A BETA INTEGRAL ASSOCIATED WITH THE ROOT SYSTEM G_2*

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Abstract. Some conjectures of Askey are proven that have to do with adding roots in the Macdonald-Morris conjecture for G_2. This is done by extending Aomoto's proof of Selberg's integral. This yields a new proof of the Macdonald-Morris root system conjecture for G_2 which should extend to other root systems.

1. Introduction. Let

(1.1)
$$G(x_1,x_2;a,b) = (1-x_1)^a\left(1-\frac{1}{x_1}\right)^a(1-x_2)^a\left(1-\frac{1}{x_2}\right)^a(1-x_1x_2)^a\left(1-\frac{1}{x_1x_2}\right)^a \cdot\left(1-\frac{x_1}{x_2}\right)^b\left(1-\frac{x_2}{x_1}\right)^b(1-x_1^2x_2)^b\left(1-\frac{1}{x_1^2x_2}\right)^b(1-x_2^2x_1)^b\left(1-\frac{1}{x_2^2x_1}\right)^b.$$

Then the Macdonald-Morris root system conjecture for G_2 is

$$\text{C.T. } G(x_1,x_2;a,b) = \frac{(3a+3b)!(3b)!(2a)!(2b)!}{(2a+3b)!(a+2b)!(a+b)!a!b!b!} = g(a,b). \tag{1.2}$$

Here C.T. means the constant term in the Laurent expansion as a polynomial in x_1, x_1^{-1}, x_2, x_2^{-1}. This has been proved independently by Habsieger [5] and Zeilberger [12]. They have also proved the q-analogue of (1.2). Although their proofs are elegant, they are special to G_2. Recently Zeilberger [13] has also proved the $G_2^\vee$ case of the Macdonald-Morris conjectures. His proof should extend to other root systems.

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In this paper we give a new proof of (1.2) which is entirely in terms of integrals and which should also extend to other root systems. Our proof is an extension of Aomoto's [1] proof of Selberg's [11] integral. See Askey [3] for a good exposition of Aomoto's proof. We were led to this by considering conjectures of Askey [3] that have to do with adding roots in the Macdonald-Morris root system conjecture for G_2. Askey conjectured

$$\text{C.T. } (1-x_1)\left(1-\frac{1}{x_1}\right)G(x_1,x_2;a,b) = \frac{2(3a+3b+1)}{2a+3b+1}g(a,b), \tag{1.3}$$

$$\begin{aligned}\text{C.T. } (1-x_1)\left(1-\frac{1}{x_1}\right)(1-x_2)\left(1-\frac{1}{x_2}\right)G(x_1,x_2;a,b) \\ = \frac{2(3a+3b+2)(3a+3b+1)}{(2a+3b+1)(a+2b+1)}g(a,b),\end{aligned} \tag{1.4}$$

$$\text{C.T. } (1-x_1^2x_2)\left(1-\frac{1}{x_1^2x_2}\right)G(x_1,x_2;a,b) = \frac{2(3a+3b+1)(3b+1)}{(2a+3b+1)(a+2b+1)}g(a,b), \tag{1.5}$$

$$\begin{aligned}\text{C.T. } (1-x_1^2x_2)\left(1-\frac{1}{x_1^2x_2}\right)(1-x_2^2x_1)\left(1-\frac{1}{x_2^2x_1}\right)G(x_1,x_2;a,b) \\ = \frac{6(3a+3b+2)(3a+3b+1)(3b+1)(3b+2)}{(2a+3b+3)(2a+3b+2)(2a+3b+1)(a+2b+1)}g(a,b).\end{aligned} \tag{1.6}$$

In §2 we give the idea of the proof of (1.2). It is well known that (1.2) may be written as a trigonometric integral formula (see, for example, Morris [10, p.46]). The starting point of our proof is to write (1.2) as

$$\begin{aligned}\frac{4^{3a+3b}}{\pi^2}\int\limits_{\mathbb{R}^2}\frac{(t_1+t_2)^{2a}(t_1-t_2)^{2b}(t_2^2+2t_1t_2-1)^{2b}(t_1^2+2t_1t_2-1)^{2b}}{(1+t_1^2)^{2a+4b+1}(1+t_2^2)^{2a+4b+1}}dt_1dt_2 \\ = g(a,b).\end{aligned} \tag{1.7}$$

The left-hand side of (1.7) is the integral referred to in the title of this paper. This is done via three changes of variables: first, by letting $x_j = e^{2i\theta_j}$ $(j=1,2)$ in (1.1) and using the orthogonality of the exponentials on $[0,\pi]$ to obtain an integral on $[0,\pi]^2$; second, by linear change of variables to obtain an integral on $[-\frac{\pi}{2},\frac{\pi}{2}]^2$; and finally by letting $t_j = \tan\theta_j$ to obtain the integral over $\mathbb{R}^2$. We note that this is the same change of variables that Morris used in transforming his constant terms formula for A_n [10, p. 95] into the Cauchy-Selberg integral [10, (6.6)]. The advantage of this integral over the trigonometric integrals is that the integrand is a rational function of t_1, t_2, which can be easily manipulated using a computer algebra package like REDUCE. In §3 we prove some preliminary results. In §4 we complete the proof of (1.2) and prove (1.3)–(1.5) as well.

