JMAA, to appear.

A COMBINATORIAL PROOF OF THE FARKAS-KRA THETA FUNCTION IDENTITIES AND THEIR GENERALIZATIONS

Frank G. Garvan

Department of Mathematics
University of Florida
Gainesville, FL 32611
Email: frank@math.ufl.edu

(Originally submitted January 6, 1994, revised November 4, 1994)

Abstract. Recently, Farkas and Kra found some cubic theta function identities from their work on automorphic forms. Shortly thereafter, Farkas and Kopeliovich were able to generalize these to p-th order theta function identities using the theory of elliptic functions. We give short, elementary proofs of the cubic identities. We show that the p-th order identities follow from more general relations between the coefficients of certain theta functions. Our proof is combinatorial and utilizes certain orthogonal transformations.

The author was supported in part by NSF Grant DMS-9208813.

1980 *Mathematics Subject Classification* (1991 *Revision*) Primary 33D10, Secondary 05A19, 11F11, 11F20

Key words and phrases. Theta functions, Dedekind's eta function, q-series, combinatorial identities, bijective proofs, orthogonal matrices

PROPOSED RUNNING HEAD

THETA FUNCTION IDENTITIES

Mailing Address of Author

Department of Mathematics
University of Florida
201 Walker Hall
Gainesville, FL 32611

1. Introduction

The Jacobian theta function $\vartheta(z,q)$ is defined by

$$\vartheta(z,q) := \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2}, \tag{1.1}$$

where $|q| < 1$ and $z \neq 0$. Recently, H. M. Farkas and I. Kra [8] discovered two cubic theta function identities:

$$\vartheta^3(\omega^2 q^{\frac{1}{2}}, q^{\frac{3}{2}}) + \omega^2 \vartheta^3(q^{\frac{1}{2}}, q^{\frac{3}{2}}) + \omega \vartheta^3(\omega q^{\frac{1}{2}}, q^{\frac{3}{2}}) = 0 \tag{1.2}$$

and

$$\vartheta^3(\omega q^{\frac{1}{2}}, q^{\frac{3}{2}}) - \vartheta^3(\omega^2 q^{\frac{1}{2}}, q^{\frac{3}{2}}) = q\vartheta^3(\omega q^{\frac{3}{2}}, q^{\frac{3}{2}}), \tag{1.3}$$

where $\omega = \exp(2\pi i/3)$. We give short elementary proofs of these identities. In §2 we will show how (1.2) follows easily from Jacobi's identity

$$\vartheta^3(q^{\frac{1}{2}}, q^{\frac{3}{2}}) = \sum_{n\geq 0} (-1)^n (2n+1) q^{n(n+1)/2}, \tag{1.4}$$

and the fact that $n(n+1)/2 \not\equiv 2 \pmod{3}$. The second identity (1.3) depends on the following identity

$$\vartheta^3(\omega q^{\frac{1}{2}}, q^{\frac{3}{2}}) - \vartheta^3(\omega^2 q^{\frac{1}{2}}, q^{\frac{3}{2}}) = -3q(\omega^2 - \omega)\vartheta^3(q^{\frac{9}{2}}, q^{\frac{27}{2}}), \tag{1.5}$$

and Jacobi's triple product identity. In §2 we give the details as well as generalizations of (1.2) and (1.5) related to the work of Newman and Kolberg. It should be noted that (1.3) can also be proved by applying Jacobi's imaginary transformation to (1.2). This observation is due to H. Farkas (private communication).

Farkas and Kopeliovich [6] have found a p-th order generalization of (1.2). It is the case $k = 1$ of the identity (1.6) below. Let p be odd and suppose $k = 1, 3, \ldots, p-2$ and $\gcd(k, 2p) = 1$. We have

$$\sum_{\ell=0}^{p-1} \zeta_p^{-r\ell} \vartheta^p(\zeta_p^{r'\ell} q^{\frac{k}{2}}, q^{\frac{p}{2}}) = 0, \tag{1.6}$$

where $\zeta_p = \exp(2\pi i/p)$, and r, r' are any integers that satisfy

$$4r \equiv -k^2 \pmod{p}, \quad 2r' \equiv k \pmod{p}. \tag{1.7}$$

When $p=3$ and $k=1$ (1.6) is precisely (1.2). In [7], Farkas and Kopeliovich apply their method to prove some of Ramanujan's modular equations. In [12], Kopeliovich shows how the methods of [6], [7] can be extended to multidimensional theta functions and finds a multidimensional analogue of the p-th power identity.

We were first led to (1.6) by observing (via MAPLE) that one residue class $\pmod p$ was missing for the exponent of q in the q-series expansion of $\vartheta^p(q^{\frac{k}{2}}, q^{\frac{p}{2}})$. The missing residue class was $r \equiv -\frac{k^2}{4} \pmod p$. We also observed other relations among the other residue classes. The general result is given in Theorem 3.4. Thus the identity (1.6) follows from a special case of this theorem. Our method of proof is combinatorial and utilizes an orthogonal affine linear transformation.

We find two different generalizations of (1.3). We have

$$\sum_{\ell=0}^{p-1} \zeta_p^{-r\ell} \vartheta^p(\zeta_p^{r'\ell} q^{\frac{k}{2}}, q^{\frac{p}{2}}) = (-1)^{(k+p)/2} q^{(p+k)(p-k)/8} \sum_{\ell=0}^{(p-1)/2} (\zeta_p^{-s\ell} - \zeta_p^{s\ell}) \vartheta^p(\zeta_p^{\ell} q^{\frac{p}{2}}, q^{\frac{p}{2}}), \tag{1.8}$$

where r, r', s are any integers that satisfy

$$8r \equiv -k^2 \pmod p, \quad 2r' \equiv k \pmod p, \quad 4s \equiv -k \pmod p. \tag{1.9}$$

We note that (1.3) follows from putting $p=3$, $k=1$ in (1.8) and using (1.2) to simplify the result.

Our second generalization is

$$\sum_{\ell=1}^{(p-1)/2} (-1)^\ell q^{\ell(\ell-1)/2} \big(\vartheta^p(\zeta_p q^{\ell-\frac{1}{2}}, q^{\frac{p}{2}}) - \vartheta^p(\overline{\zeta}_p q^{\ell-\frac{1}{2}}, q^{\frac{p}{2}})\big) \tag{1.10}$$
$$= (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} \vartheta^p(\zeta_p q^{\frac{p}{2}}, q^{\frac{p}{2}}).$$

We note that (1.3) is (1.10) with $p=3$. Our proofs of (1.6), (1.8), (1.10) and other related identities are given in §3. For p prime we show that (1.10) follows from our main Theorem 3.4. For general odd p we show that (1.10) follows from the $k=1$ case of (1.6) using Jacobi's imaginary transformation. Some concluding remarks related to the work of Borwein, Borwein and Garvan [4] are given in §4.

