A PROOF OF THE MACDONALD-MORRIS ROOT SYSTEM CONJECTURE FOR F_4

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Abstract. We give a proof of the Macdonald-Morris root system conjecture for F_4 that draws on ideas from Zeilberger's recent proof of the $G_2^\vee$ case and Kadell's proof of the q-BC_n case. Our proof depends on much computer computation. As in Zeilberger's proof the problem is reduced to solving a system of linear equations. A FORTRAN program generated the equations which were solved using the computer algebra package MAPLE.

1. Introduction.

The Macdonald-Morris root system conjecture for F_4 is

$$C.T. \prod_{1\le i<j\le 4}(1-x_ix_j)^a(1-x_i^{-1}x_j^{-1})^a(1-\frac{x_i}{x_j})^a(1-\frac{x_j}{x_i})^a \tag{1.1}$$

$$\cdot\prod_{i=1}^{4}(1-x_i^2)^b(1-x_i^{-2})^b\prod_{r_1,r_2,r_3,r_4=\pm1}(1-x_1^{r_1}x_2^{r_2}x_3^{r_3}x_4^{r_4})^b$$

$$=\frac{(6a+6b)!(4a+4b)!(2a+6b)!(4a+2b)!(2a+4b)!(4b)!(3a)!}{(5a+6b)!(3a+5b)!(3a+4b)!(3a+2b)!(2a+3b)!(a+3b)!(2a+b)!}$$

$$\cdot\frac{(3b)!(2a)!(2b)!}{(a+2b)!(a+b)!a!a!b!b!b!}=f(a,b)\ .$$

Here $C.T.$ means constant term of the Laurent polynomial in $x_1,x_1^{-1},\dots,x_4,x_4^{-1}$. We refer to the reader to Macdonald [10] for more general statements of the root system conjectures. The A_n,B_n,C_n,D_n,BC_n and G_2 cases of the Macdonald-Morris root system conjectures have been proved. The conjectures have remained open for the F_4,E_6,E_7,E_8 cases. The best proof of the A_n case is due to Good [6]. Macdonald [10] noticed that the BC_n case (and hence the B_n,C_n, and D_n cases) follow from Selberg's [12] integral. The G_2 case was proved independently by Habsieger [7] and Zeilberger [13]. There are q-analogs of these conjectures.

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We refer the reader to Macdonald [10] and Askey [2] for more details. Recently, Zeilberger [14] has proved the $G_2^\vee$ case and Kadell [9] has proved the q-BC_n case.

The goal of this paper is to prove the F_4 case, namely (1.1). Kadell's paper [9, §2] also contains a new proof of the ($q = 1$) BC_n case of the Macdonald-Morris root system conjecture that avoids integrals. This new proof is analogous to Aomoto's [1] proof of Selberg's integral in the following sense: It involves adding extra factors to the Laurent polynomial as opposed to adding extra factors to the integrand of Selberg's integral and Aomoto's integration by parts is replaced by the fact that the derivative of a Laurent polynomial has no residue. We extend Kadell's proof to the F_4 case but another idea is needed. The extra idea comes from Zeilberger's [14] proof of the $G_2^\vee$ case of the q-analog of Macdonald-Morris root system conjecture. As in Zeilberger [14] the problem is reduced to finding and solving a system of linear equations whose unknowns are constant terms of certain Laurent polynomials. These equations are generated with the aid of a FORTRAN program. Finally the equations are solved using the computer algebra package MAPLE.

After some preliminaries in §2 an idea of the proof is given in §3. The results behind the FORTRAN program that generates the desired equations are given in §§4-6. The proof is completed in §7.

We have been able to verify the results of this paper by another method. Recently we [5] have found a new proof of the G_2 case of the Macdonald-Morris root system conjecture which is solely in terms of integrals. Our proof was motivated by some conjectures of Askey [2], that have to do with adding roots to the G_2 case of the Macdonald-Morris root system conjecture, and is analogous to Aomoto's [1] proof of the Selberg's integral. We have been able to extend our integral proof of the G_2 case to the F_4 case. However, this other proof involves finding equations between certain integrals and converting these into equations involving constant terms. The proof then proceeds as usual by solving a system of linear equations. We have omitted this other proof, finding the approach of working with Laurent polynomials rather than with integrals more straightforward.

