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Shiftless Partition Identities and More Shifted Partition Identities

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Let S and T be sets of positive integers. Let a be a fixed positive identity. A *shifted partition identity* has the form

$$p(S, n) = p(T, n - a), \qquad \text{for all } n \geq a.$$

First we consider the case

$$a = 1$$

Assume $a = 1$. If S or T is finite then it is not hard to show that $S = T = \{1\}$.

$$p(\{1\}, n) = 1$$

Andrews [1987] found the following two non-trivial examples:

$$\begin{aligned} S &= \{n \mid n \text{ odd or} \\ &\qquad n \equiv \pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}\}, \\ T &= \{n \mid n \text{ odd or} \\ &\qquad n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \pmod{32}\}; \end{aligned}$$

and

$$\begin{aligned} T &= \{n \mid n \equiv \pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 10, \pm 11, \pm 14, \\ &\qquad \pm 15, \pm 16, \pm 17, \pm 19 \pmod{40}\}, \\ S &= \{n \mid n \equiv \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \\ &\qquad \pm 13, \pm 15, \pm 16, \pm 19 \pmod{40}\}. \end{aligned}$$

Proof of Andrews mod 32 and 40 shifted theorems

J.T.P:

$$
\begin{aligned}
T(z;q) &:= \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2} \\
&= \prod_{n=1}^{\infty} (1 - zq^{n-1})(1 - z^{-1}q^n)(1 - q^n) \\
&= \sum_{n \text{ even}} + \sum_{n \text{ odd}} \\
&= T(-z^2q; q^4) - zT(-z^{-2}q; q^4)
\end{aligned}
$$

$$
\boxed{\frac{T(-z^2q;q^4)}{T(z;q)} - z\frac{T(-z^{-2}q;q^4)}{T(z;q)} = 1}
$$

$\boxed{z = q, \qquad q \to q^4}$ $\to$ shifted mod 32

$\boxed{z = q, \qquad q \to q^5}$ $\to$ shifted mod 40

In these examples, each S and T is the union of arithmetic progressions modulo M for some M; namely $M = 32$ and $M = 40$. In fact, each is a union of 24 such arithmetic progressions.

Later, Kalvade [1989] found five more identities with $M = 42$, 48 and 60, each also involving the union of 24 arithmetic progressions.

In the present paper we find a further 48 identities with $M = 40$, 42, 46, 48, 54, 56, 60, 62, 66, 70 and 72. 46 of these involve unions of 24 arithmetic progressions, the remaining two involve 48 arithmetic progressions.

Modulus M	Number of Shifted Partition Identities
32	1
40	$1+1=2$
42	$1+2+1+2=6$
46	$0+4=4$
48	$1+7=8$
54	$0+11=11$
56	$0+2=2$
60	$2+10=12$
62	$0+2=2$
66	$0+1=1$
70	$0+1=1$
72	$0+5=5$

$M = 32$	1 id	[Andrews 1987]
$M = 40$	1 id	[Andrews 1987]
	1 id	[G]
$M = 42$	1 id	[Kalvade 1989] QP
	1 id	[G] QP
	2 ids	conj [Kalvade 1989]; [G]
	2 ids	[G]
$M = 46$	4 ids	[G]
$M = 48$	1 id	[Kalvade 1989] QP
	7 ids	[G]
$M = 54$	11 ids	[G]
$M = 60$	2 ids	[Kalvade 1989] Mac-BC_2
	10 ids	[G]
$M = 62$	2 ids	[G]
$M = 66$	1 ids	[G]
$M = 70$	1 ids	[G]
$M = 72$	5 conjs	[G]

Equivalent Problem

We want sets of positive integers S and T so that

$$\prod_{n \in S} \frac{1}{(1-q^n)} - q \prod_{n \in T} \frac{1}{(1-q^n)} = 1,$$

or

$$\prod_{n \in T \setminus S} (1-q^n) - q \prod_{n \in S \setminus T} (1-q^n) = \prod_{n \in S \cup T} (1-q^n).$$

Example 1.

$M = 40$ [Andrews]

$$\begin{aligned} T \setminus S &= \{n : n \equiv \pm 3, 14, 17 \pmod{40}\} \\ S \setminus T &= \{n : n \equiv \pm 6, 7, 13 \pmod{40}\} \end{aligned}$$

Example 2.