2. Cubic identities

In this section we give short elementary proofs of (1.2), (1.3). Generalizations of (1.2) and other identities related to the work of Newman [13] and Kolberg [11] are given. In the next section, different generalizations of (1.2), (1.3) are given together with elementary proofs.

We need Jacobi's triple product identity [1, p. 21],[2, p. 62],[10, p. 282]

$$\vartheta(zq^{\frac{1}{2}},q^{\frac{1}{2}}) = \prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1}), \tag{2.1}$$

Jacobi's identity [2, (3.1.14) p. 65], [10, Theorem 357 p. 285]

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2}, \tag{2.2}$$

and Euler's pentagonal number theorem [1, Corollary 1.7 p. 11], [2, (3.1.10) p. 54]

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} = \vartheta(q^{\frac{1}{2}},q^{\frac{3}{2}}). \tag{2.3}$$

It is well-known that (2.2), (2.3) follow easily from (2.1). We let

$$A(q) := \sum_{n=0}^{\infty} a_n q^n := \prod_{n=1}^{\infty}(1-q^n)^3 = \vartheta^3(q^{\frac{1}{2}},q^{\frac{3}{2}}). \tag{2.4}$$

We observe that $n(n+1)/2 \not\equiv 2 \pmod 3$, so that from (2.2) we have

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} 3a_{3n+2}q^{3n+2} = \sum_{n=0}^{\infty} a_n(1+\omega^{n-2}+\omega^{2(n-2)})q^n \\ &= A(q)+\omega A(\omega q)+\omega^2 A(\omega^2 q) \\ &= \vartheta^3(q^{\frac{1}{2}},q^{\frac{3}{2}}) + \omega\vartheta^3(\omega^2 q^{\frac{1}{2}},q^{\frac{3}{2}}) + \omega^2\vartheta^3(\omega q^{\frac{1}{2}},q^{\frac{3}{2}}). \end{aligned} \tag{2.5}$$

Equation (1.2) follows by multiplying both sides of (2.5) by ω^2.

We now prove (1.3). We define

$$\eta(q) := \vartheta(q^{\frac{1}{2}},q^{\frac{3}{2}}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} = \prod_{n=1}^{\infty}(1-q^n), \tag{2.6}$$

by (2.3). Since $n(n+1)/2 \not\equiv 2 \pmod 3$ then from (2.2) we have

$$\eta^3(q) = \epsilon_0(q^3) + q\epsilon_1(q^3), \tag{2.7}$$

for certain power series ϵ_0, ϵ_1. Now,

(2.8)

$$\begin{aligned}
&\vartheta^3(\omega q^{\frac{1}{2}}, q^{\frac{3}{2}}) - \vartheta^3(\omega^2 q^{\frac{1}{2}}, q^{\frac{3}{2}}) \\
&\quad = \eta^3(\omega^2 q) - \eta^3(\omega q) \qquad \text{(by (2.6))} \\
&\quad = (\omega^2 - \omega) q \epsilon_1(q^3) \qquad \text{(by (2.7))} \\
&\quad = (\omega^2 - \omega) \sum_{\substack{n \geq 0 \\ n \equiv 1 \pmod 3}} (-1)^n (2n+1) q^{n(n+1)/2} \\
&\quad = -3q(\omega^2 - \omega) \sum_{m \geq 0} (-1)^m (2m+1) q^{9m(m+1)/2} \qquad \text{(by substituting } n = 3m+1 \text{ in the sum)} \\
&= -3q(\omega^2 - \omega)\eta^3(q^9).
\end{aligned}$$

From (2.1) we have

$$\begin{aligned}
\vartheta(\omega q^{\frac{1}{2}}, q^{\frac{1}{2}}) &= \sum_{n=\infty}^{\infty} (-1)^n \omega^n q^{n(n+1)/2} \\
&= \prod_{n=1}^{\infty} (1-q^n)(1-\omega q^n)(1-\omega^{-1}q^{n-1}) \\
&= (1-\omega^2) \prod_{n=1} (1-q^{3n}) \\
&= (1-w^2)\eta(q^3).
\end{aligned} \tag{2.9}$$

An easy calculation gives $-3(\omega^2 - \omega) = (1-\omega^2)^3$. Thus combining (2.8), (2.9) we find that

$$\begin{aligned}
&\vartheta^3(\omega q^{\frac{1}{2}}, q^{\frac{3}{2}}) - \vartheta^3(\omega^2 q^{\frac{1}{2}}, q^{\frac{3}{2}}) \\
&\quad = -3q\,(\omega^2 - \omega)\eta^3(q^9) = q\,(1-\omega^2)^3\eta^3(q^9) \\
&\quad = q\,\vartheta^3(wq^{\frac{3}{2}}, q^{\frac{3}{2}}),
\end{aligned} \tag{2.10}$$

which is (1.3).

We now consider other cubic identities related to the work of Kolberg [11]. Let $p > 2$ be a prime. We define

$$h_s = \sum_{\substack{\frac{1}{2}n(n+1) \equiv s \pmod p \\ n \geq 0}} (-1)^n (2n+1) q^{\frac{1}{2}n(n+1)} \qquad (s = 0, 1, \ldots, p-1);$$

so that

$$\sum_{s=0}^{p-1} h_s = \eta^3(q).$$

The proof of the following lemma is analogous to that of (2.7), (2.8).

Lemma 2.11. (Kolberg) *If* $8s+1$ *is a quadratic non-residue* $\pmod p$*, then* $h_s = 0$*. If* $8s+1 \equiv 0 \pmod p$*, then*

$$h_s = (-1)^{\frac{1}{2}(p-1)}\, p\, q^{\frac{1}{8}(p^2-1)}\, \eta^3(q^{p^2}). \tag{2.12}$$

We may cast this result in terms of theta functions.

