

**CONGRUENCES FOR ANDREWS' SPT-FUNCTION
MODULO 32760 AND EXTENSION OF ATKIN'S
HECKE-TYPE PARTITION CONGRUENCES**

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Dedicated to the memory of A.J. (Alf) van der Poorten, my former teacher

ABSTRACT. New congruences are found for Andrews' smallest parts partition function $\text{spt}(n)$. The generating function for $\text{spt}(n)$ is related to the holomorphic part $\alpha(24z)$ of a certain weak Maass form $\mathcal{M}(z)$ of weight $\frac{3}{2}$. We show that a normalized form of the generating function for $\text{spt}(n)$ is an eigenform modulo 72 for the Hecke operators $T(\ell^2)$ for primes $\ell > 3$, and an eigenform modulo p for $p = 5, 7$ or 13 provided that $(\ell, 6p) = 1$. The result for the modulus 3 was observed earlier by the author and considered by Ono and Folsom. Similar congruences for higher powers of p (namely $5^6, 7^4$ and 13^2) occur for the coefficients of the function $\alpha(z)$. Analogous results for the partition function were found by Atkin in 1966. Our results depend on the recent result of Ono that $\mathcal{M}_\ell(z/24)$ is a weakly holomorphic modular form of weight $\frac{3}{2}$ for the full modular group where

$$\mathcal{M}_\ell(z) = \mathcal{M}(z)|T(\ell^2) - \left(\frac{3}{\ell}\right)(1 + \ell)\mathcal{M}(z).$$

1. INTRODUCTION

Andrews [1] defined the function $\text{spt}(n)$ as the number of smallest parts in the partitions of n . He related this function to the second rank moment and proved some surprising congruences mod 5, 7 and 13. Rank and crank moments were introduced by A. O. L. Atkin and the author [2]. Bringmann [6] studied analytic, asymptotic and congruence properties of the generating function for the second rank moment as a quasi-weak Maass form. Further congruence properties of Andrews' spt -function were found by the author [10], [11], Folsom and Ono [8] and Ono [12]. In particular, Ono [12] proved that if $\left(\frac{1-24n}{\ell}\right) = 1$ then

$$(1.1) \quad \text{spt}(\ell^2 n - \frac{1}{24}(\ell^2 - 1)) \equiv 0 \pmod{\ell},$$

for any prime $\ell \geq 5$. This amazing result was originally conjectured by the author⁽ⁱ⁾. Earlier special cases were observed by Tina Garrett [9] and her students. Recently the author [11] has proved the following congruences for powers of 5, 7 and 13. For $a, b, c \geq 3$,

$$(1.2) \quad \text{spt}(5^a n + \delta_a) + 5 \text{spt}(5^{a-2} n + \delta_{a-2}) \equiv 0 \pmod{5^{2a-3}},$$

$$(1.3) \quad \text{spt}(7^b n + \lambda_b) + 7 \text{spt}(7^{b-2} n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2) \rfloor}},$$

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⁽ⁱ⁾The congruence (1.1) was first conjectured by the author in a Colloquium given at the University of Newcastle, Australia on July 17, 2008.

$$(1.4) \quad \text{spt}(13^c n + \gamma_c) - 13 \text{spt}(13^{c-2} n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}},$$

where δ_a , λ_b and γ_c are the least nonnegative residues of the reciprocals of $24 \bmod 5^a$, 7^b and 13^c respectively.

As in [12], [11] we define

$$(1.5) \quad \mathbf{a}(n) := 12 \text{spt}(n) + (24n - 1)p(n),$$

for $n \geq 0$, and define

$$(1.6) \quad \alpha(z) := \sum_{n \geq 0} \mathbf{a}(n) q^{n - \frac{1}{24}},$$

where as usual $q = \exp(2\pi iz)$ and $\Im(z) > 0$. We note that $\text{spt}(0) = 0$ and $p(0) = 1$. Bringmann [6] showed that $\alpha(24z)$ is the holomorphic part of the weight $\frac{3}{2}$ weak Maass form $\mathcal{M}(z)$ on $\Gamma_0(576)$ with Nebentypus χ_{12} where

$$(1.7) \quad \mathcal{M}(z) := \alpha(24z) - \frac{3i}{\pi\sqrt{2}} \int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau) d\tau}{(-i(\tau + z))^{\frac{3}{2}}},$$

$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta-function, the function $\alpha(z)$ is defined in (1.6), and

$$(1.8) \quad \chi_{12}(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

Ono [12] showed that for $\ell \geq 5$ prime, the operator

$$(1.9) \quad T(\ell^2) - \chi_{12}(\ell)\ell(1 + \ell)$$

annihilates the nonholomorphic part of $\mathcal{M}(z)$, and the function $\mathcal{M}_\ell(z/24)$ is a weakly holomorphic modular form of weight $\frac{3}{2}$ for the full modular group where

$$(1.10) \quad \mathcal{M}_\ell(z) = \mathcal{M}(z) | T(\ell^2) - \chi_{12}(\ell)(1 + \ell) \mathcal{M}(z) = \alpha(24z) | T(\ell^2) - \chi_{12}(\ell)(1 + \ell) \alpha(24z).$$

In fact he obtained

Theorem 1.1 (Ono [12]). *If $\ell \geq 5$ is prime then the function*

$$(1.11) \quad \mathcal{M}_\ell(z/24) \eta(z)^{\ell^2}$$

is an entire modular form of weight $\frac{1}{2}(\ell^2 + 3)$ for the full modular group $\Gamma(1)$.

Applying this theorem Ono obtained

$$(1.12) \quad \mathcal{M}_\ell(z) \equiv 0 \pmod{\ell}.$$

The congruence (1.1) then follows easily.

Folsom and Ono [8] sketched the proof of the following

Theorem 1.2 (Folsom and Ono). *If $\ell \geq 5$ is prime then*

$$(1.13) \quad \text{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\frac{1 - 24n}{\ell} \right) \text{spt}(n) + \ell \text{spt} \left(\frac{n + s_\ell}{\ell^2} \right) \equiv \chi_{12}(\ell) (1 + \ell) \text{spt}(n) \pmod{3},$$

where

$$(1.14) \quad s_\ell = \frac{1}{24}(\ell^2 - 1).$$

This result was observed earlier by the author. In this paper we prove a much stronger result.