$M = 40$ [G]

$$\begin{aligned} T \setminus S &= \{n : n \equiv \pm 2, 9, 11, 12 \pmod{40}\} \\ S \setminus T &= \{n : n \equiv \pm 4, 6, 7, 13 \pmod{40}\} \end{aligned}$$

$$\begin{aligned} T &= \{n \mid n \equiv \pm 1, 2, 3, 5, 9, 11, 12, 15, \\ &\qquad 16, 17, 18, 19 \pmod{40}\}, \\ S &= \{n \mid n \equiv \pm 1, 3, 4, 5, 6, 7, 13, 15, \\ &\qquad 16, 17, 18, 19 \pmod{40}\}. \end{aligned}$$

Example 3.

$M = 60$ [Kalvade]

$$\begin{aligned} T \setminus S &= \{n : n \equiv \pm 5, 16, 25 \pmod{60}\} \\ S \setminus T &= \{n : n \equiv \pm 8, 11, 19 \pmod{60}\} \end{aligned}$$

Example 4.

$M = 70$ [G]

$$\begin{aligned} T \setminus S &= \{n : n \equiv \pm 2, 8, 12, 18, 21, 22, 32 \pmod{70}\} \\ S \setminus T &= \{n : n \equiv \pm 4, 6, 7, 16, 24, 26, 34 \pmod{70}\} \end{aligned}$$

$$\begin{aligned} S &= \{n : n \equiv \pm 1, 3, 4, 5, 6, 7, 9, 11, \\ &\qquad 13, 14, 15, 16, 17, 19, 23, 24, \\ &\qquad 25, 26, 27, 28, 29, 31, 33, 34 \pmod{70}\} \\ T &= \{n : n \equiv \pm 1, 2, 3, 5, 8, 9, 11, 12, \\ &\qquad 13, 14, 15, 17, 18, 19, 21, 22, \\ &\qquad 23, 25, 27, 28, 29, 31, 32, 33 \pmod{70}\} \end{aligned}$$

Truncated ST-pairs

A pair of sets $[S,T]$ is called a **truncated ST-pair** $O(q^N)$ if

$$\prod_{n\in S}\frac{1}{(1-q^n)} - q\prod_{n\in T}\frac{1}{(1-q^n)} = 1 + \mathrm{O}(q^N),$$

$$S \subset \{1,2,3,\ldots,N-1\},$$

and

$$T \subset \{1,2,3,\ldots,N-2\}.$$

Truncated ST-pairs $O(q^N)$

$N=3$	$[\{1\},\ \{1\}]$
$N=4$	$[\{1,3\},\ \{1,2\}]$ $[\{1\},\ \{1\}]$

$N=5$	$[\{1,3,4\},\ \{1,2,3\}]$ $[\{1,3\},\ \{1,2\}]$ $[\{1,4\},\ \{1,3\}]$ $[\{1\},\ \{1\}]$
$N=6$	$[\{1,3,4,5\},\ \{1,2,3\}]$ $[\{1,3,5\},\ \{1,2\}]$ $[\{1,4,5\},\ \{1,3,4\}]$ $[\{1,4\},\ \{1,3\}]$ $[\{1,5\},\ \{1,4\}]$ $[\{1\},\ \{1\}]$
$N=7$	$[\{1,3,4,5,6\},\ \{1,2,3,5\}]$ $[\{1,3,4,5\},\ \{1,2,3\}]$ $[\{1,3,5\},\ \{1,2,5\}]$ $[\{1,4,5,6\},\ \{1,3,4,5\}]$ $[\{1,4,5\},\ \{1,3,4\}]$ $[\{1,4,6\},\ \{1,3,5\}]$ $[\{1,4\},\ \{1,3\}]$ $[\{1,5,6\},\ \{1,4,5\}]$ $[\{1,5\},\ \{1,4\}]$ $[\{1,6\},\ \{1,5\}]$ $[\{1\},\ \{1\}]$

$$N = 8 \quad \begin{array}{l} [\{1,3,4,5,6,7\},\ \ \{1,2,3,5\}] \\ [\{1,3,4,5,7\},\ \ \{1,2,3\}] \\ [\{1,3,5,7\},\ \ \{1,2,5\}] \\ [\{1,4,5,6,7\},\ \ \{1,3,4,5\}] \\ [\{1,4,5,7\},\ \ \{1,3,4\}] \\ [\{1,4,6,7\},\ \ \{1,3,5\}] \\ [\{1,4,7\},\ \ \{1,3\}] \\ [\{1,5,6,7\},\ \ \{1,4,5,6\}] \\ [\{1,5,6\},\ \ \{1,4,5\}] \\ [\{1,5,7\},\ \ \{1,4,6\}] \\ [\{1,5\},\ \ \{1,4\}] \\ [\{1,6,7\},\ \ \{1,5,6\}] \\ [\{1,6\},\ \ \{1,5\}] \\ [\{1,7\},\ \ \{1,6\}] \\ [\{1\},\ \ \{1\}] \end{array}$$

