

**A GENERALIZATION OF THE
HIRSCHHORN-FARKAS-KRA SEPTAGONAL NUMBERS
IDENTITY**

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ABSTRACT. We provide elementary proofs of the Farkas-Kra septagonal numbers identity and some k -th order generalizations.

1. INTRODUCTION

Two important combinatorial identities are the Jacobi triple product identity

$$\prod_{n=1}^{\infty} (1 - zq^{n-1})(1 - z^{-1}q^n)(1 - q^n) = \sum_{n \in \mathbb{Z}} (-1)^n z^n q^{n(n-1)/2} =: \theta(z, q),$$

and the quintuple product identity

$$\begin{aligned} & \prod_{n=1}^{\infty} (1 - zq^{n-1})(1 - z^{-1}q^n)(1 - z^2q^{2n-1})(1 - z^{-2}q^{2n-1})(1 - q^n) \\ &= \sum_{n \in \mathbb{Z}} (z^{-3n} - z^{3n+1})q^{n(3n+1)/2}. \end{aligned}$$

The first combinatorial proof of the triple product identity is due to Sylvester [14]. Other more recent proofs have been found by Andrews [2] and Lewis [10]. The quintuple product identity is usually attributed to Watson [15]. However it can be found in Ramanujan's lost notebook [12, p. 207]. Also see [3, p.83], [4], [8] and [6] for more history and some proofs. Recent proofs have been found by Alladi [1], by Hirschhorn [9], and by Farkas and Kra [5]. The triple and quintuple products are special cases of the Macdonald [11] identities for affine root systems. See Stanton [13] for an elementary treatment. The triple product corresponds to the root system A_1 , and the quintuple product corresponds to the root system BC_1 .

Farkas and Kra [5] considered a generalization of the quintuple product identity different from the BC_n case of the Macdonald identities. The exponent of q in the right side of the quintuple product identity is the pentagonal number $n(3n+1)/2$. They obtained the following identity involving the septagonal number $n(5n+1)/2$:

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$$\begin{aligned}
(1.1) \quad & \prod_{n=1}^{\infty} (1 - zq^{n-1})(1 - z^{-1}q^n)(1 - z^2q^{n-1})(1 - z^{-2}q^n)(1 - q^n)^2 \\
& = J_0(q) \sum_{n \in \mathbb{Z}} (-1)^n (z^{5n+3} + z^{-5n}) q^{n(5n+3)/2} \\
& \quad - J_1(q) \sum_{n \in \mathbb{Z}} (-1)^n (z^{5n+2} + z^{1-5n}) q^{n(5n+1)/2},
\end{aligned}$$

where

$$J_0(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(5n+1)/2}, \quad J_1(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(5n+3)/2}.$$

Earlier, this identity had been discovered independently by Hirschhorn [7, (3.1)]. Observe that the left side of the identity above is the product of two theta functions. Farkas and Kra's proof of this identity involves the function theory of theta functions with rational characteristics. Foata and Han [6] have found an elementary proof. Their proof utilizes both the triple product and quintuple product identities. In the theorem below we provide a generalization of the septagonal identity and our proof depends only on the triple product identity.

Farkas and Kra found that certain k fold products of theta functions have similar expressions. In particular, for any odd $k \geq 3$ we consider the function

$$(1.2) \quad f(z, q) := \theta(z^2, q)\theta(z, q)^{k-4}.$$

Farkas and Kra [5, p.776] showed that there exist functions $c_j(q)$ such that

$$(1.3) \quad f(z, q) = \sum_{j=0}^{(k-3)/2} c_j(q) \varphi_j(z, q),$$

where each $\varphi_j(z, q)$ is the sum of two certain theta functions. Our main result below gives explicit expressions for the $c_j(q)$ in this case, provided $k \geq 5$. We note that the case $k = 3$ corresponds to the quintuple product identity.