Theorem 1.3. (i) *If $\ell \geq 5$ is prime then*

$$(1.15) \quad \text{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\frac{1 - 24n}{\ell} \right) \text{spt}(n) + \ell \text{spt} \left(\frac{n + s_\ell}{\ell^2} \right) \equiv \chi_{12}(\ell) (1 + \ell) \text{spt}(n) \pmod{72}.$$

(ii) *If $\ell \geq 5$ is prime, $t = 5, 7$ or 13 and $\ell \neq t$ then*

$$(1.16) \quad \text{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\frac{1 - 24n}{\ell} \right) \text{spt}(n) + \ell \text{spt} \left(\frac{n + s_\ell}{\ell^2} \right) \equiv \chi_{12}(\ell) (1 + \ell) \text{spt}(n) \pmod{t}.$$

Of course this implies the

Corollary 1.4. *If ℓ is prime and $\ell \notin \{2, 3, 5, 7, 13\}$ then*

$$(1.17) \quad \text{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\frac{1 - 24n}{\ell} \right) \text{spt}(n) + \ell \text{spt} \left(\frac{n + s_\ell}{\ell^2} \right) \equiv \chi_{12}(\ell) (1 + \ell) \text{spt}(n) \pmod{32760}.$$

This congruence modulo $32760 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ is the congruence referred in the title of this paper.

In 1966, Atkin [4] found a similar congruence for the partition function.

Theorem 1.5 (Atkin). *Let $t = 5, 7$, or 13 , and $c = 6, 4$, or 2 respectively. Suppose $\ell \geq 5$ is prime and $\ell \neq t$. If $\left(\frac{1-24n}{t}\right) = -1$, then*

$$(1.18) \quad \ell^3 p(\ell^2 n - s_\ell) + \ell \chi_{12}(\ell) \left(\frac{1 - 24n}{\ell} \right) p(n) + p \left(\frac{n + s_\ell}{\ell^2} \right) \equiv \gamma_t p(n) \pmod{t^c},$$

where γ_t is an integral constant independent of n .

We find that there is a corresponding result for the function $\mathbf{a}(n)$ defined in (1.5).

Theorem 1.6. *Let $t = 5, 7$, or 13 , and $c = 6, 4$, or 2 respectively. Suppose $\ell \geq 5$ is prime and $\ell \neq t$. If $\left(\frac{1-24n}{t}\right) = -1$, then*

$$(1.19) \quad \mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\frac{1 - 24n}{\ell} \right) \mathbf{a}(n) + \ell \mathbf{a} \left(\frac{n + s_\ell}{\ell^2} \right) \equiv \chi_{12}(\ell) (1 + \ell) \mathbf{a}(n) \pmod{t^c}.$$

In Section 2 we prove Theorem 1.3. The method involves reviewing the action of weight $-\frac{1}{2}$ Hecke operators $T(\ell^2)$ on the function $\eta(z)^{-1}$ and doing a careful study of the action of weight $\frac{3}{2}$ Hecke operators on the function $\frac{d}{dz} \eta(z)^{-1}$ modulo 5, 7, 13, 27 and 32. In Section 3 we prove Theorem 1.6. The method involves extending Atkin's [4] on modular functions to weight two modular forms on $\Gamma_0(t)$ for $t = 5, 7$ and 13. The proof of both Theorems 1.3 and 1.6 depend on Ono's Theorem 1.1.

2. PROOF OF THEOREM 1.3

In this section we prove Theorem 1.3. Atkin [3] showed essentially that applying certain weight $-\frac{1}{2}$ Hecke operators $T(\ell^2)$ to the function $\eta(z)^{-1}$ produces a function with the same multiplier system as $\eta(z)^{-1}$ and thus $\eta(z)$ times this function is a certain polynomial (depending on ℓ) of Klein's modular invariant $j(z)$. We review Ono's [13] recent explicit form for these polynomials. Although our proof does not depend on Ono's result it is quite useful for computational purposes. The action of the corresponding weight $\frac{3}{2}$ Hecke operators on $\frac{d}{dz} \eta(z)^{-1}$ can be given in terms of the same polynomials. See Theorem 2.3 below. To finish the proof of the theorem we need to make a careful study of the action of these operators modulo 5, 7, 13, 27 and 32.

For $\ell \geq 5$ prime we define

$$(2.1) \quad Z_\ell(z) = \sum_{n=-s_\ell}^{\infty} \left(\ell^3 p(\ell^2 n - s_\ell) + \ell \chi_{12}(\ell) \left(\frac{1 - 24n}{\ell} \right) p(n) + p \left(\frac{n + s_\ell}{\ell^2} \right) \right) q^{n - \frac{1}{24}}.$$

Proposition 2.1 (Atkin [4]). *The function $Z_\ell(z)\eta(z)$ is a modular function on the full modular group $\Gamma(1)$.*

It follows that $Z_\ell(z)\eta(z)$ is a polynomial in $j(z)$, where $j(z)$ is Klein's modular invariant

$$(2.2) \quad j(z) := \frac{E_4(z)^2}{\Delta(z)} = q^{-1} + 744 + 196884q + \cdots,$$

$E_2(z)$, $E_4(z)$, $E_6(z)$ are the usual Eisenstein series

$$(2.3) \quad E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n, \quad E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n,$$

$\sigma_k(n) = \sum_{d|n} d^k$, and $\Delta(z)$ is Ramanujan's function

$$(2.4) \quad \Delta(z) := \eta(z)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

In a recent paper, Ono [13] has found a nice formula for this polynomial. We define

$$(2.5) \quad E(q) := \prod_{n=1}^{\infty} (1 - q^n) = q^{-\frac{1}{24}} \eta(z),$$

and a sequence of polynomials $A_m(x) \in \mathbb{Z}[x]$ by

$$(2.6) \quad \sum_{m=0}^{\infty} A_m(x)q^m = E(q) \frac{E_4(z)^2 E_6(z)}{\Delta(z)} \frac{1}{j(z) - x} \\ = 1 + (x - 745)q + (x^2 - 1489x + 160511)q^2 + \cdots.$$

Theorem 2.2 (Ono [13]). *For $\ell \geq 5$ prime*

$$(2.7) \quad Z_\ell(z)\eta(z) = \ell \chi_{12}(\ell) + A_{s_\ell}(j(z)),$$

where $Z_\ell(z)$ is given in (2.1), and s_ℓ is given in (1.14).