In §8 we give some other results that involve adding extra factors to the F_4 case of the Macdonald-Morris root system conjecture. The results are analogous to Askey's [2] conjectures for G_2. Although many of these results can be written as products of factorials we are unable to generalize them to all root systems. These other results may indicate that a simpler proof of the F_4 case is possible.

All computer programs used in this paper are available from the author on request. Some preliminary calculations were done using REDUCE at the University of Wisconsin, Madison. The final FORTRAN and MAPLE programs were run on an APOLLO DN-5800 at the I.M.A., University of Minnesota, Minneapolis.

2. Some Preliminaries.

In this section we prove some properties of the root system F_4 that will be needed later. We assume that the reader is familiar with the basics of root systems and their Weyl groups. See Bourbaki [3], Carter [4] and Humphreys [8] for treatments of root systems and Weyl groups.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{R}^4$. The roots of F_4 are usually written as

$$\pm e_i \quad (1 \le i \le 4), \qquad \pm e_i \pm e_j \qquad (1 \le i < j \le 4), \qquad (2.1)$$
$$\tfrac{1}{2}\,(\pm e_1 \pm e_2 \pm e_3 \pm e_4).$$

See, for example Bourbaki [3, p. 272]. We call this set of roots $\Phi^{(1)}(F_4)$. It is clear that the long roots of F_4 are isomorphic to D_4. In this paper we shall use two other ways of writing the roots of F_4.

Firstly, we rewrite the roots of F_4 in a way that makes it clear that the short roots of F_4 are isomorphic to D_4. Let $\Phi^{(2)}(F_4)$ be the set of vectors:

$$\pm 2e_i \quad (1 \le i \le 4), \qquad \pm e_i \pm e_j \qquad (1 \le i < j \le 4), \qquad (2.2)$$
$$(\pm e_1 \pm e_2 \pm e_3 \pm e_4).$$

$\Phi^{(2)}(F_4)$ and $\Phi^{(1)}(F_4)$ are isomorphic as root systems. The isomorphism is given by

$$A:\ \Phi^{(1)}(F_4) \longrightarrow \Phi^{(2)}(F_4) \qquad (2.3)$$

where A is the transformation with matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \qquad (2.4)$$

with respect to the standard basis of $\mathbb{R}^4$. This is a root system isomorphism since $A^tA = 2I$. As an immediate consequence we have

Lemma 2.5. Short roots of $F_4 \cong D_4$.

Secondly, we may write the roots of F_4 (as given in (2.2)) as $\mathbb{Z}$-linear combinations of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ where

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_3 + e_4,\ \alpha_3 = e_3 - e_4,\ \alpha_4 = e_2 - e_3. \qquad (2.6)$$

These $\mathbb{Z}$-linear combination are given Appendix A. The α_i come from the Dynkin diagram for D_4 which is given below in Figure 1.

Figure 1 Dynkin diagram for D_4

From the symmetry of the Dynkin diagram we see that any permutation of $\alpha_1, \alpha_2, \alpha_3$ leaves the root system of D_4 invariant. It is interesting to note that any such permutation also leaves the root system of F_4 invariant. Let $\pi \in S_3$ and suppose $\alpha = \sum_{i=1}^{4} k_i\alpha_i \in \Phi^{(2)}(F_4)$ where $k_i \in \mathbb{Z}$ $(1 \le i \le 4)$. Then we define

$$\pi\alpha = k_1\alpha_{\pi(1)} + k_2\ \alpha_{\pi(2)} + k_3\alpha_{\pi(3)} + k_4\alpha_4. \tag{2.7}$$

We have

Lemma 2.8. For $\pi \in S_3$, $\qquad \pi(\Phi^{(2)}(F_4)) = \Phi^{(2)}(F_4)$.