Macdonald [8, conjecture(6.1)] has also conjectured generalizations of Mehta's integral formula for arbitary root systems [9], [8, (4.1)]. The G_2 case does not seem to be related to our integral given in (1.7). The Macdonald-Mehta integral conjecture involves parameters that are constant on root length. We note that the two parameter case of the G_2 Macdonald-Mehta integral may be proved in the same way as the one parameter case, which was proved by Macdonald [8, p.1002]. This is done by transforming to polar coordinates. The resulting integral turns out to be the product of a gamma integral and a beta integral.

We should mention that (1.3)–(1.6) and their q-analogues may be proved by other methods. Zeilberger [12] proved (1.2) using the result of Morris, mentioned above, related to A_n and Dixon's [4, §3.1] summation of a well-poised ${}_3F_2$. Equations (1.3)–(1.6) could be proved by trying to generalize Morris's results and Dixon's summation. See Kadell [6] for such generalizations of q-analogues of Morris's results and Askey [2] for some extensions of Dixon's summation. The author has proved (1.3)–(1.6), as well as other similar results, by these methods. In §4 we state these other results without proof. For the most part we restrict attention to (1.2)–(1.5), preferring to take a more direct approach.

Kadell [7] has found yet another approach to proving (1.2) which should extend to other root systems. This involves working with the function $G(x_1, x_2; a, b)$ directly rather than writing it as an integral, and using the fact that derivatives have no residues. Finally, (1.2)–(1.6) could be proved by extending Zeilberger's [13] method for the $G_2^\vee$ case and then letting $q \to 1$.

2. The idea of the proof. We label the roots of G_2 as in Fig. 1.

FIG. 1

Let

$$g'(a,b) = \text{C.T. } G(x_1,x_2;a,b). \tag{2.1}$$

Our goal is to prove that $g'(a,b) = g(a,b)$ for all $a, b \geq 0$. The idea is to proceed by induction on a. However to jump from a to $a+1$ in one step would be too much to ask for. Considering (1.3) and (1.4) we break it up into three stages:

Stage 1.

$$\text{C.T. } [\alpha_1]G = \frac{2(3a+3b+1)}{2a+3b+1} g'(a,b).$$

Stage 2.

$$\text{C.T. } [\alpha_1][\alpha_2]G = \frac{2(3a+3b+1)(3a+3b+2)}{(2a+3b+1)(a+2b+1)} g'(a,b).$$

Stage 3.

$$\text{C.T. } [\alpha_1][\alpha_2][\alpha_1+\alpha_2]G = \frac{6(3a+3b+2)(3a+3b+1)(2a+1)}{(2a+3b+2)(2a+3b+1)(a+2b+1)} g'(a,b)$$

where

$$[k_1\alpha_1 + k_2\alpha_2] = (1 - x_1^{k_1}x_2^{k_2})(1 - x_1^{-k_1}x_2^{-k_2}),$$

for $k_1, k_2 \in \mathbb{Z}$. Each stage corresponds to adding an additional pair of opposite short roots. After Stage 3 all that will remain is to prove the result for $a = 0$ since

$$\frac{g(a+1,b)}{g(a,b)} = \frac{6(3a+3b+2)(3a+3b+1)(2a+1)}{(2a+3b+2)(2a+3b+1)(a+2b+1)}. \tag{2.2}$$

The case $a = 0$ is equivalent to $b = 0$ since we have

$$\text{Long roots of } G_2 \cong \text{Short roots of } G_2 \cong A_2.$$

The result is trivially true for $a = b = 0$. The case $b = 0$ follows from (2.2) and Stage 3 by induction.

To prove Stages 1–3 we rewrite each stage as an integral and use an idea of Aomoto. To give the reader a taste of our method we work through the proof of Stage 1. If we let

(2.3)
$$w(\underset{\sim}{t}) = w(t_1, t_2; a, b) = \frac{(t_1+t_2)^{2a}(t_1-t_2)^{2b}(t_2^2+2t_1t_2-1)^{2b}(t_1^2+2t_1t_2-1)^{2b}}{(1+t_1^2)^{2a+4b+1}(1+t_2^2)^{2a+4b+1}},$$

then we may write Stage 1 as

$$\frac{4^{3a+3b}}{\pi^2}\int\limits_{\mathbb{R}^2}\frac{4}{(1+t_1^2)}\,w(\underset{\sim}{t})\,d\underset{\sim}{t} = \frac{2(3a+3b+1)}{(2a+3b+1)}\,\frac{4^{3a+3b}}{\pi^2}\int\limits_{\mathbb{R}^2}w(\underset{\sim}{t})\,d\underset{\sim}{t}, \tag{2.4}$$

using the same change of variables used to derive (1.7). It is here that we use Aomoto's idea. To get $\int\limits_{\mathbb{R}^2}\frac{1}{1+t_1^2}\,w(\underset{\sim}{t})\,d\underset{\sim}{t}$ in terms of $\int\limits_{\mathbb{R}^2}w(\underset{\sim}{t})\,d\underset{\sim}{t}$ we use