The Number of Truncated ST-pairs $O(q^N)$

Let $T(n)$ denote the number of truncated ST-pairs $O(q^N)$.

n	$T(n)$
3	1
4	2
5	4
6	6
7	11
8	15
9	26
10	41
11	67
12	96
13	138
14	197
15	300
16	431
17	636
18	893
19	1258
20	1723
21	2447
22	3425
23	4962
24	6839
25	10000
26	13989
27	21383
28	30781
29	48292
30	70456

Conjecture

$$T(n) \sim c_1 e^{c_2 n}$$

Shiftless Identities

Let S and T be two distinct sets of positive integers. Let a be a fixed positive identity. A *shiftless partition identity* has the form

$$p(S,n) = p(T,n), \qquad \text{for all } n \neq a.$$

Example 1.
$M = 40$ [G]

$$\begin{aligned} S &= \{n \,|\, n \equiv \pm 1, 2, 5, 6, 7, 8, 9, 11, \\ &\qquad\qquad 12, 13, 15, 19 \pmod{40}\} \\ T &= \{n \,|\, n \equiv \pm 1, 3, 4, 5, 6, 7, 8, 13, \\ &\qquad\qquad 14, 15, 17, 19 \pmod{40}\} \end{aligned}$$

Then

$$p(S,n) = p(T,n), \qquad \text{for all } n \neq 2.$$

Other shiftless identities exist for the moduli $M = 42$, 46, 48, 54, 56, 60, 66 and 72.

Example 2.

$M = 48$ [G]

$$\begin{aligned} S &= \{n \mid n \equiv \pm 1, 4, 6, 7, 9, 10, 11, 13 \\ &\qquad\qquad 15, 17, 20, 23 \pmod{48}\} \\ T &= \{n \mid n \equiv \pm 2, 3, 4, 5, 7, 11, 13, 17, \\ &\qquad\qquad 18, 19, 20, 21 \pmod{48}\} \end{aligned}$$

Then

$$p(S, n) = p(T, n), \qquad \text{for all } n \neq 1.$$

Example 3.

$M = 54$ [G]

$$\begin{aligned} S &= \{n \mid n \equiv \pm 1, 2, 3, 4, 5, 7, 9, 11, \\ &\qquad\qquad 16, 17, 24, 25 \pmod{54}\} \\ T &= \{n \mid n \equiv \pm 1, 2, 3, 4, 5, 7, 9, 12, \\ &\qquad\qquad 13, 20, 23, 25 \pmod{54}\} \end{aligned}$$

Then

$$p(S, n) = p(T, n), \qquad \text{for all } n \neq 11.$$

Modular Forms & Theta Products

Let $q = \exp(2\pi i\tau)$. Let n, ρ be integers, $n \geq 1$, $\rho \nmid n$.

$$\theta_{\rho;n}(\tau) := \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{8n}(2nm+2\rho-n)^2}$$
$$= q^{\frac{1}{8n}(n-2\rho)^2} \prod_{m=1}^{\infty} (1-q^{nm-\rho})(1-q^{nm-(n-\rho)}) (1-q^{nm})$$

(by J.T.P.)

Then

$$\theta_{\rho+n;n} = \theta_{-\rho;n} = \theta_{\rho;n}$$

Let $A \in \Gamma_0(n) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{n}\}$. Then

$$\theta_{\rho;n}(A\tau) = \theta_{\rho;n}((a\tau+b)/(c\tau+d))$$
$$= (-1)^{b\rho+\lfloor \rho a/n \rfloor + \lfloor \rho/n \rfloor} \exp(\rho^2 \pi i ab/n)$$
$$\nu_1(A)\sqrt{c\tau+d}\,\theta_{a\rho;n}(\tau),$$

where $\nu_1(A)$ is a certain 8th root of unity.

The Dedekind eta function is defined by

$$\begin{aligned}\eta(\tau) &:= \theta_{1;3}(\tau) = \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{24}(6m-1)^2} \\ &= q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)\end{aligned}$$

For a list of integers $\boldsymbol{\rho} = [\rho_1, \rho_2, \ldots, \rho_k]$, where each $\rho_j \nmid n$, define

$$\theta_{\boldsymbol{\rho};n}(\tau) := \prod_{j=1}^{k} \theta_{\rho_j;n}(\tau)$$

$$L(\boldsymbol{\rho}) := \sum_j \rho_j$$

$$M(\boldsymbol{\rho}, x) := \sum_j \lfloor \rho_j x/n \rfloor,$$

for $(x, n) = 1$.

$$Q(\boldsymbol{\rho}) := \sum_j \rho_j^2$$

Then for $A \in \Gamma_0(n)$,

$$\theta_{\boldsymbol{\rho};n}(A\tau) = (-1)^{L(\boldsymbol{\rho})+M(\boldsymbol{\rho},a)} \exp(Q(\boldsymbol{\rho})\pi i ab/n) \\ \nu_1^k(A)\,(c\tau+d)^{k/2}\,\theta_{a\boldsymbol{\rho};n}(\tau),$$

where $a\boldsymbol{\rho} = [a\rho_1, \ldots, a\rho_k]$.

