Department of Mathematics
University of Florida
Gainesville, Florida 32611

frank@math.ufl.edu

New relations between the rank and the crank

F. G. Garvan

May 24, 2000

DEFINITIONS

The **rank** of a partition is the *largest part* minus the *number of parts*.

For example, the rank of

$$3 + 2 + 2 + 1 + 1$$

is $3 - 5 = -2$.

Let $N(m, n)$ denote the number of partitions of n with rank m, then

$$\sum_m N(m, n) = p(n),$$

the number of partitions of n; and

$$N(m, n) = N(-m, n)$$

The **crank** of a partition is the *largest part* if the partition contains no ones, and is the *difference between the the number of parts larger than the number of ones and the number of ones*, otherwise.

For example, the crank of

$$3 + 2 + 2 + 1 + 1$$

is $1 - 2 = -1$.

Let $M(m,n)$ denote the number of partitions of n with crank m, then

$$\sum_m M(m,n) = p(n),$$

the number of partitions of n; and

$$M(m,n) = M(-m,n)$$

GENERATING FUNCTIONS

$$R(z,q) := \sum_{n\geq 0}\sum_{m} N(m,n)z^m q^n$$
$$= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}$$

$$\sum_{n\geq 0} N(m,n)q^n$$
$$= \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\frac{n}{2}(3n-1)+|m|n}(1-q^n)$$

$$C(z,q) := \sum_{n\geq 0}\sum_{m} M(m,n)z^m q^n$$
$$= \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

$$\sum_{n\geq 0} M(m,n)q^n$$

$$= \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n}{2}(n-1)+|m|n}(1-q^n)$$

RAMANUJAN's CONGRUENCES

Let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t.

Then for $p=5$ or $p=7$

$$N(k,p,n)=\frac{1}{p}p(n), \qquad 0\le k\le p-1;$$

for all n satisfying $24n\equiv 1 \pmod{p}$.

Corollary.

$$p(5n+4)\equiv 0 \pmod{5}$$

$$p(7n+5)\equiv 0 \pmod{7}$$

Let $M(m,t,n)$ denote the number of partitions of n with crank congruent to m modulo t.

Then for $p=5$, $p=7$ or $p=11$

$$M(k,p,n)=\frac{1}{p}p(n), \qquad 0\le k\le p-1;$$

for all n satisfying $24n \equiv 1 \pmod{p}$.

Corollary.

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+7) \equiv 0 \pmod{11}$$

LINEAR RELATIONS

There are more RANK RELATIONS, eg:

$$N(1,5,5n+1) = N(2,5,5n+1),$$

and others for the RANK modulo 5, 7, 8, 9 and 12.

There are similar relations for the CRANK, eg:

$$M(1,8,2n) = N(3,8,2n),$$

and others for the CRANK modulo 5, 7, 8, 9, 10 and 11.

There are relations between the RANK and the CRANK, eg:

$$M(4,9,3n) = N(4,9,3n),$$

and others modulo 5, 7, 8 and 9.

LINEAR RELATIONS MODULO p

There are modular relations for the RANK modulo 11 and 13, eg:

$$\begin{aligned} &2N(2,11,11n) + N(3,11,11n) \\ &\quad +7N(4,11,11n) + N(5,11,11n) \\ &\qquad \equiv 0 \pmod{11} \end{aligned}$$

BUT NOT for HIGHER PRIMES!!

HOWEVER, there is an analogous relation for the **CRANK FOR EACH PRIME p**.

These follow from the identity

$$\sum_{k=1}^{n} k^2 \, M(k,n) = n \, p(n),$$

due to DYSON [1989].

$$\frac{1}{p(n)} \sum_{k=1}^{n} k^2 \, N(k,n) = n - \frac{3}{\pi}\sqrt{\frac{3n}{2}} + \mathrm{O}(1).$$

Proof of $\sum_{k=1}^{n} k^2 M(k,n) = n\, p(n)$

$$\begin{aligned} C(z) = C(z,q) &= \sum M(m,n) z^m q^n \\ &= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} \end{aligned}$$

Then

$$\frac{1}{C} z \frac{\partial}{\partial z} C(z,q) = \sum_{n\geq 1} \frac{zq^n}{1-zq^n} - \frac{z^{-1}q^n}{1-z^{-1}q^n}$$

$$\frac{C''(1)}{2C(1)} = \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2}$$

Let

$$P = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \sum p(n)\, q^n$$

Then

$$\begin{aligned} \frac{1}{P} q \frac{\partial}{\partial q} P &= \sum_{n\geq 1} \frac{nq^n}{(1-q^n)} \\ &= \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2} = \sum \sigma(n)\, q^n \end{aligned}$$

But

$$C(1) = P$$

and

$$\frac{1}{2}C''(1) = q\,\frac{\partial}{\partial q}P;$$

ie.

$$\frac{1}{2}\sum_{m,n} m^2\,M(m,n)q^n = \sum n\,p(n)\,q^n.$$

EXTRA MODULAR RELATIONS

There is an extra linear modular relation for the CRANK modulo p for $p = 41$, 53, 83 and $120,667,369$.