(2.5)
$$\begin{aligned} 0 &= \int\limits_{\mathbb{R}^2}\frac{\partial}{\partial t_1}\,t_1 w(\underset{\sim}{t})\,d\underset{\sim}{t} \\ &= \int\limits_{\mathbb{R}^2}w(\underset{\sim}{t})\,d\underset{\sim}{t} - (2a+4b+1)\int\limits_{\mathbb{R}^2}\frac{2t_1^2}{1+t_1^2}\,w(\underset{\sim}{t})\,d\underset{\sim}{t} + 2a\int\limits_{\mathbb{R}^2}\frac{t_1}{t_1+t_2}\,w(\underset{\sim}{t})\,d\underset{\sim}{t} \\ &\quad + 2b\int\limits_{\mathbb{R}^2}\frac{t_1}{t_1-t_2}\,w(\underset{\sim}{t})\,d\underset{\sim}{t} + 2b\int\limits_{\mathbb{R}^2}\frac{2t_1t_2}{t_2^2+2t_1t_2-1}\,w(\underset{\sim}{t})\,d\underset{\sim}{t} \\ &\quad + 2b\int\limits_{\mathbb{R}^2}\frac{2t_1(t_1+t_2)}{t_1^2+2t_1t_2-1}\,w(\underset{\sim}{t})\,d\underset{\sim}{t}. \end{aligned}$$

We can make some progress by using the fact that $w(\underset{\sim}{t})$ is invariant under the transposition $t_1 \leftrightarrow t_2$.

$$\frac{t_1}{t_1 \pm t_2} = 1 \mp \frac{t_2}{t_1 \pm t_2}. \tag{2.6}$$

It follows that

$$\int\limits_{\mathbb{R}^2}\frac{t_1}{t_1 \pm t_2}\,w(\underset{\sim}{t})\,d\underset{\sim}{t} = \frac{1}{2}\int\limits_{\mathbb{R}^2}w(\underset{\sim}{t})\,d\underset{\sim}{t}. \tag{2.7}$$

$$\begin{aligned}(2.8)\qquad &\int_{\mathbb{R}^2} \frac{t_1t_2}{t_2^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} + \int_{\mathbb{R}^2} \frac{t_1(t_1+t_2)}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} \\ &= \int_{\mathbb{R}^2} \frac{t_1t_2}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} + \int_{\mathbb{R}^2} \frac{t_1(t_1+t_2)}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} \\ &\qquad\qquad \text{(via } t_1 \leftrightarrow t_2 \text{ on the first integral)} \\ &= \int_{\mathbb{R}^2} w(\underset{\sim}{t})\, d\underset{\sim}{t} + \int_{\mathbb{R}^2} \frac{1}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t}.\end{aligned}$$

Then (2.5) becomes

(2.9)
$$\begin{aligned}0 = (-3a-3b-1)\int_{\mathbb{R}^2} w(\underset{\sim}{t})\, d\underset{\sim}{t} + 2(2a+4b+1)\int_{\mathbb{R}^2} \frac{1}{1+t_1^2}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} \\ + 4b\int_{\mathbb{R}^2} \frac{1}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t}.\end{aligned}$$

The problem that remains is to get $\int_{\mathbb{R}^2} \frac{1}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t}$ in terms of $\int_{\mathbb{R}^2} w(\underset{\sim}{t})\, d\underset{\sim}{t}$ and $\int_{\mathbb{R}^2} \frac{1}{1+t_1^2}\, w(\underset{\sim}{t})\, d\underset{\sim}{t}$. The transformation $t_1 \leftrightarrow t_2$ is not helpful here. We need another transformation that leaves $w(\underset{\sim}{t})\, d\underset{\sim}{t}$ invariant. The real reason why $w(\underset{\sim}{t})$ is invariant under $t_1 \leftrightarrow t_2$ is that the root system G_2 is invariant under the linear transformation given by

$$\alpha_1 \longmapsto \alpha_2 \qquad \text{and} \qquad \alpha_2 \longmapsto \alpha_1,$$

which is also the reflection through the plane orthogonal to the vector $\alpha_2 - \alpha_1$. Recall that a root system is invariant under any element of the Weyl group, the group generated by the w_α, where α is a root and w_α is the reflection through the hyperplane orthogonal to α. The Weyl group for G_2 is generated by $w_{\alpha_2-\alpha_1}$ and w_{α_1}. The extra integral transformation that we need will correspond to w_{α_1}. We study the action of this reflection on the roots:

$$\begin{aligned} &\alpha_1 \leftrightarrow -\alpha_1, && \alpha_2 \leftrightarrow \alpha_1+\alpha_2, \\ &\alpha_2-\alpha_1 \leftrightarrow 2\alpha_1+\alpha_2, && \alpha_1+2\alpha_2 \leftrightarrow \alpha_1+2\alpha_2. \end{aligned}$$