Theorem 1.1. *Let $k \geq 5$ be odd, and $\ell = (k - 5)$. Then*

$$\begin{aligned}
(1.4) \quad & \prod_{n=1}^{\infty} (1 - q^n)^{k-3} (1 - z^2q^{n-1})(1 - z^{-2}q^n) [(1 - zq^{n-1})(1 - z^{-1}q^n)]^{k-4} \\
& = \sum_{r=0}^{(k-3)/2} (-1)^r J_r(q) \left(\sum_{j=-\infty}^{\infty} (-1)^j z^{kj+r} q^{j(kj+2r+2-k)/2} \right. \\
& \quad \left. + \sum_{j=-\infty}^{\infty} (-1)^j z^{kj+k-2-r} q^{j(kj-2r-2+k)/2} \right),
\end{aligned}$$

where for $r = 0, 1, 2, \dots, k-1$,

$$(1.5) \quad J_r(q) := \sum_{\vec{n} \in \mathbb{Z}^{k-4}} (-1)^{n_0} q^{Q_r(\vec{n})},$$

$$\vec{n} = (n_0, n_1, n_2, \dots, n_\ell), \quad \vec{1} = (1, 1, 1, \dots, 1),$$

$$Q_r(\vec{n}) := \Delta(r - n_0 - \vec{n} \cdot \vec{1}) + \Delta(n_0) + \Delta(n_1) + \Delta(n_2) + \dots + \Delta(n_\ell),$$

and

$$\Delta(n) := \frac{1}{2}n(n-1).$$

The case $k = 5$ corresponds to the Farkas-Kra septagonal identity (1.1).

2. PROOF OF THE MAIN RESULT

It will be useful to record some elementary properties of $\Delta(n)$:

$$(2.1) \quad \Delta(n+m) = \Delta(n) + \Delta(m) + nm$$

$$(2.2) \quad \Delta(-n+a) = \Delta(n) + \Delta(a) + n(1-a)$$

where m , n and a are any integers.

We proceed as follows:

- Step 1. First we expand the left side of (1.4) using Jacobi's triple product identity and obtain a multiple q -series.
- Step 2. We split the resulting series according the residue of the exponent of z modulo k , thus obtaining k functions of z . Then we change variables in the summation so that we can factor a single theta-series out of each of the k series.
- Step 3. We use an affine linear transformation to show that the coefficient of one of the k series is zero.
- Step 4. We use another affine linear transformation to simplify the final identity.

Step 1. Using Jacobi's triple product identity we obtain

$$\begin{aligned} & \prod_{n=1}^{\infty} (1-q^n)^{k-3} (1-z^2 q^{n-1}) (1-z^{-2} q^n) [(1-zq^{n-1})(1-z^{-1}q^n)]^{k-4} \\ &= \sum_{n \in \mathbb{Z}} (-1)^n z^{2n} q^{\Delta(n)} \sum_{m \in \mathbb{Z}} (-1)^m z^m q^{\Delta(m)} \\ & \quad \cdot \sum_{\vec{n}=(n_1, n_2, \dots, n_\ell) \in \mathbb{Z}^\ell} (-1)^{n_1+n_2+\dots+n_\ell} z^{n_1+n_2+\dots+n_\ell} q^{\Delta(n_1)+\Delta(n_2)+\dots+\Delta(n_\ell)} \end{aligned}$$

Step 2. In the summation we let $m = kj + r - 2n - n_1 - n_2 - \dots - n_\ell$ where j and r are integers with $0 \leq r \leq (k-1)$. Then we change variables in the summation:

$$n \mapsto n + 2j, \quad n_1 \mapsto n_1 + j, \quad n_2 \mapsto n_2 + j, \quad \dots, \quad n_\ell \mapsto n_\ell + j$$