Corollary 2.13. *Let* p *be an odd prime, let* $8s+1$ *be a quadratic non-residue* $\pmod p$*, let* $t = (p^2-1)/8$*, and let* $\zeta_p = \exp(2\pi i/p)$*. Then*

$$\sum_{k=0}^{p-1} \zeta_p^{-ks} \vartheta^3(\zeta_p^{k(p+1)/2} q^{\frac{1}{2}}, \zeta_p^{3k(p+1)/2} q^{\frac{3}{2}}) = 0, \tag{2.14}$$

and

$$\sum_{k=0}^{p-1} \zeta_p^{-kt} \vartheta^3(\zeta_p^{k(p+1)/2} q^{\frac{1}{2}}, \zeta_p^{3k(p+1)/2} q^{\frac{3}{2}}) = (-1)^{(p-1)/2} p^2 q^t \vartheta^3(q^{\frac{p^2}{2}}, q^{\frac{3p^2}{2}}). \tag{2.15}$$

Newman [13] has found results analogous to (2.12) or (2.15) for higher powers of $\eta(q)$. To describe Newman's results we need some notation. For an integer r we define $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^r, \tag{2.16}$$

so that $p_3(n) = a_n$, which was defined in (2.4). We note that $p_r(n)$ is defined to be zero when n is not a non-negative integer.

Theorem 2.17. (Newman) *Suppose* r *is one of the numbers* 2, 4, 6, 8, 10, 14, 26*. Let* p *be a prime* > 3 *such that* $r(p+1) \equiv 0 \pmod{24}$ *and let* $\Delta = r(p^2-1)/24$*. Then*

$$p_r(np+\Delta) = (-p)^{(r/2)-1}\, p_r(n/p). \tag{2.18}$$

3. The p-th power generalizations

In this section we prove (1.6), (1.8) and (1.10), the generalizations of (1.2) and (1.3). We also derive other analogous identities. The methods of §2 don't suffice so we use a different approach.

Throughout this section k, p are odd integers with $p \geq 3$ and $\gcd(2p, k) = 1$. We define $T(k,p,n)$ as the coefficient of q^n in the series expansion of $\vartheta^p(q^{\frac{k}{2}}, q^{\frac{p}{2}})$; ie.

$$\sum_{n \geq n_0} T(k,p,n) q^n = \sum_{\vec{n} \in \mathbb{Z}^p} (-1)^{\vec{n}\cdot\vec{1}} q^{\frac{p}{2}||\vec{n}||^2 + \frac{k}{2}\vec{n}\cdot\vec{1}} = \vartheta^p(q^{\frac{k}{2}}, q^{\frac{p}{2}}), \tag{3.1}$$

where $\vec{1} = (1,1,\ldots,1)$. By considering each of the transformations

$$\vec{n} \mapsto -\vec{n}, \tag{i}$$

and

$$\vec{n} \mapsto \vec{n} + \vec{1}, \tag{ii}$$

we have

$$T(-k,p,n) = T(k,p,n), \tag{3.2}$$

and

$$T(k,p,n) = -T(k+2p,p,n - \tfrac{p(p+k)}{2}) = -T(2p-k,p,n + \tfrac{p(k-p)}{2}). \tag{3.3}$$

Hence it is enough to consider $k = 1, 3, \ldots, p-1$ for fixed p. The main result is

Theorem 3.4. *Let p be an odd integer $p \geq 3$. Suppose $gcd(2p,k) = 1$ and $k = 1, 3, \ldots, p-2$. Then*

$$T(k,p,pn+r) = (-1)^{\ell} T(2\ell - k, p, pn + r - \tfrac{\ell(\ell-k)}{2}), \tag{3.5}$$

where ℓ is any integer that satisfies $\frac{k+1}{2} \leq \ell \leq \frac{k+p-2}{2}$, $gcd(2\ell - k, p) = 1$ and $r \equiv -\frac{k\ell}{4} \pmod{p}$; and

$$T(k,p,pn+s) = (-1)^{\ell} T(2\ell + k, p, pn + s - \tfrac{\ell(\ell+k)}{2}), \tag{3.6}$$

where ℓ is any integer that satisfies $\frac{-k+1}{2} \leq \ell \leq \frac{-k+p-2}{2}$, $gcd(2\ell + k, p) = 1$ and $s \equiv \frac{k\ell}{4} \pmod{p}$.

By taking $\ell = k$ in the theorem we obtain the following corollary which is equivalent to (1.6).

Corollary 3.7. *Let p be an odd integer $p \geq 3$. Suppose $gcd(2p, k) = 1$ and $k = 1, 3, \ldots, p-2$. Then*

$$T(k, p, pn + r) = 0, \qquad \text{where } r \equiv -\frac{k^2}{4} \pmod{p}. \tag{3.8}$$

We illustrate the theorem and its corollary for $p = 3$, 5 and 7. For $p = 3$, there is only one relation

$$T(1, 3, 3n + 2) = 0, \tag{3.9}$$

which is equivalent to (1.2). For $p = 5$, there are four relations:

$$T(1, 5, 5n + 1) = 0, \tag{3.10}$$

$$T(1, 5, 5n + 2) = T(3, 5, 5n + 1), \tag{3.11}$$

$$T(1, 5, 5n + 4) = -T(3, 5, 5n + 3), \tag{3.12}$$

$$T(3, 5, 5n + 4) = 0. \tag{3.13}$$

For $p = 7$, there are 9 relations:

$$T(1, 7, 7n + 1) = -T(5, 7, 7n - 2), \tag{3.14}$$

$$T(1, 7, 7n + 2) = -T(3, 7, 7n + 1), \tag{3.15}$$

$$T(1, 7, 7n + 3) = T(3, 7, 7n + 2), \tag{3.16}$$

$$T(1, 7, 7n + 4) = T(5, 7, 7n + 1), \tag{3.17}$$

$$T(1, 7, 7n + 5) = 0, \tag{3.18}$$

$$T(3, 7, 7n + 3) = 0, \tag{3.19}$$

$$T(3, 7, 7n + 4) = T(5, 7, 7n + 2), \tag{3.20}$$

$$T(3, 7, 7n + 6) = -T(5, 7, 7n + 4), \tag{3.21}$$

$$T(5, 7, 7n + 6) = 0. \tag{3.22}$$

Our proof of Theorem 3.4 is combinatorial and utilizes an orthogonal affine linear transformation L_ℓ given below in (3.30). For fixed odd $p \geq 3$ we define the $p \times p$ matrix Q as the matrix whose columns are

