \omega(\pi_k)$$

<u>THEOREM</u> Let $t > 3$ be prime

* The residue of multicrank I mod t divides the extended multipartitions of n from $\mathcal{E}^{\frac{1}{2}(t-1)} \times \mathcal{P}^{\frac{1}{2}(t-1)}$ into t classes of equal weight provided $24n+1$ is a q.n.r. mod t

* The residue of the multicrank II mod t divides the extended multipartition of n from $\mathcal{E}^{\frac{1}{2}(t-3)} \times \mathcal{P}^{\frac{1}{2}(t-3)}$ into t ~~equal~~ classes of equal weight provided $8n+1$ is not a q.r. mod t.

* The residue of the multicrank II mod t divides the extended multipartitions of n from $\mathcal{E}^{2} \times \mathcal{P}^{t-5}$ into t ~~equal~~ classes of equal weight provided $8n+1$ is ~~[illegible]~~ q.n.r (mod t).