We need a transformation

$$(2.10)\qquad f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad (t_1,t_2) \longmapsto (f_1(t_1,t_2), f_2(t_1,t_2))$$

with the following action:

$$\frac{1}{1+t_1^2} \leftrightarrow \frac{1}{1+t_1^2}, \quad \frac{1}{1+t_2^2} \leftrightarrow \frac{(t_1+t_2)^2}{(1+t_1^2)(1+t_2)^2},$$

(2.11)

$$\frac{(t_1-t_2)^2}{(1+t_1^2)(1+t_2^2)} \leftrightarrow \frac{(t_1^2+2t_1t_2-1)^2}{(1+t_1^2)^2(1+t_2^2)}, \quad \frac{(t_2^2+2t_1t_2-1)^2}{(1+t_1^2)(1+t_2^2)^2} \leftrightarrow \frac{(t_2^2+2t_1t_2-1)^2}{(1+t_1^2)(1+t_2^2)^2}.$$

The transformation that does the job is

$$f_1(t_1,t_2) = t_1, \qquad f_2(t_1,t_2) = \frac{1-t_1t_2}{t_1+t_2} \qquad (t_1 \neq -t_2). \tag{2.12}$$

In §3 we show that

$$\int_{\mathbb{R}^2} g(\underset{\sim}{t}) w(\underset{\sim}{t})\, d\underset{\sim}{t} = \int_{\mathbb{R}^2} g(f(\underset{\sim}{t})) w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{2.13}$$

for a certain restricted class of functions g.

Now we return to the problem of evaluating $\int_{\mathbb{R}^2} \frac{1}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t}$. A routine calculation shows that

$$\frac{1}{t_1^2+2t_1t_2-1} \xrightarrow{f} \frac{(t_1+t_2)}{(1+t_1^2)(t_1-t_2)}, \tag{2.14}$$

$$\frac{(t_1+t_2)}{(1+t_1^2)(t_1-t_2)} = -\frac{(t_1+t_2)^2}{(1+t_1^2)(1+t_2^2)} + \frac{(t_1+t_2)}{(1+t_2^2)(t_1-t_2)}. \tag{2.15}$$

It follows that

(2.16)

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{1}{t_1^2+2t_1t_2-1}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} &= \int_{\mathbb{R}^2} \frac{(t_1+t_2)}{(1+t_1^2)(t_1-t_2)}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\qquad \text{(by (2.13) and (2.14))} \\
&= -\frac{1}{2} \int_{\mathbb{R}^2} \frac{(t_1+t_2)^2}{(1+t_1^2)(1+t_2^2)}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\qquad \text{(by applying } t_1 \leftrightarrow t_2 \text{ and using (2.15))} \\
&= -\frac{1}{2} \int_{\mathbb{R}^2} \frac{1}{1+t_1^2}\, w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\qquad \text{(by applying } f \text{, then } t_1 \leftrightarrow t_2).
\end{aligned}$$

Substituting this into (2.9) gives

$$(3a+3b+1)\int\limits_{\mathbb{R}^2} w(\underset{\sim}{t})\,d\underset{\sim}{t} = 2(2a+3b+1)\int\limits_{\mathbb{R}^2} \frac{1}{1+t_1^2}\,w(\underset{\sim}{t})\,d\underset{\sim}{t}$$

and Stage 1 follows. The proof of Stages 2–3 is analogous and will be given in §4.

3. Preliminary results. The main result of this section is Lemma 3.6. It contains a list of integrals that we will need in the proof of Stages 2–3. In the proof of this lemma we will use the transformation formula (2.13) for f. A more formal statement of this formula is given in Lemma 3.2. The idea of the proof of Lemma 3.2 is to write both sides of (2.13) as an integral over $[0,\pi]^2$, using the same change of variables mentioned after (1.7) in the introduction, use the transformation $T(\theta_1,\theta_2) = (-\theta_1, \theta_1+\theta_2)$ and apply Lemma 3.1. The proofs of Lemmas 3.1 and 3.2 are omitted. In the proof of Lemma 3.6 we will also need to calculate the image of certain rational functions in t_1, t_2 under f. This was done using the computer algebra package REDUCE.

Lemma 3.1. *Let* $h : [0,\pi]^2 \longrightarrow \mathbb{R}$ *be continuous; then*

$$\int\limits_{[0,\pi]^2} h(\theta_1,\theta_2)\,d\theta_1 d\theta_2 = \int\limits_{[0,\pi]^2} h^*(\theta_1,\theta_2)\,d\theta_1 d\theta_2$$

where

$$h^*(\theta_1,\theta_2) = \begin{cases} h(\pi-\theta_1, \theta_1+\theta_2) & \text{if} \quad 0 \le \theta_1+\theta_2 \le \pi, \\ h(\pi-\theta_1, \theta_1+\theta_2-\pi) & \text{if} \quad \pi < \theta_1+\theta_2 \le 2\pi. \end{cases}$$

Let

$$w_0(\underset{\sim}{t}) = (1+t_1^2)(1+t_2^2)w(\underset{\sim}{t}).$$

Lemma 3.2. *Suppose* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is a continuous function that satisfies*