We define a sequence of polynomials $C_\ell(x) \in \mathbb{Z}[x]$ by

$$(2.8) \quad C_\ell(x) := \ell \chi_{12}(\ell) + A_{s_\ell}(x), \\ = \sum_{n=0}^{s_\ell} c_{n,\ell} x^n,$$

so that

$$(2.9) \quad Z_\ell(z)\eta(z) = C_\ell(j(z)).$$

We define

$$(2.10) \quad d(n) := (24n - 1)p(n),$$

so that

$$(2.11) \quad \sum_{n=0}^{\infty} d(n)q^{24n-1} = q \frac{d}{dq} \frac{1}{\eta(24z)} = -\frac{E_2(24z)}{\eta(24z)},$$

and

$$(2.12) \quad \mathbf{a}(n) = 12\text{spt}(n) + d(n).$$

For $\ell \geq 5$ prime we define

$$(2.13) \quad \Xi_\ell(z) = \sum_{n=-s_\ell}^{\infty} \left(d(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell} \right) - 1 - \ell \right) d(n) + \ell d \left(\frac{n + s_\ell}{\ell^2} \right) \right) q^{n - \frac{1}{24}}.$$

We then have the following analogue of Theorem 2.2.

Theorem 2.3. *For $\ell \geq 5$ prime we have*

$$(2.14) \quad \ell \Xi_\ell(z) \eta(z) \Delta(z)^{s_\ell} = - \sum_{n=0}^{s_\ell} c_{n,\ell} E_4(z)^{3n-1} \Delta(z)^{s_\ell-n} (24nE_6(z) + E_4(z)E_2(z)) + \chi_{12}(\ell)\ell(1+\ell)E_2(z) \Delta(z)^{s_\ell},$$

where the coefficients $c_{n,\ell}$ are defined by (2.6) and (2.8).

Proof. Suppose $\ell \geq 5$ is prime. In equation (2.9) we replace z by $24z$, apply the operator $q \frac{d}{dq}$ and replace z by $\frac{1}{24}z$ to obtain

$$(2.15) \quad \ell \Xi_\ell(z) \eta(z) = 24C'_\ell(j(z)) q \frac{d}{dq}(j(z)) + (\chi_{12}(\ell)\ell(1+\ell) - C_\ell(j(z))) E_2(z)$$

The result then follows easily from the identities

$$(2.16) \quad j(z) \Delta(z) = E_4(z)^3, \quad q \frac{d}{dq}(\Delta(z)) = \Delta(z) E_2(z), \quad \text{and} \quad q \frac{d}{dq}(j(z)) \Delta(z) = -E_4(z)^2 E_6(z),$$

which we leave as an easy exercise. □

We are now ready to prove Theorem 1.3. A standard calculation gives the following congruences.

$$(2.17) \quad E_4(z)^3 - 720 \Delta(z) \equiv 1 \pmod{65520}, \quad \text{and} \quad E_2(z) \equiv E_4(z)^2 E_6(z) \pmod{65520}.$$

We now use (2.17) to reduce (2.15) modulo 65520.

$$(2.18)$$

$$\begin{aligned} & \ell \Xi_\ell(z) \eta(z) \Delta(z)^{s_\ell} \\ & \equiv - \sum_{n=0}^{s_\ell} c_{n,\ell} E_4(z)^{3n-1} \Delta(z)^{s_\ell-n} (24nE_6(z)(E_4(z)^3 - 720 \Delta(z)) + E_4(z)^3 E_6(z)) \\ & \quad + \chi_{12}(\ell)\ell(1+\ell) E_4(z)^2 E_6(z) \Delta(z)^{s_\ell} \pmod{65520} \\ & \equiv - \sum_{n=0}^{s_\ell} (24n+1)c_{n,\ell} E_4(z)^{3n+2} E_6(z) \Delta(z)^{s_\ell-n} \\ & \quad + \sum_{n=0}^{s_\ell} 720 \cdot 24nc_{n,\ell} E_4(z)^{3n-1} E_6(z) \Delta(z)^{s_\ell-n+1} + \chi_{12}(\ell)\ell(1+\ell) E_4(z)^2 E_6(z) \Delta(z)^{s_\ell} \pmod{65520} \\ & \equiv (720 c_{1,\ell} - c_{0,\ell} + \chi_{12}(\ell)\ell(1+\ell)) E_4(z)^2 E_6(z) \Delta(z)^{s_\ell} \\ & \quad + \sum_{n=1}^{s_\ell-1} (720 \cdot 24(n+1)c_{n+1,\ell} - (24n+1)c_{n,\ell}) E_4(z)^{3n+2} E_6(z) \Delta(z)^{s_\ell-n} \\ & \quad - (24s_\ell + 1)c_{s_\ell} E_4(z)^{3s_\ell+2} E_6(z) \pmod{65520}. \end{aligned}$$

We define

$$(2.19) \quad \mathcal{A}_\ell(z) := \sum_{n=-s_\ell}^{\infty} \left(\mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell} \right) - 1 - \ell \right) \mathbf{a}(n) + \ell \mathbf{a} \left(\frac{n+s_\ell}{\ell^2} \right) \right) q^{n-\frac{1}{24}}$$

and

(2.20)

$$\mathcal{S}_\ell(z) := \sum_{n=1}^{\infty} \left(\text{spt}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell} \right) - 1 - \ell \right) \text{spt}(n) + \ell \text{spt} \left(\frac{n+s_\ell}{\ell^2} \right) \right) q^{n-\frac{1}{24}},$$

so that

$$(2.21) \quad \mathcal{A}_\ell(z) = 12 \mathcal{S}_\ell(z) + \Xi_\ell(z) = \mathcal{M}_\ell(z/24).$$

By Theorem 1.1 and equation (1.10) we see that the function

$$(2.22) \quad \ell \mathcal{A}_\ell(z) \eta(z) \Delta(z)^{s_\ell} \in M_{\frac{1}{2}(\ell^2+3)}(\Gamma(1)),$$

the space of entire modular forms of weight $\frac{1}{2}(\ell^2+3)$ on $\Gamma(1)$. Since $\frac{1}{2}(\ell^2+3) = 2 + 12s_\ell$ the set

$$(2.23) \quad \{E_4(z)^{3n-1} E_6(z) \Delta(z)^{s_\ell-n} : 1 \leq n \leq s_\ell\}$$

is a basis. Hence there are integers $b_{n,\ell}$ ($1 \leq n \leq s_\ell$) such that

$$(2.24) \quad \mathcal{A}_\ell(z) \eta(z) \Delta(z)^{s_\ell} = \sum_{n=1}^{s_\ell} b_{n,\ell} E_4(z)^{3n-1} E_6(z) \Delta(z)^{s_\ell-n}.$$