For a root system R we denote its Weyl group by $W(R)$. We need a nice way to code the elements of $W(F_4)$. By Bourbaki [3, p. 257] $W(D_4)$ consists of all signed permutations with an even number of signs, that act on the coordinates e_1, e_2, e_3, e_4. Let H denote the set of all signed permutations that act on the coordinates e_1, e_2, e_3, e_4. For $\alpha \in R$ we denote by w_α the reflection through the hyperplane orthogonal to α. Since $w_{2e_i} \in W(F_4)$ $\quad (1 \le i \le 4)$ we have

$$W(D_4)\ \subset\ H\ \subset\ W(F_4). \tag{2.9}$$

We introduce some notation to describe the elements of H. We denote the permutations on the coordinates e_1, e_2, e_3, e_4 using the usual cycle notation. We define the sign changes as follows: For $1 \le i \le 4$ let s_i denote the transformation given by

$$s_i : \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \qquad e_i \longmapsto -e_i \qquad \text{and} \qquad e_j \longmapsto e_j \quad (j \ne i). \tag{2.10}$$

For example,

$$s_1(34)\ (e_1 + e_2 + 2e_3 - e_4) = -e_1 + e_2 - e_3 + 2e_4.$$

Lemma 2.11. Every $w \in W(F_4)$ can be written

$$w = (\tau\sigma)^k\ h$$

where $k = 0, 1, 2$ $\quad h \in H$ and $\tau = w_{2e_4}$ and $\sigma = w_{e_1-e_2-e_3-e_4}$.

Proof.

$$| H | = 2^4.4! = 2^7.3$$
$$| W(F_4) | = 2^7.3^2 \qquad \text{(Bourbaki [3, p.273 (X)])}.$$

The result follows since $\tau\sigma \notin H$ and $(\tau\sigma)^3 = I$. $\square$

Let,

(2.12)
$$\overline{C} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R} : \quad x_1 \ge x_2 \ge x_3 \ge x_4 \ge 0,$$
$$x_2 + x_3 + x_4 \ge x_1,\ x_1 + x_4 \ge x_2 + x_3\}.$$

Lemma 2.13. For every $\underset{\sim}{x} \in \mathbb{R}^4$ there is a unique $\underset{\sim}{v} \in \overline{C}$ such that $\underset{\sim}{x}$ can be transformed into $\underset{\sim}{v}$ by some element of $W(F_4)$.

Proof. From Bourbaki [3, p. 272 (II)] the following vectors form a base for F_4:

$$e_2 - e_3,\ e_3 - e_4,\ e_4,\ \tfrac{1}{2}\,(e_1 - e_2 - e_3 - e_4), \tag{2.14}$$

with the roots written in the usual way (i.e: as elements of $\Phi^{(1)}(F_4)$). In our representation of F_4 (i.e: as elements of $\Phi^{(2)}(F_4)$) this corresponds to

$$e_1 - e_2 + e_3 - e_4,\ -e_1 + e_2 - e_3 - e_4,\ e_1 + e_4,\ -e_1 + e_3. \tag{2.15}$$

By applying the relevant signed permutation we find that the following vectors also form a base for F_4:

$$-e_1 + e_2 + e_3 + e_4,\ e_1 - e_2 - e_3 + e_4,\ e_3 - e_4,\ -e_3 + e_2, \tag{2.16}$$

by Carter [4, Theorem 2.2.4]. The closure of the chamber, C, corresponding to this base is given in (2.12). The result follows from Carter [4, Proposition 2.3.4]. □

Let,

$$\text{funch}:\ \mathbb{R}^4 \longrightarrow \overline{C} \qquad \text{funch}\,(\underset{\sim}{x}) = \underset{\sim}{v} \tag{2.17}$$

with $\underset{\sim}{x}, \underset{\sim}{v}$ as in Lemma (2.13) above. For $\alpha = \sum_{i=1}^{4} k_i e_i$ $(k_i \in \mathbb{Z})$ we let

$$x^\alpha = \prod_{i=1}^{4} x_i^{k_i}. \tag{2.18}$$

The elements w of the Weyl group act on monomials by

$$w(x^\alpha) = x^{w(\alpha)}, \tag{2.19}$$

and by linearity on Laurent polynomials that are linear combinations of the x^α.

Let

$$F(\underset{\sim}{x}; a, b) = \prod_{\alpha \in \Phi^{(2)}(F_4)} (1 - x^\alpha)^{k_\alpha} \tag{2.20}$$

$$\text{where} \qquad k_\alpha = \begin{cases} a, & \text{if } \alpha \text{ is a short root,} \\ b, & \text{if } \alpha \text{ is a long root.} \end{cases}$$

We note that $F(\underset{\sim}{x}; a, b)$ is the Laurent polynomial on the left hand side of (1.1).

Lemma 2.21. For $w \in W(F_4)$,

$$C.T.\ x^\alpha\ F = C.T.\ x^{w(\alpha)} F$$

where F is defined in (2.20).