For each prime $p > 13$ there are **SEVEN EXTRA** modular relations between the **RANK** and the **CRANK** modulo p. For example,

$$\begin{aligned}
&6N(0,29,29n+23)+17N(1,29,29n+23)\\
&+24N(2,29,29n+23)+18N(3,29,29n+23)\\
&+17N(4,29,29n+23)+14N(5,29,29n+23)\\
&+22N(6,29,29n+23)+24N(7,29,29n+23)\\
&+2N(9,29,29n+23)+15N(10,29,29n+23)\\
&+19N(11,29,29n+23)+18N(12,29,29n+23)\\
&+20N(13,29,29n+23)+16N(14,29,29n+23)\\
&\equiv 11M(0,29,29n+23)+17M(1,29,29n+23)\\
&\quad +28M(2,29,29n+23)+26M(4,29,29n+23)\\
&\quad +6M(5,29,29n+23)+28M(8,29,29n+23)\\
&\qquad\qquad \pmod{29}
\end{aligned}$$

NEW EXACT RELATIONS BETWEEN THE RANK AND THE CRANK

These EXTRA congruences come from certain exact relations wetween the RANK and the CRANK.

For $j \geq 2$ and EVEN, we define

$$N_j(n) = \sum_{k=1}^{n} k^j \, N(k,n)$$

$$M_j(n) = \sum_{k=1}^{n} k^j \, M(k,n)$$

Then

$$N_4(n) = -(2n+\tfrac{2}{3}) \, M_2(n) + \tfrac{8}{3} M_4(n) + (1-12n) \, N_2(n)$$

In fact, there are poynomials $P_k(n)$ and $Q_{j,k}(n)$ such that

$$N_{2k}(n) = P_k(n) N_2(n) + \sum_{j=1}^{k} Q_{j,k}(n) \, M_{2j}(n)$$

for $k = 1, 2, 3, 4, 5$.

$P_k(n)$ has degree $k - 1$

and $Q_{k,j}(n)$ has degree $(k - j)$.

$k = 6$ NO RELATION

$k = 7$ There is a similar relation but with an extra term $N_{12}(n)$.

THE RANK-CRANK PDE

Let

$$R^*(z,q) := \frac{R(z,q)}{(1-z)},$$

and

$$C^*(z,q) := \frac{C(z,q)}{(1-z)}.$$

Then

$$\begin{aligned} z\,(q)_\infty^2\,[C^*(z,q)]^3 &= \frac{1}{2}\left(z\,\frac{\partial}{\partial z}\right)^2 R^*(z,q) \\ &\quad + z\,\frac{\partial}{\partial z} R^*(z,q) \\ &\quad + 3q\,\frac{\partial}{\partial q} R^*(z,q) \end{aligned}$$

Proof of PDE

Define

$$P(z,q) := \prod_{n\geq 1}(1-z^{-1}q^n)(1-zq^{n-1})$$

and

$$\Sigma(z,\zeta,q) := \sum_{n}\frac{(-1)^n\zeta^2q^{3n(n+1)/2}}{1-zq^n}.$$

Then

(Elliptic ASWD)

$$\begin{aligned}\zeta^3\Sigma(z\zeta,\zeta^3,q)&+\Sigma(z\zeta^{-1},\zeta^{-3},q)\\ &-\zeta\frac{P(\zeta^2,q)}{P(\zeta,q)}\Sigma(z,1,q)\\ &=\frac{P(\zeta,q)P(\zeta^2,q)(q)_\infty^2}{P(\zeta z,q)P(z,q)P(z\zeta^{-1},q)}\end{aligned}$$

We let $g(\zeta)$ be the rhs of the equation above. Then g has a double zero at $\zeta=1$;

$$g''(1)=4(q)_\infty^3\left[C^*(z,q)\right]^3.$$

There is a simple relation between the function $\Sigma(z,1,q)$ and the generating function for the rank $R(z,q)$.

$$z\Sigma(z,1,q) = (q)_\infty \left(-1 + \frac{R(z,q)}{1-z}\right).$$

Now let $h(\zeta)$ be the sum of the first two terms on the lhs of (Elliptic ASWD). Then

$$\begin{aligned} h''(1) &= 3q\frac{\partial}{\partial q}\Sigma(z,1,q) + 6\Sigma(z,1,q) \\ &\quad + 8z\frac{\partial}{\partial z}\Sigma(z,1,q) \\ &\quad + 2z^2\left(\frac{\partial}{\partial z}\right)^2\Sigma(z,1,q) \end{aligned}$$

Let $j(\zeta)$ be the third function on the lhs of (Elliptic-ASWD). Then

$$j''(1) = -2\left(1 - 6\sum_{n\geq 1}\sigma(n)\,q^n\right)\Sigma(z,1,q).$$

The PDE folows from

$$h''(1) = j''(1) = g''(1)$$

and the identity

$$(q)_\infty q \frac{\partial}{\partial q} R(z,q) = q\,\frac{\partial}{\partial q}\Sigma(z,1,q) + \Sigma(z,1,q)\sum \sigma(n)\,q^n.$$