Using (2.17) we find that

$$(2.25) \quad \begin{aligned} \mathcal{A}_\ell(z) \eta(z) \Delta(z)^{s_\ell} &\equiv -720b_{1,\ell} E_4(z)^2 E_6(z) \Delta(z)^{s_\ell} \\ &\quad + \sum_{n=1}^{s_\ell-1} (b_{n,\ell} - 720b_{n+1,\ell}) E_4(z)^{3n+2} E_6(z) \Delta(z)^{s_\ell-n} \\ &\quad + b_{s_\ell,\ell} E_4(z)^{3s_\ell+2} E_6(z) \pmod{65520}. \end{aligned}$$

By (2.18), (2.21) and (2.24) we deduce that there are integers $a_{n,\ell}$ ($0 \leq n \leq s_\ell$) such that

$$(2.26) \quad 12 \ell \mathcal{S}_\ell(z) \eta(z) \Delta(z)^{s_\ell} \equiv \sum_{n=0}^{s_\ell} a_{n,\ell} E_4(z)^{3n+2} E_6(z) \Delta(z)^{s_\ell-n} \pmod{65520}.$$

It follows that

$$(2.27) \quad 12 \ell \mathcal{S}_\ell(z) \equiv 0 \pmod{65520},$$

since

$$(2.28) \quad \begin{aligned} \text{ord}_{i_\infty} (12 \ell \mathcal{S}_\ell(z) \eta(z) \Delta(z)^{s_\ell}) &= s_\ell + 1, \\ 0 \leq \text{ord}_{i_\infty} (E_4(z)^{3n+2} E_6(z) \Delta(z)^{s_\ell-n}) &\leq s_\ell, \\ E_4(z)^{3n+2} E_6(z) \Delta(z)^{s_\ell-n} &= q^{s_\ell-n} + \dots, \end{aligned}$$

for $0 \leq n \leq s_\ell$ and all functions have integral coefficients. Since $65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, the congruence (2.27) implies Part (ii) of Theorem 1.3. To prove Part (i) we need to work a little harder. We note that the congruence (2.27) does imply

$$(2.29) \quad \mathcal{S}_\ell(z) \equiv 0 \pmod{12}.$$

We need to show this congruence actually holds modulo 72.

First we show the congruence holds modulo 8 by studying $\Xi_\ell(z)$ modulo 32. We need the congruences,

$$(2.30) \quad E_2(z) \equiv E_4(z)E_6(z) + 16\Delta(z) \pmod{32}, \quad \text{and} \quad E_4(z)^2 \equiv 1 \pmod{32},$$

which are routine to prove. We proceed as in the proof of (2.18) to find that

$$(2.31) \quad \begin{aligned} & \ell \Xi_\ell(z) \eta(z) \Delta(z)^{s_\ell} \\ & \equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell} - 16c_{1,\ell}) E_2(z) \Delta(z)^{s_\ell} \\ & \quad - \sum_{n=1}^{s_\ell-1} ((24n+1)c_{n,\ell} + 16c_{n+1,\ell}) E_4(z)^{3n-1} E_6(z) \Delta(z)^{s_\ell-n} \\ & \quad - (24s_\ell + 1)c_{s_\ell} E_4(z)^{3s_\ell-1} E_6(z) \pmod{32}. \end{aligned}$$

By (2.31), (2.21) and (2.24) we deduce that there are integers $a'_{n,\ell}$ ($0 \leq n \leq s_\ell$) such that

$$(2.32) \quad 12 \ell \mathcal{S}_\ell(z) \eta(z) \Delta(z)^{s_\ell} \equiv \sum_{n=1}^{s_\ell} a'_{n,\ell} E_4(z)^{3n-1} E_6(z) \Delta(z)^{s_\ell-n} + a'_{0,\ell} E_2(z) \Delta(z)^{s_\ell} \pmod{32}.$$

Arguing as before, it follows that

$$(2.33) \quad 12\mathcal{S}_\ell(z) \equiv 0 \pmod{32}, \quad \text{and} \quad \mathcal{S}_\ell(z) \equiv 0 \pmod{8}.$$

To complete the proof, we need to study $\Xi_\ell(z)$ modulo 27. We need the congruences,

$$(2.34) \quad E_2(z) \equiv E_4(z)^5 + 18\Delta(z) \pmod{27} \quad \text{and} \quad E_6(z) \equiv E_4(z)^6 \pmod{27},$$

which are routine to prove. We proceed as in the proof of (2.18) and (2.31) to find that

$$(2.35) \quad \begin{aligned} & \ell \Xi_\ell(z) \eta(z) \Delta(z)^{s_\ell} \\ & \equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell} - 18c_{1,\ell}) E_2(z) \Delta(z)^{s_\ell} \\ & \quad - \sum_{n=1}^{s_\ell-1} ((24n+1)c_{n,\ell} + 18c_{n+1,\ell}) E_4(z)^{3n-1} E_6(z) \Delta(z)^{s_\ell-n} \\ & \quad - (24s_\ell + 1)c_{s_\ell} E_4(z)^{3s_\ell-1} E_6(z) \pmod{27}. \end{aligned}$$

By (2.35), (2.21) and (2.24) we deduce that there are integers $a''_{n,\ell}$ ($0 \leq n \leq s_\ell$) such that

$$(2.36) \quad 12 \ell \mathcal{S}_\ell(z) \eta(z) \Delta(z)^{s_\ell} \equiv \sum_{n=1}^{s_\ell} a''_{n,\ell} E_4(z)^{3n-1} E_6(z) \Delta(z)^{s_\ell-n} + a''_{0,\ell} E_2(z) \Delta(z)^{s_\ell} \pmod{27}.$$

Arguing as before, it follows that

$$(2.37) \quad 12\mathcal{S}_\ell(z) \equiv 0 \pmod{27}, \quad \text{and} \quad \mathcal{S}_\ell(z) \equiv 0 \pmod{9}.$$

The congruences (2.33) and (2.37) give (1.15) and this completes the proof of Theorem 1.3.

3. PROOF OF THEOREM 1.6

In this section we prove Theorem 1.6. Atkin [4] proved Theorem 1.5 by constructing certain special modular functions on $\Gamma_0(t)$ and $\Gamma_0(t^2)$ for $t = 5, 7$ and 13 . We attack the problem by extending Atkin's results to the corresponding weight 2 case.