(i) gw_0 *is bounded on* $\mathbb{R}^2$,

(ii) $(g\circ f)w_0$ *can be extended to a continuous function on* $\mathbb{R}^2$,

where f *is defined in* (2.10) *and* (2.12). *Then*

$$\int\limits_{\mathbb{R}^2} g(\underset{\sim}{t})\,w(\underset{\sim}{t})\,d\underset{\sim}{t} = \int\limits_{\mathbb{R}^2} g(f(\underset{\sim}{t}))\,w(\underset{\sim}{t})\,d\underset{\sim}{t}. \tag{3.3}$$

For notational convenience we let

$$<\alpha_1> = \frac{1}{1+t_1^2}, \quad <\alpha_2> = \frac{1}{1+t_2^2}, \quad <\alpha_1+\alpha_2> = \frac{(t_1+t_2)^2}{(1+t_1^2)(1+t_2^2)}. \tag{3.4}$$

This notation is related to the notation introduced in Stages 1–3. Using the same change of variables used to derive (1.7) we have

(3.5)
$$\text{C.T.}[\alpha_1]^{k_1}[\alpha_2]^{k_2}[\alpha_1+\alpha_2]^{k_3}G = \frac{4^{3a+3b+k_1+k_2+k_3}}{\pi^2}\int_{\mathbb{R}^2} <\alpha_1>^{k_1}<\alpha_2>^{k_2}<\alpha_1+\alpha_2>^{k_3} w(\underset{\sim}{t})\,d\underset{\sim}{t},$$

where $w(\underset{\sim}{t})$ is defined in (2.3) and k_1, k_2, k_3 are nonnegative integers.

Lemma 3.6.

$$\int_{\mathbb{R}^2} (t_2^2+2t_1t_2-1)<\alpha_1> w(\underset{\sim}{t})\,d\underset{\sim}{t} = 0, \tag{3.7}$$

$$\int\limits_{\mathbb{R}^2} \frac{t_1}{t_1+t_2} <\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t} = \frac{5}{4}\int\limits_{\mathbb{R}^2} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} - \int\limits_{\mathbb{R}^2} <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{3.8}$$

$$\int\limits_{\mathbb{R}^2} \frac{t_1}{t_1-t_2} <\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t} = \frac{3}{4}\int\limits_{\mathbb{R}^2} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{3.9}$$

$$\int\limits_{\mathbb{R}^2} \frac{t_2(t_1+t_2)}{t_2^2+2t_1t_2-1} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} = \frac{1}{2}\int\limits_{\mathbb{R}^2} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{3.10}$$

$$\int\limits_{\mathbb{R}^2} \frac{1-t_1t_2}{t_1^2+2t_1t_2-1} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} = -\frac{3}{4}\int\limits_{\mathbb{R}^2} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{3.11}$$

$$\int\limits_{\mathbb{R}^2} \frac{1}{t_1^2+2t_1t_2-1} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} = -\int\limits_{\mathbb{R}^2} <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{3.12}$$

$$\int\limits_{\mathbb{R}^2} <\alpha_1><\alpha_2><\alpha_1+\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t} = -\frac{3}{4}\int\limits_{\mathbb{R}^2} <\alpha_1>^2 w(\underset{\sim}{t})\, d\underset{\sim}{t} + \frac{3}{2}\int\limits_{\mathbb{R}^2} <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{3.13}$$

$$\int\limits_{\mathbb{R}^2} \frac{1}{t_2^2+2t_1t_2-1} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} = -\frac{1}{2}\int\limits_{\mathbb{R}^2} <\alpha_1>^2 w(\underset{\sim}{t})\, d\underset{\sim}{t}, \tag{3.14}$$

$$\int\limits_{\mathbb{R}^2} \frac{t_1t_2}{t_2^2+2t_1t_2-1} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} = \frac{1}{2}\int\limits_{\mathbb{R}^2} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} - \frac{1}{2}\int\limits_{\mathbb{R}^2} <\alpha_1>^2 w(\underset{\sim}{t})\, d\underset{\sim}{t}. \tag{3.15}$$

Proof. Equation (3.7) follows from (3.3) and

$$(t_2^2+2t_1t_2-1) <\alpha_1><\alpha_2> \xrightarrow{\ f\ } -(t_2^2+2t_1t_2-1) <\alpha_1><\alpha_2> .$$

We note that $(t_2^2+2t_1t_2-1) <\alpha_1><\alpha_2>$ satisfies the conditions of Lemma 3.2 since it is equal to $<\alpha_1+\alpha_2> - <\alpha_2>$. Whenever we apply f to a rational function in the remainder of the proof we leave it to the reader to verify that the function involved satisfies the conditions of Lemma 3.2.