Proof. The result follows from the fact that F is symmetric with respect to the Weyl group and w does not change the constant term. □

3. The idea of the proof.

Let,

$$f'(a,b) = C.T. F(\underset{\sim}{x}; a, b) \tag{3.1}$$

where F is defined in (2.20). Our goal is to prove that $f'(a,b) = f(a,b)$ for all $a, b \geq 0$. The idea is to proceed by induction on a. That is, we want to prove that

$$\frac{f'(a+1,b)}{f'(a,b)} = \frac{f(a+1,b)}{f(a,b)} . \tag{3.2}$$

This will be enough because the case $a = 0$ is already known since

$$\text{Long roots of } F_4 \cong D_4. \tag{3.3}$$

The flavor of our proof is similar to Zeilberger's [14] proof of the $G_2^\vee$ case of the q-version of the Macdonald-Morris root system conjecture. Let L be the lattice generated by α ($\alpha \in F_4$). Now,

(3.4)

$$\begin{aligned}
f'(a+1,b) &= C.T.\ F(\underset{\sim}{x}; a+1, b) \\
&= C.T. \prod_{\alpha \in \text{ short } F_4} (1 - x^\alpha)\ F(\underset{\sim}{x}; a, b) \\
&= C.T. \prod_{\alpha \in D_4} (1 - x^\alpha) F(\underset{\sim}{x}; a, b) \qquad \text{(by Lemma (2.5))} \\
&= C.T. \sum_{\alpha \in L'} a_\alpha x^\alpha F(\underset{\sim}{x}; a, b) \qquad \text{(for some } L' \subset L) \\
&= C.T. \sum_{\alpha \in L'} a_\alpha x^{\text{funch}(\alpha)}\ F(\underset{\sim}{x}; a, b) \qquad \text{(by Lemma (2.13), (2.17) and Lemma (2.21))} \\
&= C.T. \sum_{\alpha \in S} a'_\alpha x^\alpha\ F(\underset{\sim}{x}; a, b)
\end{aligned}$$

for some finite subset S of $\overline{C}$, which is defined in (2.12). $\prod_{\alpha \in D_4} (1 - x^\alpha)$ was multiplied at utilizing a FORTRAN program, funch(α) was calculated for each monomial x^α that arose in each stage of the multiplication. The 37 vectors that arose, $v(i)$ $(1 \leq i \leq 37)$, are listed in Appendix B. $S = \{v(i)\ :\ 1 \leq i \leq 37\}$ and

$$\sum_{\alpha \in S} a'_\alpha x^\alpha = 192 x^{v(1)} - 768 x^{v(2)} + \cdots + 192 x^{v(37)}. \tag{3.5}$$

The complete list of coefficients in (3.5) is given in Appendix B. Let,

$$an(i) = C.T.\ x^{v(i)} F(\underset{\sim}{x}; a, b) \qquad (1 \leq i \leq 37). \tag{3.6}$$

The problem is to get $an(i)$ $(2 \leq i \leq 37)$ in terms of $an(1) = C.T.\ x^{v(1)} F(\underset{\sim}{x}; a, b) = C.T.\ F(\underset{\sim}{x}; a, b) = f'(a, b)$. Once we have done this (3.2) should follow from (3.4) and (3.5).

Hence we need to find 36 independent equations in the unknowns $an(i)$ $(1 \leq i \leq 37)$. Our goal is to write a FORTRAN program that will generate equations. The input of this program is a 4-tuple $\underset{\sim}{k} = (k_1, k_2, k_3, k_4) \in \mathbb{N}^4$ and the output will be either a homogeneous linear equation in the $an(i)$ $(1 \leq i \leq 37)$ or an error message, which says such an equation is not possible. It will turn out that when $\underset{\sim}{k} = v(i)$ $(2 \leq i \leq 37)$ that the corresponding outputs will be the required 36 independent equations in the $an(i)$ $(1 \leq i \leq 37)$. See §6 for more details. We describe how we came by this program in 4 steps:

STEP 1: Use

$$C.T.\ x_1 \frac{\partial}{\partial x_1}\, x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4}\ F(\underset{\sim}{x}; a, b) = 0\ . \tag{3.7}$$

As noted before this idea was used by Kadell [9] in his proof of the q-BC_n case. This gives rise to an equation involving constant terms of rational functions in x_1, x_2, x_3, x_4 times F.