$$\vec{v}_i = \vec{e}_i - \frac{2}{p}\vec{1}, \qquad (1 \leq i \leq p), \tag{3.23}$$

where the $\vec{e}_i$ are the columns of the identity matrix so that

$$Q = I - \frac{2}{p}J, \tag{3.24}$$

where J is the $p \times p$ matrix all of whose entries are one. The following properties of Q are easily proved:

$$Q^T = Q, \tag{3.25}$$

$$Q^T Q = I \qquad (Q \text{ is an orthogonal matrix}), \tag{3.26}$$

$$Q\vec{1} = -\vec{1}. \tag{3.27}$$

It should be noted that Q is an example of a Householder matrix. The use of Householder matrices is important in numerical linear algebra [L, p. 358]. For $\vec{n} \in \mathbb{Z}^p$, we define

$$F(\vec{n}) := F(\vec{n}, k) = \frac{p}{2}||\vec{n}||^2 + \frac{k}{2}\vec{n} \cdot \vec{1}, \tag{3.28}$$

so that

$$\sum_{\vec{n} \in \mathbb{Z}^p} (-1)^{\vec{n} \cdot \vec{1}} q^{F(\vec{n})} = \vartheta^p(q^{\frac{k}{2}}, q^{\frac{p}{2}}). \tag{3.29}$$

For any integer ℓ we define the affine linear transformation

$$L_\ell : \mathbb{R}^p \longrightarrow \mathbb{R}^p \quad \text{by} \quad L_\ell(\vec{n}) = Q\vec{n} - \frac{\ell}{p}\vec{1}. \tag{3.30}$$

Then the following properties of L_ℓ are easily proved:

$$F(L_\ell(\vec{n}), k) = F(\vec{n}, 2\ell - k) + \frac{\ell(\ell - k)}{2}, \tag{3.31}$$

$$L_\ell(\vec{n}) = \vec{n} - \frac{1}{p}(2(\vec{n} \cdot \vec{1}) + \ell)\vec{1}, \tag{3.32}$$

$$L_\ell(\vec{n}) \cdot \vec{1} = -(\vec{n} \cdot \vec{1}) - \ell, \tag{3.33}$$

$$L_\ell^2 = I. \tag{3.34}$$

For $\epsilon = +1$ or -1, we define

$$\mathcal{S}_{k,p}^{\epsilon}(N) = \{\vec{n} \in \mathbb{Z}^p \,:\, N = \frac{p}{2}||\vec{n}||^2 + \frac{k}{2}\vec{n} \cdot \vec{1} \quad \text{and} \quad (-1)^{\vec{n} \cdot \vec{1}} = \epsilon\}. \tag{3.35}$$

We turn to the proof of Theorem 3.4. First we prove (3.5) by showing that

$$L_\ell : \mathcal{S}^{\epsilon}_{k,p}(N) \longrightarrow \mathcal{S}^{\epsilon'}_{k',p}(N + \tfrac{\ell(\ell-k)}{2}) \tag{3.36}$$

is a bijection if $N \equiv -\frac{k\ell}{4} \pmod p$, $\gcd(k, 2p) = 1$ and $\gcd(2\ell - k, p) = 1$, and where $\epsilon' = (-1)^\ell \epsilon$ and $k' = 2\ell - k$. Suppose $N \equiv -\frac{k\ell}{4} \pmod p$, $\gcd(k, 2p) = 1$ and $\vec{n} \in \mathcal{S}^{\epsilon}_{k,p}(N)$. Then

$$\frac{k}{2}\,\vec{n}\cdot\vec{1} \equiv -\frac{k\ell}{4} \pmod p,$$

and

$$\vec{n}\cdot\vec{1} \equiv -\frac{\ell}{2} \pmod p \qquad \text{since } \gcd(k,p) = 1.$$

From (3.32) it follows that

$$L_\ell(\vec{n}) \in \mathbb{Z}^p.$$

If we let $\vec{m} := L_\ell(\vec{n})$ then

$$\begin{aligned}(-1)^{\vec{m}\cdot\vec{1}} &= (-1)^{-\vec{n}\cdot\vec{1}-\ell} \qquad \text{by (3.33)}\\ &= (-1)^\ell\epsilon\\ &= \epsilon',\end{aligned}$$

and

$$\begin{aligned}\frac{p}{2}\vec{m}\cdot\vec{m} + \frac{k'}{2}\vec{m}\cdot\vec{1} &= F(L_\ell(\vec{n}), 2\ell - k)\\ &= F(\vec{n}, k) + \frac{\ell(k-\ell)}{2} \qquad \text{by (3.31)}\\ &= N + \frac{\ell(k-\ell)}{2}.\end{aligned}$$

It follows that $L_\ell(\vec{n}) \in \mathcal{S}^{\epsilon'}_{k',p}(N + \frac{\ell(k-\ell)}{2})$, and the map in (3.36) is well-defined. If in addition, $\gcd(2\ell - k, p) = 1$, then analogously we find that

$$L_\ell(\vec{m}) \in \mathcal{S}^{\epsilon}_{k,p}(N)$$

where $\vec{m}$ is any element of $\mathcal{S}^{\epsilon'}_{k',p}(N + \frac{\ell(k-\ell)}{2})$. Since $L_\ell^2 = I$, we see that the transformation L_ℓ in (3.36) defines a bijection and (3.5) follows. Equation (3.6) follows by replacing k by $-k$ in (3.5) and using (3.2). This completes the proof of Theorem 3.4.