Equation (3.8) follows from (3.7) and

$$\begin{aligned}\frac{t_1}{t_1+t_2} <\alpha_2> + \frac{t_2}{t_1+t_2} <\alpha_1> \\ = \frac{1}{2} <\alpha_1><\alpha_2> \{-(t_2^2+2t_1t_2-1)+3(1+t_2^2)+2(1+t_1^2)-4\}.\end{aligned}$$

Similarly (3.9) follows from

$$\begin{aligned}\frac{t_1}{t_1-t_2} <\alpha_2> + \frac{t_2}{t_2-t_1} <\alpha_1> \\ = \frac{1}{2} <\alpha_1><\alpha_2> \{(t_2^2+2t_1t_2-1)+(1+t_2^2)+2(1+t_1^2)\}.\end{aligned}$$

We have

$$\frac{t_2(t_1+t_2)}{t_2^2+2t_1t_2-1} <\alpha_1> = <\alpha_1> + \frac{(1-t_1t_2)}{t_2^2+2t_1t_2-1} <\alpha_1>, \tag{3.16}$$

$$\frac{(1-t_1t_2)}{t_2^2+2t_1t_2-1} <\alpha_1> \xrightarrow{f} -\frac{t_2(t_1+t_2)}{t_2^2+2t_1t_2-1} <\alpha_1>$$

and (3.10) follows.

$$\frac{(1-t_1t_2)}{t_1^2+2t_1t_2-1} <\alpha_1> \xrightarrow{f} \xrightarrow{t_1\leftrightarrow t_2} -\frac{t_1}{t_1-t_2} <\alpha_2>$$

and (3.11) follows by (3.9).

$$\begin{aligned}\frac{<\alpha_1>}{t_1^2+2t_1t_2-1} + <\alpha_1><\alpha_2> \\ = \frac{<\alpha_1+\alpha_2>}{t_1^2+2t_1t_2-1} \xrightarrow{t_1\leftrightarrow t_2} \xrightarrow{f} \xrightarrow{t_1\leftrightarrow t_2} -\frac{<\alpha_1+\alpha_2>}{t_1^2+2t_1t_2-1}\end{aligned}$$

and (3.12) follows.

We observe that (3.13) is equivalent to

$$\int_{\mathbb{R}^2} A(\underset{\sim}{t})w(\underset{\sim}{t})\,d\underset{\sim}{t} = 0,$$

where

$$A(\underset{\sim}{t}) = 4I(\underset{\sim}{t}) + 3<\alpha_1+\alpha_2>^2 - 6<\alpha_1><\alpha_1+\alpha_2>,$$

$$I(\underset{\sim}{t}) = <\alpha_1><\alpha_2><\alpha_1+\alpha_2>,$$

since

$$<\alpha_1>^2 \xrightarrow{t_1\leftrightarrow t_2} \xrightarrow{f} <\alpha_1+\alpha_2>^2,$$

$$<\alpha_1><\alpha_2> \xrightarrow{f} <\alpha_1><\alpha_1+\alpha_2>.$$

$$A(\underset{\sim}{t}) = I(\underset{\sim}{t})(-2+3t_1^2+6t_1t_2-3t_2^2)$$

so that

$$\int\limits_{\mathbb{R}^2} A(\underset{\sim}{t})w(\underset{\sim}{t})\,d\underset{\sim}{t} = -2\int\limits_{\mathbb{R}^2} I(\underset{\sim}{t})(1-3t_1t_2)w(\underset{\sim}{t})\,d\underset{\sim}{t},$$

since $I(\underset{\sim}{t})$ is invariant under $t_1 \leftrightarrow t_2$.

$$(1-t_1t_2) \xrightarrow{\ f\ } \frac{t_2(1+t_1^2)}{(t_1+t_2)},$$
$$\frac{t_2(1+t_1^2)}{(t_1+t_2)} + \frac{t_1(1+t_2^2)}{(t_1+t_2)} = (1+t_1t_2).$$

It follows that

$$2\int\limits_{\mathbb{R}^2} I(\underset{\sim}{t})(1-t_1t_2)w(\underset{\sim}{t})\,d\underset{\sim}{t} = \int\limits_{\mathbb{R}^2} I(\underset{\sim}{t})(1+t_1t_2)w(\underset{\sim}{t})\,d\underset{\sim}{t},$$

since $I(\underset{\sim}{t})$ is invariant under f and $t_1 \leftrightarrow t_2$. Therefore,

$$\int\limits_{\mathbb{R}^2} I(\underset{\sim}{t})(1-3t_1t_2)w(\underset{\sim}{t})\,d\underset{\sim}{t} = 0,$$

as required. This completes the proof of (3.13).

Equation (3.14) follows from

$$\frac{<\alpha_1>}{t_2^2+2t_1t_2-1} \xrightarrow{\ f\ } -\frac{<\alpha_1>}{t_2^2+2t_1t_2-1} - <\alpha_1>^2 .$$

Finally, (3.15) follows easily from (3.10), (3.14), and (3.16). □

4. Proof of Stages 2–3 and (1.2)–(1.5). In this section we use Lemma 3.6 to prove Stages 2–3, thus completing the proof of (1.2)–(1.4). Finally, we show how