STEP 2: Use the Weyl group to reduce the number of types of terms arising in STEP 1.

STEP 3: Use the Weyl group to write the constant terms that arise in STEP 2 as constant terms of Laurent polynomials times F.

STEP 4: Use the Weyl group to write the constant terms that arise in STEP 3 in terms of the $an(i)$ $(1 \leq i \leq 37)$, if possible.

In this way, for certain $\underset{\sim}{k}$, (3.7) can be written as a linear homogeneous equation in the $an(i)$ $(1 \leq i \leq 37)$.

4. Steps 1 and 2: An Equation involving constant terms with denominators.

In this section we describe the first two steps, mentioned in §3, that are needed in turning the equation (3.7) into an equation involving the $an(i)$. (3.7) is

(4.1)

$$\begin{aligned} 0 = C.T.\Bigg\{ \Bigg(& k_1 + a \sum_{j=2}^{4} \left(\frac{-x_1 x_j}{1 - x_1 x_j} + \frac{x_1^{-1} x_j^{-1}}{1 - x_1^{-1} x_j^{-1}} - \frac{x_1/x_j}{1 - x_1/x_j} + \frac{x_j/x_1}{1 - x_j/x_1} \right) \\ & + b \left(\frac{-2x_1^2}{1 - x_1^2} + \frac{2x_1^{-2}}{1 - x_1^{-2}} \right) \\ & + b \sum_{r_2, r_3, r_4 = \pm 1} \left(\frac{-x_1 x_2^{r_2} x_3^{r_3} x_4^{r_4}}{1 - x_1 x_2^{r_2} x_3^{r_3} x_4^{r_4}} + \frac{x_1^{-1} x_2^{r_2} x_3^{r_3} x_4^{r_4}}{1 - x_1^{-1} x_2^{r_2} x_3^{r_3} x_4^{r_4}} \right) \Bigg) \\ & . \, x^{\underset{\sim}{k}}\ F(\underset{\sim}{x}; a, b) \Bigg\}, \end{aligned}$$

where $\underset{\sim}{k} = (k_1, k_2, k_3, k_4)$. Hence,

$$\text{(4.2)} \qquad C.T.\ k_1\ x^{\underset{\sim}{k}}\ F$$
$$= C.T. \left(a \sum_{j=2}^{4} \left(\frac{1 + x_1 x_j}{1 - x_1 x_j} + \frac{1 + \frac{x_1}{x_j}}{1 - \frac{x_1}{x_j}} \right) \right.$$
$$+ 2b\, \frac{(1 + x_1^2)}{(1 - x_1^2)}$$
$$\left. + b \sum_{r_2, r_3, r_4 = \pm 1} \frac{1 + x_1 x_2^{r_2} x_3^{r_3} x_4^{r_4}}{1 - x_1 x_2^{r_2} x_3^{r_3} x_4^{r_4}} \right) .\ x^{\underset{\sim}{k}}\ F.$$

This completes STEP 1.

In STEP 2 we reduce the number of different denominators appearing in STEP 1 to two (one for each root length). The reason this can be done is that each term on the right hand side of (4.2) is of the form $\dfrac{1 + x^\alpha}{1 - x^\alpha}$ for some $\alpha \in F_4$, and so can be converted to one of two types by using the fact that the Weyl group acts transitively on roots of equal length and by using Lemma (2.21). Hence for each $\alpha \in F_4$ we need to find a $w \in W(F_4)$ such that

$$w(\alpha) = \begin{cases} e_1 - e_3, & \alpha \text{ short}, \\ e_1 - e_2 + e_3 + e_4, & \alpha \text{ long}. \end{cases}$$

It is clear that for α of the form $e_i \pm e_j,\ e_1 \pm e_2 \pm e_3 \pm e_4$ we may take $w \in H$, the set of signed permutations (defined in §2), and w is easy to calculate. All that remains is to find a $w \in H$ such that $w(2e_1) = e_1 - e_2 + e_3 + e_4$. Let SYM denote the reflection through the hyperplane orthogonal to $e_1 - e_2 - e_3 - e_4$ then

$$2e_1 \xrightarrow{(13)} 2e_3 \xrightarrow{SYM} e_1 - e_2 + e_3 - e_4 \xrightarrow{s_4} e_1 - e_2 + e_3 + e_4\ .$$