Let $GL_2^+(\mathbb{R})$ denote the group of all real 2×2 matrices with positive determinant. $GL_2^+(\mathbb{R})$ acts on the complex upper half plane \mathcal{H} by linear fractional transformations. We define the slash

operator for modular forms of integer weight. Let $k \in \mathbb{Z}$. For a function $f : \mathcal{H} \rightarrow \mathbb{C}$ and $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ we define

$$(3.1) \quad f(z) \mid_k L = f \mid_k L = f \mid L = (\det L)^{\frac{k}{2}} (cz + d)^{-k} f(Lz).$$

Let $\Gamma' \subset \Gamma(1)$ (a subgroup of finite index). We say $f(z)$ is a *weakly holomorphic modular form* of weight k on Γ' if $f(z)$ is holomorphic on the upper half plane \mathcal{H} , $f(z) \mid_k L = f(z)$ for all L in Γ' , and $f(z)$ has at most polar singularities in the local variables at the cusps of the fundamental region of Γ' . We say $f(z)$ is a *weakly holomorphic modular function* if it is a weakly holomorphic modular form of weight 0. We say $f(z)$ is an *entire modular form* of weight k on Γ' if it is a *weakly holomorphic modular form* that is holomorphic at the cusps of the fundamental region of Γ' . We denote the space of entire modular forms of weight k on Γ' by $M_k(\Gamma')$.

Suppose that $t \geq 5$ is prime. We need

$$W_t = W = \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad V_a = \begin{pmatrix} a & \lambda \\ t & a' \end{pmatrix}, \quad B_t = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix},$$

$$T_{b,t} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad Q_{b,t} = \begin{pmatrix} 1/t & b/t \\ 0 & 1 \end{pmatrix},$$

where for $1 \leq a \leq t-1$, a' is uniquely defined by $1 \leq a' \leq t-1$, and $a'a - \lambda t = 1$. We have

$$(3.2) \quad B_t R^{at} = W_t V_a T_{-a'/t}$$

$$(3.3) \quad R^{at} W_t = W_{t^2} Q_{a,t}.$$

We define

$$(3.4) \quad \Phi_t(z) = \Phi(z) := \frac{\eta(z)}{\eta(t^2 z)}.$$

Then $\Phi_t(z)$ is a modular function of $\Gamma_0(t)$,

$$(3.5) \quad \Phi_t(z) \mid W_{t^2} = t \Phi_t(z)^{-1} \quad ([4, (24)]),$$

and

$$(3.6) \quad \Phi_t(z) \mid R^{at} = \sqrt{t} e^{\pi i(t-1)/4} e^{-\pi i a' t/12} \left(\frac{a'}{t}\right) \frac{\eta(z)}{\eta(z - a'/t)} \quad ([4, (25)]).$$

Although $E_2(z)$ is not a modular form, it well-known that

$$(3.7) \quad \mathcal{E}_{2,t}(z) := \frac{1}{t-1} (t E_2(tz) - E_2(z)),$$

is an entire modular form of weight 2 on $\Gamma_0(t)$ and

$$(3.8) \quad \mathcal{E}_{2,t}(z) \mid W_t = -\mathcal{E}_{2,t}(z).$$

Proposition 3.1. *Suppose $t \geq 5$ is prime, $K(z)$ is a weakly holomorphic modular function on $\Gamma_0(t)$, and*

$$(3.9) \quad S(z) = \mathcal{E}_{2,t}(tz) K^*(tz) \frac{\eta(z)}{\eta(t^2 z)} - \chi_{12}(t) \eta(z) \sum_{n=m}^{\infty} \left(\frac{1-24n}{t}\right) \beta_t(n) q^{n-\frac{1}{24}},$$

where

$$(3.10) \quad \mathcal{E}_{2,t}(z) \frac{K(z)}{\eta(z)} = \sum_{n=m}^{\infty} \beta_t(n) q^{n-\frac{1}{24}},$$

and

$$(3.11) \quad K^*(z) = K(z) \mid W_t.$$

Then $S(z)$ is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t)$.

Proof. Suppose $t \geq 5$ is prime and $K(z)$, $K^*(z)$, $S(z)$ are defined as in the statement of the proposition. The function

$$(3.12) \quad H(z) := \mathcal{E}_{2,t}(tz) \Phi_t(z) K^*(tz)$$

is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t^2)$. As in [4, Lemma1] the function

$$(3.13) \quad S_1(z) := \sum_{a=0}^{t-1} H(z) \mid R^{at}$$

is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t)$. Utilizing (3.2), (3.6), (3.8) and the evaluation of a quadratic Gauss sum [5, (1.7)] we find that

$$(3.14) \quad S_1(z) = \mathcal{E}_{2,t}(tz) K^*(tz) \frac{\eta(z)}{\eta(t^2z)} - \frac{1}{\sqrt{t}} e^{\pi i(t-1)/4} \eta(z) \sum_{a'=1}^{t-1} e^{-\pi i a' t/12} \left(\frac{a'}{t}\right) \mathcal{E}_{2,t}(z - a'/t) \frac{K(z - a'/t)}{\eta(z - a'/t)}$$

Here we have also used the fact that

$$(3.15) \quad \mathcal{E}_{2,t}(z) \mid R^{at} = -\frac{1}{t} \mathcal{E}_{2,t}(z - a'/t),$$

where $a a' \equiv 1 \pmod{t}$. Hence

$$S_1(z) = S(z).$$

This gives the result. □

We illustrate Proposition 3.1 with two examples:

$$(3.16) \quad S(z) = \mathcal{E}_{2,5}(z) \left(\frac{\eta(z)}{\eta(5z)}\right)^6 \quad (K(z) = 1 \text{ and } t = 5)$$

and

$$(3.17) \quad S(z) = \mathcal{E}_{2,7}(z) \left(\left(\frac{\eta(z)}{\eta(7z)}\right)^8 + 3 \left(\frac{\eta(z)}{\eta(7z)}\right)^4 \right) \quad (K(z) = 1 \text{ and } t = 7).$$

Corollary 3.2. *Suppose $t \geq 5$ is prime and $S(z)$, $K(z)$ and the sequence $\beta_t(n)$ are defined as in Proposition 3.1. Then*

$$(3.18) \quad S(z) \mid W_t = -\eta(tz) \sum_{tn-s_t \geq m} \beta_t(tn - s_t) q^{n - \frac{t}{24}}.$$