Example

$M = 40$ [G]

$$\begin{aligned} T \setminus S &= \{n : n \equiv \pm 2, 9, 11, 12 \pmod{40}\} \\ S \setminus T &= \{n : n \equiv \pm 4, 6, 7, 13 \pmod{40}\} \end{aligned}$$

$$\begin{aligned} T &= \{n \,|\, n \equiv \pm 1, 2, 3, 5, 9, 11, 12, 15, \\ &\qquad 16, 17, 18, 19 \pmod{40}\}, \\ S &= \{n \,|\, n \equiv \pm 1, 3, 4, 5, 6, 7, 13, 15, \\ &\qquad 16, 17, 18, 19 \pmod{40}\}. \end{aligned}$$

Proof of Shifted Identity (with ($a = 1$)

$$\boldsymbol{\rho}_1 := \{1, 3, 4, 5, 6, 7, 13, 15, 16, 17, 18, 19\}$$

We define a map $\mathbb{Z} \longrightarrow \{m : 0 \leq m \leq 19\}$ by $S(\mathfrak{z}) \equiv \pm\mathfrak{z} \pmod{40}$. $\mathbb{Z}$ acts on sets of integers $\boldsymbol{\rho}$ by

$$a\,\boldsymbol{\rho} := \{S(ar) : r \in \boldsymbol{\rho}\}.$$

For example, consider the set

$$\boldsymbol{\rho}_1 := \{1, 3, 4, 5, 6, 7, 13, 15, 16, 17, 18, 19\}$$

Then

$$3\,\boldsymbol{\rho}_1 = \{3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 17\},$$

since

$$\begin{aligned}
&3\,\{\pm 1, \pm 2, \pm 3, \pm 5, \pm 9, \pm 11, \pm 12, \pm 15, \pm 16,\\
&\qquad\qquad \pm 17, \pm 18, \pm 19\} \pmod{40}\\
&\equiv \{\pm 3, \pm 6, \pm 9, \pm 15, \pm 27, \pm 33, \pm 36, \pm 45,\\
&\qquad\qquad \pm 48, \pm 51, \pm 54, \pm 57\} \pmod{40}\\
&\equiv \{\pm 3, \pm 6, \pm 9, \pm 15, \pm 13, \pm 7, \pm 4, \pm 5,\\
&\qquad\qquad \pm 8, \pm 11, \pm 14, \pm 17\} \pmod{40}\\
&\equiv \{\pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 11,\\
&\qquad\qquad \pm 13, \pm 14, \pm 15, \pm 17\} \pmod{40}
\end{aligned}$$

The orbit of $\boldsymbol{\rho}_1$ under $\mathbb{Z}_{40}^{\times}$ is

$$\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_3, \boldsymbol{\rho}_7, \boldsymbol{\rho}_9\},$$

where

$$\boldsymbol{\rho}_j = j\boldsymbol{\rho}_1.$$

We may extend the definition of $\boldsymbol{\rho}_j$ for all $(j, 40) = 1$, by

$$\boldsymbol{\rho}_m = \boldsymbol{\rho}_{-m}, \qquad \boldsymbol{\rho}_{m+20} = \boldsymbol{\rho}_m.$$

In this way,

$$a\,\boldsymbol{\rho}_j = \boldsymbol{\rho}_{aj},$$

for $a \in \mathbb{Z}_{40}^{\times}$.