We now turn to (1.8) and consider L_ℓ where $\ell = \frac{k+p}{2}$. In this case we have $2\ell - k = p$ so that Theorem 3.4 does not apply. However we are still able to identify the image of $\mathcal{S}^\epsilon_{k,p}(N)$ under L_ℓ when $N \equiv -\frac{k^2}{8} \pmod{p}$. For $\epsilon = +1$ or -1 we define

(3.37)

$$\mathcal{R}^\epsilon_{k,p}(M) = \{\vec{m} \in \mathbb{Z}^p \,:\, M = \frac{p}{2}||\vec{m}||^2 + \frac{p}{2}\vec{m}\cdot\vec{1},\ (-1)^{\vec{m}\cdot\vec{1}} = \epsilon, \quad \text{and} \quad \vec{m}\cdot\vec{1} \equiv -\frac{k}{4} \pmod{p}\}.$$

For $\gcd(k, 2p) = 1$ and $N \equiv -\frac{k^2}{8} \pmod{p}$ we will prove that

$$L_{\frac{k+p}{2}} \,:\, \mathcal{S}^\epsilon_{k,p}(N) \longrightarrow \mathcal{R}^{\epsilon'}_{k,p}(N + \tfrac{(k+p)(k-p)}{8}) \tag{3.38}$$

is a bijection, where $\epsilon' = (-1)^{\frac{k+p}{2}}\epsilon$. We suppose $N \equiv -\frac{k^2}{8} \pmod{p}$, $\gcd(k, 2p) = 1$ and $\vec{n} \in \mathcal{S}^\epsilon_{k,p}(N)$. If we let $\vec{m} = L_{\frac{k+p}{2}}(\vec{n})$ then, as in the proof of (3.36), we find that

$$\vec{m} = L_{\frac{k+p}{2}}(\vec{n}) \in \mathbb{Z}^p, \quad (-1)^{\vec{m}\cdot\vec{1}} = \epsilon', \quad \frac{p}{2}\vec{m}\cdot\vec{m} + \frac{p}{2}\vec{m}\cdot\vec{1} = N + \frac{(k+p)(k-p)}{8}.$$

In addition we find that

$$\vec{m}\cdot\vec{1} = L_{\frac{k+p}{2}}(\vec{n})\cdot\vec{1} = -(\vec{n}\cdot\vec{1}) - \frac{k+p}{2} \equiv -\frac{k}{4} \pmod{p},$$

since $N \equiv -\frac{k^2}{8} \pmod{p}$ and $\gcd(k, p) = 1$ imply $\vec{n}\cdot\vec{1} \equiv -\frac{k}{4} \pmod{p}$. Hence,

$$\vec{m} = L_{\frac{k+p}{2}}(\vec{n}) \in \mathcal{R}^{\epsilon'}_{k,p}(N + \frac{(k+p)(k-p)}{8}).$$

Now suppose $N \equiv -\frac{k^2}{8} \pmod{p}$, $\gcd(k, 2p) = 1$ and $\vec{m} \in \mathcal{R}^{\epsilon'}_{k,p}(N + \frac{(k+p)(k-p)}{8})$. If we let $\vec{n} := L_{\frac{k+p}{2}}(\vec{m})$ and $\epsilon = (-1)^{\frac{k+p}{2}}\epsilon'$ then we find that

$$\vec{n} = L_{\frac{k+p}{2}}(\vec{m}) \in \mathbb{Z}^p, \quad (-1)^{\vec{n}\cdot\vec{1}} = \epsilon, \quad \frac{p}{2}\vec{n}\cdot\vec{n} + \frac{k}{2}\vec{n}\cdot\vec{1} = N.$$

Hence $\vec{n} \in \mathcal{S}^\epsilon_{k,p}(N)$. Since $L^2_{\frac{k+p}{2}} = I$, it follows that the map in (3.38) is a bijection.

We now show how the bijection in (3.38) may be interpreted in terms of theta-functions and hence prove (1.8). If $\zeta_p = \exp(2\pi i/p)$ then

$$\sum_{j=0}^{p-1} \zeta_p^{jk} = \begin{cases} p & \text{if } k \equiv 0 \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 \leq j \leq p-1$, we define

$$G_j := G_j(q) = \left(\sum_{m=-\infty}^{\infty} (-1)^m \zeta_p^{jm} q^{pm(m+1)/2} \right)^p \tag{3.39}$$
$$= \vartheta^p(\zeta_p^j q^{\frac{p}{2}}, q^{\frac{p}{2}}),$$

and

$$F_j := F_j(q) = \sum_{\substack{\vec{m}\in\mathbb{Z}^p \\ \vec{m}\cdot\vec{1}\equiv j \pmod p}} (-1)^{\vec{m}\cdot\vec{1}} q^{\frac{p}{2}||\vec{m}||^2 + \frac{p}{2}\vec{m}\cdot\vec{1}}.$$

We observe that $m(m+1)$ is invariant under $m \mapsto -1-m$. We have

$$(-\vec{1}-\vec{m})\cdot\vec{1} = -(\vec{m}\cdot\vec{1}) - p$$

so that by considering $\vec{m} \mapsto -\vec{1}-\vec{m}$ we find that

$$F_j = -F_{p-j}.$$

Hence

$$G_j = \sum_{k=1}^{p-1} \zeta_p^{kj} F_k \tag{3.40}$$
$$= \sum_{k=1}^{(p-1)/2} (\zeta_p^{kj} - \zeta_p^{-kj}) F_k.$$

We claim that

$$F_j = \frac{1}{p} \sum_{k=1}^{(p-1)/2} (\zeta_p^{-kj} - \zeta_p^{kj}) G_k. \tag{3.41}$$

Now

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} (\zeta_p^{-kj} - \zeta_p^{kj}) G_k \\
&= \sum_{k=1}^{(p-1)/2} (\zeta_p^{-kj} - \zeta_p^{kj}) \sum_{\ell=1}^{(p-1)/2} (\zeta_p^{\ell k} - \zeta_p^{-\ell k}) F_\ell \\
&= \sum_{\ell=1}^{(p-1)/2} \left(\sum_{k=1}^{(p-1)/2} \zeta_p^{k(\ell-j)} - \zeta_p^{k(j+\ell)} - \zeta_p^{-k(j+\ell)} + \zeta_p^{k(j-\ell)} \right) F_\ell \\
&= \left(p - 1 - \sum_{k=1}^{p-1} \zeta_p^{2kj} \right) F_j \qquad \text{(since the inner sum above is zero when } \ell \neq j) \\
&= pF_j,
\end{aligned}$$

and (3.41) follows. This means we may now interpret the bijection (3.38) as a theta-function identity. If $r \equiv -\frac{k^2}{8} \pmod p$ then we have

$$T(k,p,pN+r) = (-1)^{\frac{k+p}{2}} \text{ Coefficient of } q^{pN+r+\frac{(k+p)(k-p)}{8}} \text{ in } F_s(q), \tag{3.42}$$

where $s \equiv -\frac{k}{4} \pmod p$. The equation (1.8) follows easily via (3.41).