and obtain

$$\begin{aligned}
& \prod_{n=1}^{\infty} (1-q^n)^{k-3} (1-z^2 q^{n-1}) (1-z^{-2} q^n) [(1-zq^{n-1})(1-z^{-1}q^n)]^{k-4} \\
&= \sum_{r=0}^{k-1} \sum_{j \in \mathbb{Z}} (-1)^{n+j+r} z^{kj+r} \\
&\quad \cdot \sum_{n, n_1, n_2, \dots, n_\ell \in \mathbb{Z}} q^{\Delta(kj+r-2n-n_1-n_2-\dots-n_\ell)+\Delta(n)+\Delta(n_1)+\Delta(n_2)+\dots+\Delta(n_\ell)} \\
&= \sum_{r=0}^{k-1} \sum_{j \in \mathbb{Z}} (-1)^{n+j+r} z^{kj+r} \\
&\quad \cdot \sum_{n, n_1, n_2, \dots, n_\ell \in \mathbb{Z}} q^{\Delta(j+r-2n-n_1-n_2-\dots-n_\ell)+\Delta(n+2j)+\Delta(n_1+j)+\Delta(n_2+j)+\dots+\Delta(n_\ell+j)}
\end{aligned}$$

Now using (2.1) the exponent of q in the summation above may be written

$$\begin{aligned}
& \Delta(j+r-2n-n_1-n_2-\dots-n_\ell) + \Delta(n+2j) \\
&+ \Delta(n_1+j) + \Delta(n_2+j) + \dots + \Delta(n_\ell+j) \\
&= (\ell+1)\Delta(j) + \Delta(2j) + jr + \Delta(r-2n-n_1-n_2-\dots-n_\ell) + \Delta(n) \\
&\quad + \Delta(n_1) + \Delta(n_2) + \dots + \Delta(n_\ell), \\
&= j(kj+2r+2-k)/2 + Q(n, n_1, n_2, \dots, n_\ell).
\end{aligned}$$

In this way we obtain

$$\begin{aligned}
& \prod_{n=1}^{\infty} (1-q^n)^{k-3} (1-z^2 q^{n-1}) (1-z^{-2} q^n) [(1-zq^{n-1})(1-z^{-1}q^n)]^{k-4} \\
&= \sum_{r=0}^{k-1} \sum_{j \in \mathbb{Z}} (-1)^{j+r} z^{kj+r} q^{j(kj+2r+2-k)/2} J_r(q),
\end{aligned}$$

where $J_r(q)$ is defined in (1.5).

Step 3. We show

$$(2.3) \quad J_{k-1} = 0.$$

Now

$$J_{k-1}(q) = \sum_{\vec{n}=(n_0, n_1, n_2, \dots, n_\ell) \in \mathbb{Z}^{\ell+1}} (-1)^{n_0} q^{Q_{k-1}(\vec{n})}.$$

In the summation we change variables:

$$n_0 \mapsto -n_0 + 3, \quad n_1 \mapsto -n_1 + 2, \quad n_2 \mapsto -n_2 + 2, \quad \dots, \quad n_\ell \mapsto -n_\ell + 2$$

Under this transformation

$$(2.4) \quad (-1)^{n_0} \mapsto -(-1)^{n_0}$$

$$(2.5) \quad Q_{k-1}(\vec{n}) \mapsto Q_{k-1}(\vec{n})$$

(2.4) is immediate. (2.5) follows from a calculation using (2.2). From (2.4) and (2.5) it follows that

$$J_{k-1}(q) = -J_{k-1}(q),$$

and hence $J_{k-1}(q) = 0$.

Step 4. We show that

$$(2.6) \quad J_r(q) = -J_{k-2-r}(q),$$

for $r = 0, 1, \dots, (k-3)/2$. This time we consider the transformation

$$\vec{n} \longmapsto -\vec{n} + \vec{1}.$$

The result follows by showing under this transformation

$$(2.7) \quad (-1)^{n_0} \longmapsto -(-1)^{n_0}$$

$$(2.8) \quad Q_r(\vec{n}) \longmapsto Q_{k-2-r}(\vec{n})$$

(2.7) is immediate. (2.8) follows from an easy calculation utilizing (2.2) with $a = 1$.

This completes the proof of (1.4).

3. REMARKS

What makes our result interesting are the two properties of $J_r(q)$, namely (2.3) and (2.6) and the fact that we were able to explicitly identify the coefficient functions. We only considered products of the form given in (1.2). Farkas and Kra [5] considered more general products. It would be interesting to investigate which other theta products had nice evaluations and if there are any connections to the Macdonald identities [11].

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