Define

$$\boldsymbol{\sigma}_1 = \{1, 2, 3, 5, 9, 11, 12, 15, 16, 17, 18, 19\}.$$

Then the orbit of $\boldsymbol{\sigma}_1$ under $\mathbb{Z}_{40}^{\times}$ is

$$\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_3, \boldsymbol{\sigma}_7, \boldsymbol{\sigma}_9\},$$

where

$$\boldsymbol{\sigma}_j = j\boldsymbol{\sigma}_1.$$

The identity

$$\prod_{n\in S} \frac{1}{(1-q^n)} - q \prod_{n\in T} \frac{1}{(1-q^n)} = 1,$$

is equivalent to

$$\left(\frac{1}{\theta_{\boldsymbol{\rho}_1;40}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_1;40}(\tau)}\right) \eta^{12}(40\tau) = 1.$$

We define

$$f_1(\tau) := \left(\frac{1}{\theta_{\boldsymbol{\rho}_1}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_1}(\tau)}\right) \eta^{12}(40\tau),$$

$$f_3(\tau) := \left(\frac{1}{\theta_{\boldsymbol{\sigma}_3}(\tau)} - \frac{1}{\theta_{\boldsymbol{\rho}_3}(\tau)}\right) \eta^{12}(40\tau),$$

$$f_7(\tau) := \left(\frac{1}{\theta_{\boldsymbol{\rho}_7}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_7}(\tau)}\right) \eta^{12}(40\tau),$$

$$f_9(\tau) := \left(\frac{1}{\theta_{\boldsymbol{\rho}_9}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_9}(\tau)}\right) \eta^{12}(40\tau),$$

NOTE: The *shiftless* (mod 40) result is equivalent to showing $f_7 \equiv 1$.

Theorem: For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(40)$,

$$f_k(A\tau) = f_{ak}(\tau),$$

where $\pm ak$ is reduced $\pmod{20}$.

Corollary: Any symmetric polynomial in f_1, f_3, f_7, f_9 is a modular function on $\Gamma_0(40)$.

Let

$$F := (f_1 - 1)(f_3 - 1)(f_7 - 1)(f_9 - 1).$$

Then $F(\tau)$ is a modular function of $\Gamma_0(40)$. Either $F \equiv 0$ or

$$\sum_{s \in \mathcal{F}} \mathrm{ORD}_s(F) = 0.$$

We know

$$\sum_{s \neq i\infty} \mathrm{ORD}_s(F) \geq -5.$$

We want to show that

$$\mathrm{ORD}_{i\infty}(F) \geq 6.$$

Now,

$$\begin{aligned} f_1 &= \frac{1}{(1-q)(1-q^3)\cdots} - q\frac{1}{(1-q)(1-q^2)\cdots} \\ &= 1 + O(q^3) \end{aligned}$$

and

$$\mathrm{ORD}_{i\infty}(f_1 - 1) \geq 3.$$

$$\begin{aligned} f_3 &= \frac{1}{(1-q^3)\cdots} - q^3\frac{1}{(1-q)\cdots} \\ &= 1 + O(q^4) \end{aligned}$$

and

$$\mathrm{ORD}_{i\infty}(f_3 - 1) \geq 4.$$

$$
\begin{aligned}
f_7 &= \frac{1}{q^2(1-q)(1-q^2)\cdots} - \frac{1}{q^2(1-q)(1-q^3)\cdots} \\
&= O(q^{-1})
\end{aligned}
$$

and

$$\mathrm{ORD}_{i\infty}(f_7 - 1) \geq -1.$$

$$
\begin{aligned}
f_9 &= \frac{1}{(1-q^2)(1-q^3)\cdots} - q^2 \frac{1}{(1-q)(1-q^2)\cdots} \\
&= 1 + O(q^3)
\end{aligned}
$$

and

$$\mathrm{ORD}_{i\infty}(f_9 - 1) \geq 3.$$

Hence

$$\mathrm{ORD}_{i\infty}(F1) \geq 3 + 4 - 1 + 3 = 9,$$

and

$$F \equiv 0,$$

which implies $f_j = 1$ for some j. But $\Gamma_0(40)$ acts transitively on the $\{f_j\}$, and so

$$f_1 = f_3 = f_7 = f_9 = 1.$$

Andrews's Questions

Question 1. Are there other pairs S and T besides ... such that $p(S,n) = p(T,n-1)$ for all $n \geq 1$?

Question 2. Apart from ... , for what pairs of positive integers S and T is it true that $p(S,n) = p(T,n-a)$ for all $n \geq a$ (where a is fixed)?

Question 3. Are there any instances of the equation

$$\prod_{n \in S} \frac{1}{1-q^n} = 1 + q^a \prod_{n \in T} \frac{1}{1-q^n},$$

in which the infinite products appearing are not essentially modular forms?

Question 4. For each pair S and T which answers Question 2 (or 1) can a bijection be found between the partitions of n into elements of S and the partitions of $n - a$ into elements of T?

Other Questions

Question 5. For a *periodic* ST-pair (ie $a = 1$) what is the smallest modulus M that occurs?

Question 6. Are there any *periodic* ST-pairs (ie $a = 1$) with odd modulus M?

Question 7. Are there infinitely many ST-pairs?

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