Our proof of (3.41) leads one to consider the matrix

$$S = \big(\sin(2\pi ij/p)\big)_{1 \le i,j \le \frac{p-1}{2}}. \tag{3.43}$$

Our result (3.41) is equivalent to

$$S^2 = \frac{p}{4} I. \tag{3.44}$$

For

$$C = \big(\cos(2\pi ij/p)\big)_{1 \le i,j \le \frac{p-1}{2}} \tag{3.45}$$

we find

$$C^2 = \frac{p}{4} I - \frac{1}{2} J. \tag{3.46}$$

We note that the matrices S and C occurred in a problem in the American Mathematical Monthly proposed by Ron Evans and Jerrold Griggs [5]. See page 64 of the January 1991 issue for the solution.

We now turn to the proof of (1.10). First we show how (1.10) follows from Theorem 3.4 when p is a prime. Then we show how, for general odd p, (1.10) follows from the $k = 1$ case of (1.6) using Jacobi's imaginary transformation.

We assume p is an odd prime. First we write (1.10) in terms of the $T(k, p, n)$ defined in (3.1). For $k \not\equiv 0 \pmod{p}$ we define $k' \pmod{p}$ by

$$k' \equiv \frac{2}{k} \pmod{p}. \tag{3.47}$$

By replacing ℓ by $\frac{k+1}{2}$ and using (3.1), (3.39) we find that (1.10) may be written as

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{p-2} (-1)^{\frac{k+1}{2}} q^{\frac{k^2-1}{8}} \sum_n T(k,p,n) \big(\zeta_p^{k'n} - \overline{\zeta}_p^{k'n}\big) q^n = (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} G_1, \tag{3.48}$$

or

(3.49)
$$\sum_n \sum_{\substack{k=1 \\ k \text{ odd}}}^{p-2} \sum_{r=0}^{p-1} (-1)^{\frac{k+1}{2}} \big(\zeta_p^{k'(r-(\frac{k^2-1}{8}))} - \overline{\zeta}_p^{k'(r-(\frac{k^2-1}{8}))}\big) T(k,p,pn+r-\frac{k^2-1}{8}) q^{pn+r}$$
$$= (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} G_1.$$

Since G_1 is a function of q^p we must first show that

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{p-2} (-1)^{\frac{k+1}{2}} \big(\zeta_p^{k'(r-(\frac{k^2-1}{8}))} - \overline{\zeta}_p^{k'(r-(\frac{k^2-1}{8}))}\big) T(k,p,pn+r-\frac{k^2-1}{8}) = 0, \tag{3.50}$$

for $0 \le r \le p-1$ with $r \not\equiv -\frac{1}{8} \pmod{p}$. We prove (3.50) by showing that each term on the left side is either zero or can be paired with another term of opposite sign.

Let $0 \le r \le p-1$ be fixed and suppose $r \not\equiv -\frac{1}{8} \pmod{p}$. We consider each of the integers $1 \le k \le p-2$ with k odd. Each such integer k is one of two types: *Type(I)* or *Type(II)*. The types correspond to the two parts of Theorem 3.4 namely, (3.5) and (3.6). More explicitly the types are defined as follows. First we consider such an integer k. The sequence of consecutive integers

$$\frac{k-(p-2)}{2}, \dots, \frac{k+(p-2)}{2} \tag{3.51}$$

represent all residue classes (mod p) except the class congruent to $\frac{k}{2}$ (mod p). Since $r \not\equiv -\frac{1}{8}$ (mod p) we may choose an integer ℓ_1 from the sequence (3.51) such that

$$\ell_1 \equiv \frac{1}{2k}(k^2 - (8r+1)) \pmod{p}, \tag{3.52}$$

so that

$$r - \left(\frac{k^2-1}{8}\right) \equiv -\frac{k\ell_1}{4} \pmod{p}. \tag{3.53}$$

If $\frac{k+1}{2} \le \ell_1 \le \frac{k+(p-2)}{2}$ we say k is of Type(I) and we let $\ell = \ell_1$. On the other hand if $\frac{k-(p-2)}{2} \le \ell_1 \le \frac{k-1}{2}$ we say k is of Type(II) and we let $\ell = -\ell_1$.

We now fix n. We define the set

$$\mathcal{S} := \{1, 3, \ldots, p-2\}.$$

For $k \in \mathcal{S}$ we define the weight $\omega(k)$ as

$$\omega(k) := (-1)^{\frac{k+1}{2}}\big(\zeta_p^{k'(r-(\frac{k^2-1}{8}))} - \overline{\zeta}_p^{k'(r-(\frac{k^2-1}{8}))}\big)T(k,p,pn+r-\frac{k^2-1}{8}),$$

so that (3.50) may be written as

$$\sum_{k\in\mathcal{S}} \omega(k) = 0. \tag{3.54}$$

Equation (3.54) and hence (3.50) will follow by constructing a sign-reversing involution Φ

$$\Phi : \mathcal{S} \longrightarrow \mathcal{S}; \quad \omega(\Phi(k)) = -\omega(k).$$

Our map Φ is defined simply as $\Phi(k) = 2\ell - k$ where $\ell = \ell(k)$ is defined above. If k is of Type(I) then an easy calculation shows that

$$k'\big(r - \left(\frac{k^2-1}{8}\right)\big) \equiv (2\ell-k)'\big(r - \left(\frac{(2\ell-k)^2-1}{8}\right)\big) \pmod{p}, \tag{3.55}$$

$$\frac{k^2-1}{8} + \frac{\ell(\ell-k)}{2} = \frac{(2\ell-k)^2-1}{8}, \tag{3.56}$$

$$(-1)^{\frac{k+1}{2}}(-1)^{\ell} = -(-1)^{\frac{2\ell-k+1}{2}}. \tag{3.57}$$

Also we have

$$r - \left(\frac{k^2-1}{8}\right) \equiv -\frac{k\ell}{4} \pmod p, \tag{3.58}$$

by (3.52). Hence by (3.5) we have

$$\omega(\Phi(k)) = \omega(2\ell - k) = -\omega(k). \tag{3.59}$$

We find that

$$1 \le 2\ell - k \le p-2, \tag{3.60}$$

$$\frac{2\ell - k}{2} \le \ell \le \frac{(2\ell-k)+p-2}{2}, \tag{3.61}$$

and

$$r - \left(\frac{(2\ell-k)^2-1}{8}\right) \equiv -\frac{\ell}{4}(2\ell-k) \pmod p. \tag{3.62}$$

It follows that $\Phi(k) \in \mathcal{S}$, $\Phi(k)$ is also Type(I) with the same value of ℓ as k, so that

$$\Phi(\Phi(k)) = 2\ell - (2\ell - k) = k.$$

Similarly, when k is of Type(II) we find that

$$\omega(\Phi(k)) = -\omega(k)$$

using (3.6), $\Phi(k) \in \mathcal{S}$, $\Phi(k)$ is of Type(II) and $\Phi(\Phi(k)) = k$. Hence Φ is a sign-reversing involution on $\mathcal{S}$ as required.