(1.5) follows from (1.3) and (1.4). For $k_1, k_2 \geq 0$ we have

$$
\begin{aligned}
(4.1)\qquad 0 &= \int_{\mathbb{R}^2} \frac{\partial}{\partial t_1}\, t_1 <\alpha_1>^{k_1}<\alpha_2>^{k_2}\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&= -(4a+8b+2k_1+1) \int_{\mathbb{R}^2} <\alpha_1>^{k_1}<\alpha_2>^{k_2}\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\quad + 2(2a+4b+k_1+1) \int_{\mathbb{R}^2} <\alpha_1>^{k_1+1}<\alpha_2>^{k_2}\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\quad + 2a \int_{\mathbb{R}^2} \frac{t_1}{t_1+t_2} <\alpha_1>^{k_1}<\alpha_2>^{k_2}\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\quad + 2b \int_{\mathbb{R}^2} \frac{t_1}{t_1-t_2} <\alpha_1>^{k_1}<\alpha_2>^{k_2}\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\quad + 4b \int_{\mathbb{R}^2} \frac{t_1(t_1+t_2)}{t_1^2+2t_1t_2-1} <\alpha_1>^{k_1}<\alpha_2>^{k_2}\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&\quad + 4b \int_{\mathbb{R}^2} \frac{t_1t_2}{t_2^2+2t_1t_2-1} <\alpha_1>^{k_1}<\alpha_2>^{k_2}\ w(\underset{\sim}{t})\, d\underset{\sim}{t}.
\end{aligned}
$$

Letting $k_1 = 0$, $k_2 = 1$ in (4.1) and using (3.8)–(3.12), we find that

$$
(4.2)\qquad \int_{\mathbb{R}^2} <\alpha_1><\alpha_2>\ w(\underset{\sim}{t})\, d\underset{\sim}{t} = \frac{(3a+3b+2)}{4(a+2b+1)} \int_{\mathbb{R}^2} <\alpha_2>\ w(\underset{\sim}{t})\, d\underset{\sim}{t}.
$$

Hence, by (3.5) we have

$$
\begin{aligned}
\text{C.T.}\,[\alpha_1][\alpha_2]G &= \frac{4^{3a+3b}}{\pi^2} \int_{\mathbb{R}^2} 16 <\alpha_1><\alpha_2>\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&= \frac{(3a+3b+2)}{(a+2b+1)}\, \frac{4^{3a+3b}}{\pi^2} \int_{\mathbb{R}^2} 4 <\alpha_1>\ w(\underset{\sim}{t})\, d\underset{\sim}{t} \\
&= \frac{(3a+3b+2)}{(a+2b+1)}\, \text{C.T.}\,[\alpha_1]G \\
&= \frac{2(3a+3b+1)(3a+3b+2)}{(2a+3b+1)(a+2b+1)} g'(a,b) \qquad \text{(by Stage 1)}.
\end{aligned}
$$

This completes the proof of Stage 2.

We cannot prove Stage 3 directly. Instead we use (4.1) to get $\int_{\mathbb{R}^2} <\alpha_1>^2 w(\underset{\sim}{t})\, d\underset{\sim}{t}$ in terms of $\int_{\mathbb{R}^2} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t}$ and $\int_{\mathbb{R}^2} <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t}$. Then we show how Stage 3 will follow from Stages 1 and 2 using (3.13). Letting $k_1 = 1$, $k_2 = 0$ in (4.1) and using (3.8), (3.9), (3.11), and (3.15) we find that

(4.3)

$$
\begin{aligned}
2(2a+3b+2)\int_{\mathbb{R}^2} <\alpha_1>^2 w(\underset{\sim}{t})\, d\underset{\sim}{t}
&= \frac{3}{2}(3a+3b+2)\int_{\mathbb{R}^2} <\alpha_1> w(\underset{\sim}{t})\, d\underset{\sim}{t} - 2a\int_{\mathbb{R}^2} <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t}\\
&= 2(2a+6b+3)\int_{\mathbb{R}^2} <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t} \qquad \text{(by (4.2))}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\text{C.T. } [\alpha_1][\alpha_2][\alpha_1+\alpha_2]G\\
&= \frac{4^{3a+3b}}{\pi^2}\int_{\mathbb{R}^2} 64 <\alpha_1><\alpha_2><\alpha_1+\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t}\\
&= -3\cdot\frac{4^{3a+3b}}{\pi^2}\int_{\mathbb{R}^2} 16 <\alpha_1>^2 w(\underset{\sim}{t})\, d\underset{\sim}{t}\\
&\qquad + 6\cdot\frac{4^{3a+3b}}{\pi^2}\int_{\mathbb{R}^2} 16 <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t} \qquad \text{(by (3.13))}\\
&= \frac{(-3(2a+6b+3)+6(2a+3b+2))}{(2a+3b+2)}\,\frac{4^{3a+3b}}{\pi^2}\int_{\mathbb{R}^2} 16 <\alpha_1><\alpha_2> w(\underset{\sim}{t})\, d\underset{\sim}{t}\\
&\qquad\qquad \text{(by (4.3))}\\
&= \frac{3(2a+1)}{(2a+3b+2)}\text{C.T. } [\alpha_1][\alpha_2]G\\
&= \frac{6(3a+3b+2)(3a+3b+1)(2a+1)}{(2a+3b+2)(2a+3b+1)(a+2b+1)}g'(a,b) \qquad \text{(by Stage 2)}.
\end{aligned}
$$

This completes the proof of Stage 3.

