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More Shifted Partition Identities

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Let S and T be sets of positive integers. A *shifted partition identity* has the form

$$p(S, n) = p(T, n-1), \qquad \text{for all } n \geq 1.$$

If S or T is finite then it is not hard to show that $S = T = \{1\}$.

$$p(\{1\}, n) = 1$$

Andrews [1987] found the following two non-trivial examples:

$$\begin{aligned} S &= \{n \mid n \text{ odd or} \\ &\quad n \equiv \pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}\}, \\ T &= \{n \mid n \text{ odd or} \\ &\quad n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \pmod{32}\}; \end{aligned}$$

and

$$\begin{aligned} T &= \{n \mid n \equiv \pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 10, \pm 11, \pm 14, \\ &\qquad \pm 15, \pm 16, \pm 17, \pm 19 \pmod{40}\}, \\ S &= \{n \mid n \equiv \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \\ &\qquad \pm 13, \pm 15, \pm 16, \pm 19 \pmod{40}\}. \end{aligned}$$

In these examples, each S and T is the union of arithmetic progressions modulo M for some M; namely $M = 32$ and $M = 40$.

Later, Kalvade [1989] found five more identities with 42, 48 and 60.

In the present paper we find a further 48 identities with $M = 40$, 42, 46, 48, 54, 56, 60, 62, 66, 70 and 72.

Modulus M	Number of Shifted Partition Identities
32	1
40	$1+1=2$
42	$1+2+1+2=6$
46	$0+4=4$
48	$1+7=8$
54	$0+11=11$
56	$0+2=2$
60	$2+10=12$
62	$0+2=2$
66	$0+1=1$
70	$0+1=1$
72	$0+5=5$

$M = 32$	1 id	[Andrews 1987]
$M = 40$	1 id	[Andrews 1987]
	1 conj	[G]
$M = 42$	1 id	[Kalvade 1989] QP
	1 id	[G] QP
	2 conjs	[Kalvade 1989]
	2 conjs	[G]
$M = 46$	4 conjs	[G]
$M = 48$	1 id	[Kalvade 1989] QP
	7 conjs	[G]
$M = 54$	11 conjs	[G]
$M = 60$	2 ids	[Kalvade 1989] Mac-BC_2
	10 conjs	[G]
$M = 62$	2 conjs	[G]
$M = 66$	1 conj	[G]
$M = 70$	1 conj	[G]
$M = 72$	5 conjs	[G]

Notation

Let S, T be sets of positive integers. Define F_S by

$$F_S := F_S(q) = \sum_{n \geq 0} p(S,n) q^n = \prod_{n \in S} \frac{1}{(1-q^n)}.$$

The Problem

We want sets of positive integers S and T so that

$$F_S - q\, F_T = 1.$$

The pair $[S,T]$ is called an ST-**pair**.

Equivalent Problem

We want sets of positive integers S and T so that

$$\prod_{n\in T\setminus S}(1-q^n)-q\prod_{n\in S\setminus T}(1-q^n)=\prod_{n\in S\cup T}(1-q^n).$$

Example 1.

$M=40$ [Andrews]

$$\begin{aligned} T\setminus S &= \{n: n\equiv \pm 3, 14, 17 \pmod{40}\} \\ S\setminus T &= \{n: n\equiv \pm 6, 7, 13 \pmod{40}\} \end{aligned}$$

Example 2.

$M=40$ [G] Conjecture

$$\begin{aligned} T\setminus S &= \{n: n\equiv \pm 2, 9, 11, 12 \pmod{40}\} \\ S\setminus T &= \{n: n\equiv \pm 4, 6, 7, 13 \pmod{40}\} \end{aligned}$$

Example 3.

$M=60$ [Kalvade]

$$\begin{aligned} T\setminus S &= \{n: n\equiv \pm 5, 16, 25 \pmod{60}\} \\ S\setminus T &= \{n: n\equiv \pm 8, 11, 19 \pmod{60}\} \end{aligned}$$

Example 4.