From (3.49) we now see that the proof of (1.10) is reduced to proving

(3.63)

$$\begin{aligned}\sum_n \sum_{\substack{k=1\\k\text{ odd}}}^{p-2} &(-1)^{\frac{k+1}{2}}\left(\zeta_p^{k'(r_0-(\frac{k^2-1}{8}))} - \overline{\zeta}_p^{k'(r_0-(\frac{k^2-1}{8}))}\right)T(k,p,pn+r_0-\frac{k^2-1}{8})q^{pn+r_0}\\ = \quad &(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} G_1,\end{aligned}$$

where $1 \le r_0 \le p-1$ and $r_0 \equiv \frac{p^2-1}{8} \pmod{p}$. From (3.42) we find that the left side of (3.63) may be written as

(3.64)

$$(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{p-2} \left(\zeta_p^{k/4} - \zeta_p^{-k/4}\right) F_{(-k/4)}(q) \qquad \text{(where each } -k/4 \text{ is reduced modulo } p\text{)},$$

$$= (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} \sum_{k=1}^{p-1} \zeta_p^k F_k(q) \qquad \text{(since } \pm k \ (k \text{ odd}, k = 1, 3, \ldots, p-2) \text{ forms a complete non-zero residue system modulo } p\text{)},$$

$$= (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} G_1 \qquad \text{(by (3.40))},$$

and this gives the result (3.63) and this completes the proof of (1.10) for p prime.

We now show how, for general odd p, (1.10) follows from the $k=1$ case of (1.6) using Jacobi's imaginary transformation formula [2, Eq. (2.2.5) p.38]

(3.65)
$$\sum_{n=-\infty}^{\infty} e^{-\frac{1}{s}(n+x)^2\pi} = \sqrt{s} \sum_{k=-\infty}^{\infty} e^{2\pi i k x} e^{-\pi k^2 s},$$

where $\mathrm{re}(s) > 0$. If we let $x = \frac{m}{2p} + \frac{\ell\tau}{2p}$ and $s = -i\tau$ then (3.65) becomes

(3.66)
$$\sqrt{-i\tau} \sum_{k=-\infty}^{\infty} e^{\frac{\pi i k m}{p}} q^{\frac{1}{2}(k^2 + \frac{\ell k}{p}) + \frac{\ell^2}{8p^2}}$$
$$= e^{-\frac{\pi i m \ell}{2p^2}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n \ell}{p}} q'^{\frac{1}{2}(n^2 + \frac{nm}{p}) + \frac{m^2}{8p^2}},$$

where $q = \exp(\pi i \tau)$ $q' = q(-1/\tau) = \exp(-\frac{\pi i}{\tau})$ and $\mathrm{im}(\tau) > 0$. The $k=1$ case of (1.6) may be written as

(3.67)
$$\sum_{\ell=0}^{p-1} \zeta_p^{\frac{\ell(1-p^2)}{4}} \left(\sum_{n=-\infty}^{\infty} (-1)^n \zeta_p^{\frac{n\ell(1+p)}{2}} q^{\frac{n}{2}(pn+1)} \right)^p = 0.$$

We observe that

$$(-1)^n \zeta_p^{\frac{n\ell(1+p)}{2}} = \exp(n\pi i\{\ell - 1 + \tfrac{\ell}{p}\}) = \begin{cases} \exp(\frac{\pi i n \ell}{p}), & \ell \text{ odd}, \\ \exp(\frac{\pi i n(\ell - p)}{p}), & \ell \text{ even}. \end{cases}$$

If we multiply both sides of (3.67) by $\sqrt{-i\tau}q^{\frac{1}{8p^2}}$, apply the transformation (3.66) with $\ell = 1$, and replace q' by q^p we obtain

(3.68)
$$\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{p-2} e^{\frac{-\pi i\ell p}{2}} q^{\frac{\ell^2}{8}} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n}{p}} q^{\frac{n}{2}(pn+\ell)} \right)^p + e^{\frac{\pi i}{2}} \sum_{\substack{\ell=1\\ \ell \text{ even}}}^{p-2} e^{\frac{-\pi i\ell p}{2}} q^{\frac{(\ell-p)^2}{8}} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n}{p}} q^{\frac{n}{2}(pn+\ell-p)} \right)^p$$
$$+ e^{-\frac{\pi i}{2}} q^{\frac{p^2}{8}} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n}{p}} q^{\frac{pn}{2}(n+1)} \right)^p = 0.$$

On replacing ℓ by $p-\ell$ in the outer sum in the second term, then simplifying and reversing the order of summation in the inner sum we have

(3.69)
$$\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{p-2} e^{\frac{-\pi i\ell p}{2}} q^{\frac{\ell^2}{8}} \left\{ \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n}{p}} q^{\frac{n}{2}(pn+\ell)} \right)^p - \left(\sum_{n=-\infty}^{\infty} e^{\frac{\pi i n}{p}} q^{\frac{n}{2}(pn+\ell)} \right)^p \right\}$$
$$= -e^{-\frac{\pi i}{2}} q^{\frac{p^2}{8}} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n}{p}} q^{\frac{pn}{2}(n+1)} \right)^p.$$

On multiplying both sides by $(-1)^{\frac{p+1}{2}} e^{\pi i/2} q^{-\frac{1}{8}}$ and simplifying we have

$$\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{p-2} (-1)^{\frac{\ell+1}{2}} q^{\frac{\ell^2-1}{8}} \left\{ \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n}{p}} q^{\frac{n}{2}(pn+\ell)} \right)^p - \left(\sum_{n=-\infty}^{\infty} e^{\frac{\pi i n}{p}} q^{\frac{n}{2}(pn+\ell)} \right)^p \right\}$$
$$= (-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} \left(\sum_{n=-\infty}^{\infty} e^{-\frac{\pi i n}{p}} q^{\frac{pn}{2}(n+1)} \right)^p,$$

which is (1.10) with ζ_p replaced by $\zeta_p^{-(p-1)/2}$ which is a primitive p-th root of unity, so that (1.10) clearly follows.