At the beginning of §2 we showed how Stage 3 implies (1.2). We remark that (1.3) and (1.4) follow from Stages 1 and 2 together with (1.2). We have been unable to find a proof of (1.5) or (1.6) in terms of integrals. However (1.5) follows easily from (1.3) and (1.4). In order to show this we need to recall how the Weyl group acts on polynomials. For $\alpha = k_1\alpha_1 + k_2\alpha_2$, where k_1, $k_2 \in \mathbb{Z}$ and α_1, α_2 are the roots from the root system G_2 as in Fig. 1, we let

$$x^\alpha = x_1^{k_1} x_2^{k_2}.$$

The elements w of the Weyl group W act on monomials by

$$w(x^\alpha) = x^{w(\alpha)},$$

and by linearity on Laurent polynomials that are linear combinations of the x^α. For w in the Weyl group W we have

$$\text{C.T.}\, x^\alpha G = \text{C.T.}\, x^{w(\alpha)} G, \tag{4.4}$$

since G is symmetric with respect to the Weyl group and w does not change the constant term. Utilizing (4.4) we find that the left-hand sides of (1.3)–(1.5) can be written as

$$\text{C.T.}\, [\alpha_1] G = 2\text{C.T.} (1 - x_1) G, \tag{4.5}$$

$$\text{C.T.}\, [\alpha_1][\alpha_2] G = 2\text{C.T.} \left(2 - 3x_1 + \frac{x_1}{x_2}\right) G, \tag{4.6}$$

$$\text{C.T.}\, [2\alpha_1 + \alpha_2] G = 2\text{C.T.} \left(1 - \frac{x_1}{x_2}\right) G. \tag{4.7}$$

Hence,

$$\begin{aligned}
\text{C.T.}\, [2\alpha_1 + \alpha_2] G &= 3\text{C.T.}\, [\alpha_1] G - \text{C.T.}\, [\alpha_1][\alpha_2] G \\
&= \left\{ \frac{6(3a+3b+1)}{(2a+3b+1)} - \frac{2(3a+3b+2)(3a+3b+1)}{(2a+3b+1)(a+2b+1)} \right\} g(a,b) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (1.3), (1.4))} \\
&= \frac{2(3a+3b+1)(3b+1)}{(2a+3b+1)(a+2b+1)} g(a,b),
\end{aligned}$$

which is (1.5).

5. Other results. In this section we state other results that are similar to (1.3)–(1.6). These may be proved by extending Zeilberger's [12] proof of the ordinary G_2 case as mentioned in §1.

(5.1)

$$\text{C.T.}\, [\alpha_1][\alpha_1 - \alpha_2] G = \frac{4(3a+3b+2)(3a+3b+1)(3b+1)}{(2a+3b+2)(2a+3b+1)(a+2b+1)} g(a,b),$$

(5.2)

$$\text{C.T.}\,[\alpha_1][\alpha_1+\alpha_2][\alpha_1-\alpha_2]G = \frac{6(4a+3b+4)(3a+3b+2)(3a+3b+1)(3b+1)}{(2a+3b+3)(2a+3b+2)(2a+3b+1)(a+2b+1)}g(a,b),$$

(5.3)

$$\text{C.T.}\,[\alpha_1][\alpha_2][\alpha_1-\alpha_2]G = 6\left\{\frac{(3a+3b+2)(3a+3b+1)(3b+1)}{(2a+3b+1)(a+2b+2)(a+2b+1)} - \frac{b(4a+3b+4)(3a+3b+2)(3a+3b+1)(3b+1)}{(2a+3b+3)(2a+3b+2)(2a+3b+1)(a+2b+2)(a+2b+1)}\right\}g(a,b),$$

(5.4)

$$\text{C.T.}\,[\alpha_1][\alpha_1-\alpha_2][\alpha_1+2\alpha_2]G = \frac{6(3a+4b+4)(3a+3b+2)(3a+3b+1)(3b+2)(3b+1)}{(2a+3b+3)(2a+3b+2)(2a+3b+1)(a+2b+2)(a+2b+1)}g(a,b),$$

(5.5)

$$\text{C.T.}\,[\alpha_1][\alpha_1-\alpha_2][2\alpha_1+\alpha_2]G = \frac{6(6a^2+20ab+23a+12b^2+28b+16)}{(2a+3b+4)(2a+3b+3)(2a+3b+2)} \cdot \frac{(3a+3b+2)(3a+3b+1)(3b+2)(3b+1)}{(2a+3b+1)(a+2b+2)(a+2b+1)}g(a,b),$$

(5.6)

$$\text{C.T.}[\alpha_1][\alpha_1+\alpha_2][\alpha_1-\alpha_2][\alpha_1+2\alpha_2]G = \frac{18(3a+3b+4)(3a+3b+2)(3a+3b+1)}{(2a+3b+4)(2a+3b+3)(2a+3b+2)} \cdot \frac{(2a+2b+3)(3b+2)(3b+1)}{(2a+3b+1)(a+2b+2)(a+2b+1)}g(a,b).$$

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