Proof. The result follows easily from (3.3), (3.5) and (3.8). □

We illustrate the corollary by applying W to both sides of the equations (3.16)–(3.17):

$$(3.19) \quad \sum_{n=1}^{\infty} \beta_5(5n - 1) q^{n - \frac{5}{24}} = 5^3 \frac{\mathcal{E}_{2,5}(z)}{\eta(5z)} \left(\frac{\eta(5z)}{\eta(z)}\right)^6 \quad (K(z) = 1 \text{ and } t = 5)$$

and

(3.20)

$$\sum_{n=1}^{\infty} \beta_7(7n-2)q^{n-\frac{7}{24}} = 7^2 \frac{\mathcal{E}_{2,7}(z)}{\eta(7z)} \left(3 \left(\frac{\eta(7z)}{\eta(z)} \right)^4 + 7^2 \left(\frac{\eta(7z)}{\eta(z)} \right)^8 \right) \quad (K(z) = 1 \text{ and } t = 7).$$

For t and $K(z)$ as in Proposition 3.1 we define

$$(3.21) \quad \Psi_{t,K}(z) = \Psi_t(z) = \mathcal{E}_{2,t}(tz) \frac{K^*(tz)}{\eta(t^2z)} - \chi_{12}(t) \sum_{n=m}^{\infty} \left(\frac{1-24m}{t} \right) \beta_t(n) q^{n-\frac{1}{24}} \\ - \sum_{t^2n-s_t \geq m} \beta_t(t^2n-s_t) q^{n-\frac{1}{24}},$$

where $K^*(z)$ and the sequence $\beta_t(n)$ is defined in (3.10)–(3.11). We have the following analogue of 2.1.

Corollary 3.3. *The function $\Psi_{t,K}(z)\eta(z)$ is a weakly holomorphic modular form of weight 2 on the full modular group $\Gamma(1)$.*

Proof. Let $S(z)$ be defined as in (3.9), so that $S(z)$ is a weakly holomorphic modular form of weight 2 on $\Gamma_0(t)$. By [14, Lemma 7], the function

$$(3.22) \quad S(z) + S(z) | W_t | U$$

is a modular form of weight 2 on $\Gamma(1)$. Here $U = U_t$ is the Atkin operator

$$(3.23) \quad g(z) | U_t = \frac{1}{t} \sum_{a=0}^{t-1} g\left(\frac{z+a}{t}\right).$$

The result then follows from applying the U -operator to equation (3.18). \square

We illustrate the $K(z) = 1$ case of Corollary 3.3 with two examples:

$$(3.24) \quad \Psi_5(z) = \frac{E_4(z)^2 E_6(z)}{\eta(z)^{25}}$$

and

$$(3.25) \quad \Psi_7(z) = \frac{1}{\eta(z)^{49}} (E_4(z)^5 E_6(z) - 745 E_4(z)^2 E_6(z) \Delta(z)).$$

We need a weight 2 analogue of [4, Lemma 3]. For $t = 5, 7$ or 13 the genus of $\Gamma_0(t)$ is zero, and a Hauptmodul is

$$(3.26) \quad G_t(z) := \left(\frac{\eta(z)}{\eta(tz)} \right)^{24/(t-1)}.$$

This function satisfies

$$(3.27) \quad G_t\left(\frac{-1}{tz}\right) = t^{12/(t-1)} G_t(z)^{-1}.$$

Proposition 3.4. *Suppose $t = 5, 7$ or 13 , and let m be any negative integer such that $24m \not\equiv 1 \pmod{t}$. Suppose constants k_j ($1 \leq j \leq -m$) are chosen so that*

$$(3.28) \quad \beta_t(n) = 0, \quad \text{for } m+1 \leq n \leq -1,$$

where

$$(3.29) \quad K(z) = G_t(z)^{-m} + \sum_{k=1}^{-m-1} k_j G_t(z)^j$$

and

$$(3.30) \quad \sum_{n=m}^{\infty} \beta_t(n)q^{n-\frac{1}{24}} = \mathcal{E}_{2,t}(z) \frac{K(z)}{\eta(z)}.$$

Then

$$(3.31) \quad \beta_t(n) = 0, \quad \text{for } \left(\frac{1-24n}{t}\right) = -\left(\frac{1-24m}{t}\right).$$

Proof. Suppose $t = 5, 7$ or 13 , and m is a negative integer such that $24m \not\equiv 1 \pmod{t}$. Suppose $K(z)$ is chosen so that (3.28) holds. Let $S(z)$ be defined as in (3.9), and define

$$(3.32) \quad B(z) := S(z) + \chi_{12}(t) \left(\frac{1-24m}{t}\right) \mathcal{E}_{2,t}(z) K(z),$$

so that

$$(3.33) \quad B(z) | W_t = S^*(z) - \chi_{12}(t) \left(\frac{1-24m}{t}\right) \mathcal{E}_{2,t}(z) K^*(z),$$

where

$$(3.34) \quad S^*(z) = S(z) | W_t.$$

Since $24m \not\equiv 1 \pmod{t}$, we see that

$$(3.35) \quad \text{ord}_0(S(z)) = \text{ord}_{i\infty}(S^*(z)) > 0.$$

From (3.27) and (3.29) we see that

$$(3.36) \quad \text{ord}_0(K(z)) = \text{ord}_{i\infty}(K^*(z)) > 0$$

and hence

$$(3.37) \quad \text{ord}_0(B(z)) > 0.$$

Now

$$(3.38) \quad \text{ord}_{i\infty} \left(\mathcal{E}_{2,t}(tz) K^*(tz) \frac{\eta(z)}{\eta(t^2z)} \right) \geq t - \frac{1}{24}(t^2 - 1) > 0,$$

for $t = 5, 7, 13$. By construction the coefficient of q^m in $B(z)$ is zero and so (3.28), (3.38) imply that

$$(3.39) \quad \text{ord}_{i\infty}(B(z)) \geq 0.$$

Therefore $B(z)$ is an entire modular form of weight 2 and hence a multiple of $\mathcal{E}_{2,t}(z)$ since there are no nontrivial cusp forms of weight 2 on $\Gamma_0(t)$ for $t = 5, 7$ or 13 by [7]. This implies that $B(z)$ is identically zero by (3.37). Hence

$$(3.40) \quad \frac{B(z)}{E(q)} = q^{-st} \mathcal{E}_{2,t}(tz) K^*(tz) \frac{1}{E(q^{t^2})} - \chi_{12}(t) \sum_{n=m}^{\infty} \left(\left(\frac{1-24n}{t}\right) - \left(\frac{1-24m}{t}\right) \right) \beta_t(n)q^n = 0.$$