4. Concluding remarks

The following classical identity follows easily from Theorem 4 and Corollary 1 in [8]

$$\vartheta^4(-1, q) = \vartheta^4(1, q) + q\vartheta^4(-q, q). \tag{4.1}$$

It is of interest that the two theta functions $\vartheta^2(-1, q)$ and $\vartheta^2(1, q)$ parametrize the arithmetic-geometric mean iteration of Gauss [2, Chapter 1] whose limit is identified with the hypergeometric function ${}_2F_1(\frac{1}{2},\frac{1}{2};1;\cdot)$. Recently, Borwein and Borwein [3] were able to parametrize a cubic mean iteration whose limit is identified with the the hypergeometric function ${}_2F_1(\frac{1}{3},\frac{2}{3};1;\cdot)$. For a survey of recent related results see [9]. Some of the properties of the Borweins' cubic iteration bear some striking similarities to the classical case of Gauss. This time the parametrization is in terms of

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{4.2}$$

and

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{m^2+mn+n^2} \quad \text{where } \omega = \exp(2\pi i/3). \tag{4.3}$$

The analogue of (4.1) is

$$a^3(q) = b^3(q) + c^3(q), \tag{4.4}$$

where

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}. \tag{4.5}$$

Equation (4.4) bears a striking resemblance to (1.2). It is however a different identity. It would be interesting to investigate whether the theta-functions in (1.2) occurred in the parametrization of some two-term iteration whose limit could be explicitly identified. There are connections with (1.2), (1.3). From (2.7), (2.8) we have

$$\eta^3(q) = \epsilon_0(q^3) + q\epsilon_1(q^3) + q^2\epsilon_2(q^3) = \epsilon_0(q^3) - 3\,q\eta^3(q^9), \tag{4.6}$$

for certain power series $\epsilon_i(q)$. The fact that $\epsilon_2(q) = 0$ led to identity (1.2). The identity $\epsilon_1(q) = -3\,\eta^3(q^3)$ led to identity (1.3). Below we identify $\epsilon_0(q)$ which leads analogously to a third identity. We find that

$$\epsilon_0(q) = \eta(q)\,a(q), \tag{4.7}$$

where $a(q)$ is defined in (4.2). From [4] we have

$$b(q) = \frac{\eta^3(q)}{\eta(q^3)}, \tag{4.8}$$

$$c(q) = 3\,q^{\frac{1}{3}}\,\frac{\eta^3(q^3)}{\eta(q)}, \tag{4.9}$$

$$b(q) = a(q^3) - c(q^3). \tag{4.10}$$

If we multiply both sides of (4.10) by $\eta(q^3)$ we obtain

$$\eta^3(q) = \eta(q^3)\, a(q^3) - 3\, q\eta^3(q^9), \tag{4.11}$$

which gives (4.7). In [4] we showed how the cubic identity (4.4) follows easily from (4.8) and (4.10).

Using (1.2) and (2.9) we find that (4.7) may be written as the identity

$$\vartheta^3(\omega^2 q^{\frac{1}{2}}, q^{\frac{3}{2}}) - \omega^2 \vartheta^3(\omega q^{\frac{1}{2}}, q^{\frac{3}{2}}) = \vartheta(\omega q^{\frac{1}{2}}, q^{\frac{1}{2}})\, a(q^3). \tag{4.12}$$

It should be pointed out that Kopeliovich [12] has found a multidimensional generalization of the p-th power identity (1.6). It would interesting to see whether the methods of this paper can be extended to the multidimensional case.

Acknowledgments

I would first like to thank Bruce Berndt and Jon Borwein for communicating the Farkas-Kra cubic identities to me. When I wrote the original version of this paper I was unaware of the work of Farkas and Kopeliovich. I had independently discovered and proved the p-th power generalization (1.6) of (1.3). I was also unaware that Kopeliovich had found a multidimensional analogue. I would like to thank the referee and others for pointing out these oversights. I would also like to thank Farkas, Kra and Kopeliovich for sending preprints of their papers and for useful discussions. In the original version I had found only one generalization of (1.3), namely (1.8). I tried to find a more straightforward generalization after H. Farkas pointed out to me that (1.3) follows from (1.2) via a modular transformation. Thus I found (1.10) by applying a similar transformation to (1.6). But my goal was then to find a more combinatorial proof.

References

1. G. E. Andrews, "The Theory of Partitions," Encyclopedia of Mathematics and Its Applications, Vol.2 (G.-C. Rota, ed.), Cambridge Univ. Press, N.Y., 1985.
2. J. M. Borwein and P. B. Borwein, "Pi and the AGM – A Study in Analytic Number Theory and Computational Complexity," Wiley, N.Y., 1987.
3. J. M. Borwein and P. B. Borwein, A cubic counterpart of Jacobi's identity and the AGM, *Trans. Amer. Math. Soc.* **323** (1991), 691–701.
4. J. M. Borwein, P. B. Borwein and F. G. Garvan, Some cubic modular identities of Ramanujan, *Trans. Amer. Math. Soc.* **343** (1994), 35–47.
5. R. J. Evans and J. R. Griggs, Problem 6591, *Amer. Math. Monthly*, **96** (1989), 66.

6. H. M. Farkas and Y. Kopeliovich, New theta constant identities, *Israel J. Math.* **82** (1993), 133–141.
7. H. M. Farkas and Y. Kopeliovich, New theta constant identities II, *Proc. Amer. Math. Soc*, to appear.
8. H. M. Farkas and I. Kra, Automorphic forms for subgroups of the modular group, *Israel J. Math.* **82** (1993), 87–131.
9. F. G. Garvan, Cubic modular identities of Ramanujan, hypergeometric functions and analogues of the arithmetic-geometric mean iteration, Proceedings of the Rademacher Centenary Conference (G. E. Andrews, D. M. Bressoud and L. A. Parsons, eds), *Contemp. Math.* **166** (1994), 245–264.
10. G.H. Hardy and E.M. Wright, "An Introduction to the Theory of Numbers," Oxford Univ. Press, London, 1979.
11. O. Kolberg, Some identities involving the partition function, *Math. Scand.* **5** (1957), 77-92.
12. Y. Kopeliovich, Multidimensional theta constant identities, preprint.
13. Morris Newman, An identity for the coefficients of certain modular forms, *J. London Math. Soc.* **30** (1955), 488–493.