$M = 70$ [G] Conjecture

$$\begin{aligned} T \setminus S &= \{n : n \equiv \pm 2, 8, 12, 18, 21, 22, 32 \pmod{70}\} \\ S \setminus T &= \{n : n \equiv \pm 4, 6, 7, 16, 24, 26, 34 \pmod{70}\} \end{aligned}$$

Truncated ST-pairs

A pair of sets $[S, T]$ is called a **truncated** ST-**pair** $O(q^N)$ if

$$\prod_{n \in S} \frac{1}{(1-q^n)} - q \prod_{n \in T} \frac{1}{(1-q^n)} = 1 + \mathrm{O}(q^N),$$

$$S \subset \{1, 2, 3, \ldots, N-1\},$$

and

$$T \subset \{1, 2, 3, \ldots, N-2\}.$$

Truncated ST-pairs $O(q^N)$

$N = 3$ $[\{1\}, \ \{1\}]$

$N = 4$
$[\{1,3\}, \ \{1,2\}]$
$[\{1\}, \ \{1\}]$

$N = 5$
$[\{1,3,4\}, \ \{1,2,3\}]$
$[\{1,3\}, \ \{1,2\}]$
$[\{1,4\}, \ \{1,3\}]$
$[\{1\}, \ \{1\}]$

$N = 6$
$[\{1,3,4,5\}, \ \{1,2,3\}]$
$[\{1,3,5\}, \ \{1,2\}]$
$[\{1,4,5\}, \ \{1,3,4\}]$
$[\{1,4\}, \ \{1,3\}]$
$[\{1,5\}, \ \{1,4\}]$
$[\{1\}, \ \{1\}]$

$$N = 7 \quad \begin{array}{l} [\{1,3,4,5,6\},\ \ \{1,2,3,5\}] \\ [\{1,3,4,5\},\ \ \{1,2,3\}] \\ [\{1,3,5\},\ \ \{1,2,5\}] \\ [\{1,4,5,6\},\ \ \{1,3,4,5\}] \\ [\{1,4,5\},\ \ \{1,3,4\}] \\ [\{1,4,6\},\ \ \{1,3,5\}] \\ [\{1,4\},\ \ \{1,3\}] \\ [\{1,5,6\},\ \ \{1,4,5\}] \\ [\{1,5\},\ \ \{1,4\}] \\ [\{1,6\},\ \ \{1,5\}] \\ [\{1\},\ \ \{1\}] \end{array}$$

$$
N = 8 \quad
\begin{array}{l}
[\{1,3,4,5,6,7\},\ \ \{1,2,3,5\}] \\
[\{1,3,4,5,7\},\ \ \{1,2,3\}] \\
[\{1,3,5,7\},\ \ \{1,2,5\}] \\
[\{1,4,5,6,7\},\ \ \{1,3,4,5\}] \\
[\{1,4,5,7\},\ \ \{1,3,4\}] \\
[\{1,4,6,7\},\ \ \{1,3,5\}] \\
[\{1,4,7\},\ \ \{1,3\}] \\
[\{1,5,6,7\},\ \ \{1,4,5,6\}] \\
[\{1,5,6\},\ \ \{1,4,5\}] \\
[\{1,5,7\},\ \ \{1,4,6\}] \\
[\{1,5\},\ \ \{1,4\}] \\
[\{1,6,7\},\ \ \{1,5,6\}] \\
[\{1,6\},\ \ \{1,5\}] \\
[\{1,7\},\ \ \{1,6\}] \\
[\{1\},\ \ \{1\}]
\end{array}
$$

The Number of Truncated ST-pairs $O(q^N)$

Let $T(n)$ denote the number of truncated ST-pairs $O(q^N)$.

n	$T(n)$
3	1
4	2
5	4
6	6
7	11
8	15
9	26
10	41
11	67
12	96
13	138
14	197
15	300
16	431
$\vdots$	
35	591648
36	876405
37	1354888

Conjecture

$$T(n) \sim c_1 e^{c_2 n}$$

Andrews's Questions

Question 1. Are there other pairs S and T besides ... such that $p(S, n) = p(T, n-1)$ for all $n \geq 1$?

Question 2. Apart from ..., for what pairs of positive integers S and T is it true that $p(S, n) = p(T, n-a)$ for all $n \geq a$ (where a is fixed)?

Question 3. Are there any instances of the equation

$$\prod_{n \in S} \frac{1}{1-q^n} = 1 + q^a \prod_{n \in T} \frac{1}{1-q^n},$$

in which the infinite products appearing are not essentially modular forms?

Question 4. For each pair S and T which answers Question 2 (or 1) can a bijection be found between the partitions of n into elements of S and the partitions of $n-a$ into elements of T?

Other Questions

Question 5. For a *periodic* ST-pair (ie $a = 1$) what is the smallest modulus M that occurs?

Question 6. Are there any *periodic* ST-pairs (ie $a = 1$) with odd modulus M?

Question 7. Are there infinitely many ST-pairs?