Since $-24st - 1 \equiv 0 \pmod{t}$, this implies that $\beta_t(n) = 0$ whenever $\left(\frac{1-24n}{t}\right) = -\left(\frac{1-24m}{t}\right)$. □

We illustrate Proposition 3.4 with two examples:

$$(3.41) \quad \sum_{n=-2}^{\infty} \beta_5(n)q^n = \frac{\mathcal{E}_{2,5}(z)}{E(q)} (G_5(z)^2 + 5G_5(z)) \\ = q^{-2} + 1 - 379q^3 + 625q^4 + 869q^5 - 20125q^8 + 23125q^9 + 25636q^{10} - 329236q^{13} + \dots$$

In this example, $t = 5$ and $m = -2$, and we see that $\beta_5(n) = 0$ for $n \equiv 1, 2 \pmod{5}$. In our second example, $t = 7$ and $m = -1$.

(3.42)

$$\sum_{n=-1}^{\infty} \beta_7(n) q^n = \frac{\mathcal{E}_{2,7}(z)}{E(q)} G_7(z)$$

(3.43)

$$= q^{-1} + 1 - 15q^2 + 49q^5 - 24q^6 + 88q^7 - 311q^9 + 392q^{12} - 182q^{13} + 811q^{14} - 1886q^{16} + \dots$$

In this example we see that $\beta_7(n) = 0$ for $n \equiv 1, 3, 4 \pmod{7}$.

The function

(3.44)

$$\begin{aligned} \frac{E_4(z)^2 E_6(z)}{\Delta(z)} &= \frac{E_6(z)}{E_4(z)} j(z) \\ &= q^{-1} - 196884q - 42987520q^2 - 2592899910q^3 - 80983425024q^4 - 1666013203000q^5 + \dots \end{aligned}$$

is a modular form of weight 2 on $\Gamma(1)$. As a modular form on $\Gamma_0(t)$ it has a simple pole at $i\infty$ and a pole of order t at $z = 0$. When $t = 5, 7$ or 13 , it is straightforward to show that there are integers $a_{j,t}$ ($-1 \leq j \leq t$) such that

(3.45)

$$\frac{E_6(z)}{E_4(z)} j(z) = \mathcal{E}_{2,t}(z) \sum_{j=-1}^t a_{j,t} G_t(z)^j.$$

For example,

(3.46)

$$\begin{aligned} \frac{E_6(z)}{E_4(z)} j(z) &= \mathcal{E}_{2,5}(z) (G_5(z) - 3^2 \cdot 5^5 \cdot 7 G_5(z)^{-1} - \cdot 2^3 \cdot 5^8 \cdot 13 G_5(z)^{-2} - \cdot 3^3 \cdot 5^{10} \cdot 7 G_5(z)^{-3} \\ &\quad - 3 \cdot 2^3 \cdot 5^{13} G_5(z)^{-4} - 5^{16} G_5(z)^{-5}). \end{aligned}$$

Reducing (3.45) mod t^c we obtain a weight 2 analogue of [4, Lemma 4].

Lemma 3.5. *We have*

(3.47)

$$\frac{E_6(z)}{E_4(z)} j(z) \equiv \mathcal{E}_{2,5}(z) (G_5(z) + 2 \cdot 31 \cdot 5^5 G_5(z)^{-1}) \pmod{5^8},$$

(3.48)

$$\frac{E_6(z)}{E_4(z)} j(z) \equiv \mathcal{E}_{2,7}(z) G_7(z) \pmod{7^4},$$

(3.49)

$$\frac{E_6(z)}{E_4(z)} j(z) \equiv \mathcal{E}_{2,13}(z) G_{13}(z) \pmod{13^2}.$$

We also need [4, Lemma 4].

Lemma 3.6 (Atkin [4]). *[Atkin [4]] We have*

(3.50)

$$j(z) \equiv G_5(z) + 750 + 3^2 \cdot 7 \cdot 5^5 G_5(z)^{-1} \pmod{5^8},$$

(3.51)

$$j(z) \equiv G_7(z) + 748 \pmod{7^4},$$

(3.52)

$$j(z) \equiv G_{13}(z) + 70 \pmod{13^2}.$$

Remark 3.7. In equation (3.50) we have corrected a misprint in [4, Lemma 4].

To handle the $(t, c) = (5, 6)$ case of Theorem 1.6 we will need

Lemma 3.8.

$$(3.53) \quad \frac{E_6(z)}{E_4(z)} j(z) \equiv \mathcal{E}_{2,5}(z) (G_5(z) + 2 \cdot 5^5 G_5(z)^{-1}) \pmod{5^6},$$

$$(3.54) \quad \frac{E_6(z)}{E_4(z)} j(z)^2 \equiv \mathcal{E}_{2,5}(z) (2 \cdot 3 \cdot 5^3 G_5(z) + G_5(z)^2) \pmod{5^6},$$

and

$$(3.55) \quad \frac{E_6(z)}{E_4(z)} j(z)^a \equiv \mathcal{E}_{2,5}(z) (\varepsilon_{1,a} G_5(z)^{a-2} + \varepsilon_{2,a} G_5(z)^{a-1} + G_5(z)^a) \pmod{5^6},$$

for $a \geq 3$, where $\varepsilon_{1,a}, \varepsilon_{2,a}$ are integers satisfying

$$(3.56) \quad \varepsilon_{1,a} \equiv 0 \pmod{5^5} \quad \text{and} \quad \varepsilon_{2,a} \equiv 0 \pmod{5^3}.$$

Proof. The result can be proved from Lemmas 3.5 and 3.6, some calculation and an easy induction argument. \square

We need bases for $M_{2+12s_\ell}(\Gamma_0(t))$ for $t = 5, 7, 13$. The following result follows from [7] and by checking the modular forms involved are holomorphic at the cusps $i\infty$ and 0 .

Lemma 3.9. *Suppose $t = 5, 7$ or 13 and $\ell > 3$ is prime. Then*

$$(3.57) \quad \dim M_{2+12s_\ell}(\Gamma_0(t)) = 1 + (1+t)s_\ell,$$

and the set

$$(3.58) \quad \{\mathcal{E}_{2,t}(z) \Delta(z)^{s_\ell} G_t(z)^a : -ts_\ell \leq a \leq s_\ell\}$$

is a basis for $M_{2+12s_\ell}(\Gamma_0(t))$.

We are now ready to prove Theorem 1.6. We have two cases:

Case 1. In the first case we assume that $(t, c) = (5, 5), (7, 4)$ or $(13, 2)$. Suppose $\ell > 3$ is prime and $\ell \neq t$. By (2.24) we have

$$(3.59) \quad \mathcal{A}_\ell(z) = \sum_{n=1}^{s_\ell} b_{n,\ell} \frac{E_6(z)}{E_4(z)} \frac{j(z)^n}{\eta(z)}.$$

By Theorem 1.1, equation (2.21) and Lemma 3.9 we have

$$(3.60) \quad \mathcal{A}_\ell(z) = \sum_{n=-ts_\ell}^{s_\ell} d_{n,\ell} \frac{\mathcal{E}_{2,t}(z)}{\eta(z)} G_t(z)^n,$$

for some integers $d_{n,\ell}$ ($-ts_\ell \leq n \leq s_\ell$). Now let

$$(3.61) \quad K(z) = \sum_{n=1}^{s_\ell} d_{n,\ell} G_t(z)^n.$$

By using Lemmas 3.5 and 3.6 to reduce equation (3.59) modulo t^c and comparing the result with (3.60) we deduce that

$$(3.62) \quad \mathcal{A}_\ell(z) \equiv \mathcal{E}_{2,t}(z) \frac{K(z)}{\eta(z)} \pmod{t^c}$$

and that

$$(3.63) \quad d_{n,\ell} \equiv 0 \pmod{t^c},$$

for $-ts_\ell \leq n \leq 0$. By examining (3.60) we see that

$$(3.64) \quad \mathcal{A}_\ell(z) = \mathcal{E}_{2,t}(z) \frac{K(z)}{\eta(z)} + O(q^{-\frac{1}{24}}).$$

So if we let

$$(3.65) \quad \mathcal{E}_{2,t}(z) \frac{K(z)}{\eta(z)} = \sum_{n=-s_\ell}^{\infty} \beta_{t,\ell}(n) q^{n-\frac{1}{24}},$$

then (3.62) may be rewritten as

$$(3.66) \quad \mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell} \right) - 1 - \ell \right) \mathbf{a}(n) + \ell \mathbf{a} \left(\frac{n+s_\ell}{\ell^2} \right) \equiv \beta_{t,\ell}(n) \pmod{t^c}$$

and from (3.64) we have

$$(3.67) \quad \mathbf{a}(\ell^2 n - s_\ell) + \chi_{12}(\ell) \left(\left(\frac{1-24n}{\ell} \right) - 1 - \ell \right) \mathbf{a}(n) + \ell \mathbf{a} \left(\frac{n+s_\ell}{\ell^2} \right) = \beta_{t,\ell}(n)$$

for $-s_\ell \leq n \leq -1$. Equation (3.67) implies that

$$(3.68) \quad \beta_{t,\ell}(-s_\ell) = -\ell$$

and

$$(3.69) \quad \beta_{t,\ell}(n) = 0,$$

for $-s_\ell \leq n \leq -1$. We can now apply Proposition 3.4 with $m = -s_\ell$ since $1 - 24m = \ell^2$ and $t \neq \ell$. Hence

$$(3.70) \quad \beta_{t,\ell}(n) = 0, \quad \text{provided} \quad \left(\frac{1-24n}{t} \right) = -1.$$

This gives Theorem 1.6 when $(t, c) = (5, 5)$, $(7, 4)$ or $(13, 2)$ by (3.66).

Case 2. We consider the remaining case $(t, c) = (5, 6)$ and assume $\ell > 5$ is prime. We proceed as in Case 1. This time when we use Lemma 3.8 to reduce (2.24) modulo 5^6 we see that the only extra term occurs when $n = 1$. We find that with $K(z)$ as before we have

$$(3.71) \quad \mathcal{A}_\ell(z) \equiv \mathcal{E}_{2,t}(z) \frac{K(z)}{\eta(z)} + b_{1,5} \cdot 2 \cdot 5^5 \frac{\mathcal{E}_{2,5}(z)}{\eta(z)} G_5(z)^{-1} \pmod{5^6}.$$

All that remains is to show that

$$(3.72) \quad b_{1,5} \equiv 0 \pmod{5},$$

since then (3.62) actually holds modulo 5^6 and the rest of the proof proceeds as in Case 1. Since $E_4(z) \equiv 1 \pmod{5}$ we may reduce (2.18) modulo 5 to obtain

$$(3.73) \quad \ell \Xi_\ell \equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell}) \frac{E_6(z)}{E_4(z)} \frac{1}{\eta(z)} - \sum_{n=1}^{s_\ell} (24n+1) c_{n,\ell} \frac{E_6(z)}{E_4(z)} \frac{j(z)^n}{\eta(z)} \pmod{5}.$$

But

$$(3.74) \quad \mathcal{S}_\ell(z) \equiv 0 \pmod{5},$$

by Theorem 1.3 (ii). Hence

$$(3.75) \quad \ell \mathcal{A}_\ell(z) \equiv (\chi_{12}(\ell)\ell(1+\ell) - c_{0,\ell}) \frac{E_6(z)}{E_4(z)} \frac{1}{\eta(z)} - \sum_{n=1}^{s_\ell} (24n+1) c_{n,\ell} \frac{E_6(z)}{E_4(z)} \frac{j(z)^n}{\eta(z)} \pmod{5}$$

and we see that $b_{1,5}$ the coefficient of $\frac{E_6(z)}{E_4(z)} \frac{j(z)}{\eta(z)}$ is divisible by 5 as required. This completes the proof of Theorem 1.6.

We close by illustrating Theorem 1.6 when $t = 5$ and $\ell = 7$. In this case the theorem predicts that

$$\mathbf{a}(49n - 2) - \left(\frac{1 - 24n}{7}\right) \mathbf{a}(n) + 7\mathbf{a}\left(\frac{n + 2}{49}\right) \equiv -8\mathbf{a}(n) \pmod{5^6},$$

when $n \equiv 1, 2 \pmod{5}$. When $n = 1$ this says

$$149077845 \equiv -280 \pmod{5^6},$$

which is easy to check.

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