Hence we find that (4.2) may be written as

$$\text{(4.3)} \qquad 0 = C.T. \left\{ k_1 x^{\underset{\sim}{k}} \right.$$
$$+ a\, \frac{(x_1 + x_3)}{(x_1 - x_3)}\, (x^{\underset{\sim}{k}} + (23) x^{\underset{\sim}{k}} + (34) x^{\underset{\sim}{k}}$$
$$+ s_3 x^{\underset{\sim}{k}} + (23) s_2 x^{\underset{\sim}{k}} + (34) s_4 x^{\underset{\sim}{k}})$$
$$+ b\, \frac{(x_2 + x_1 x_3 x_4)}{(x_1 x_3 x_4 - x_2)}\, (2\ s_4\ SYM\ (13) x^{\underset{\sim}{k}}$$
$$+ s_2 x^{\underset{\sim}{k}} + s_4 s_2 x^{\underset{\sim}{k}} + s_2 s_3 x^{\underset{\sim}{k}}$$
$$+ s_2 s_3 s_4 x^{\underset{\sim}{k}} + x^{\underset{\sim}{k}} + s_4 x^{\underset{\sim}{k}}$$
$$\left. + s_3 x^{\underset{\sim}{k}} + s_3 s_4 x^{\underset{\sim}{k}}) \right\} F(\underset{\sim}{x}; a, b).$$

5. STEP 3: Getting rid of denominators.

The constant term expressions that arise in STEP 2 can be written as either

$$C.T. \frac{p_1(\underset{\sim}{x})}{x_1 - x_3} F(\underset{\sim}{x}; a, b) \tag{5.1}$$

or

$$C.T. \frac{p_2(\underset{\sim}{x})}{x_1x_3x_4 - x_2} F(\underset{\sim}{x}; a, b), \tag{5.2}$$

where $p_i(\underset{\sim}{x})$ $(i = 1, 2)$ are Laurent polynomials. In this section we show how each of these expressions can be written in the form

$$C.T. p(\underset{\sim}{x}) \; F(\underset{\sim}{x}; a, b),$$

for some Laurent polynomial $p(\underset{\sim}{x})$, and how such an expression can be computed.

(5.1) is easy to handle. Since $F(\underset{\sim}{x}; a, b)$ is symmetric in x_1, x_3 we have

$$C.T. \frac{p_1(\underset{\sim}{x})}{x_1 - x_3} F(\underset{\sim}{x}; a, b) = \frac{1}{2} \, C.T. \left(\frac{p_1(\underset{\sim}{x}) - (13)p_1(\underset{\sim}{x})}{x_1 - x_3} \right) F(\underset{\sim}{x}; a, b) \tag{5.3}$$

and it is clear that $\dfrac{p_1(\underset{\sim}{x}) - (13)p_1(\underset{\sim}{x})}{x_1 - x_3}$ is a polynomial.

Before we can handle (5.2) we need to define an algorithm, FUN, whose input is a vector given in terms of the e_i $(1 \le i \le 4)$ and whose output is the same vector given in terms of the α_i $(1 \le i \le 4)$, defined in (2.6). FUN : $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is a linear transformation whose matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \tag{5.4}$$

with respect to the bases $\{e_i : \; 1 \le i \le 4\}, \{\alpha_i : 1 \le i \le 4\}$. The inverse of FUN is UNFUN with matrix

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \tag{5.5}$$

For $\alpha = \sum_{i=1}^{4} \ell_i \alpha_i \quad (\ell_i \in \mathbb{Z})$ we let

$$y^{\alpha} = \prod_{i=1}^{4} y_i^{\ell_i} \ . \tag{5.6}$$

FUN acts on monomials by

$$\text{FUN } (x^{\alpha}) = y^{\text{FUN } (\alpha)}, \tag{5.7}$$

and by linearity on Laurent polynomials that are linear combinations of the x^{α}. For example,

$$\begin{aligned} \text{FUN } \left(\frac{x_1 x_3 x_4}{x_2} \right) &= \text{FUN } (x^{e_1 - e_2 + e_3 + e_4}) \\ &= y^{\text{FUN } (e_1 - e_2 + e_3 + e_4)} \\ &= y^{\alpha_1 + \alpha_2} \\ &= y_1 y_2 \ . \end{aligned}$$

Alternatively, we could describe the action of FUN as replacing x_1 by y_1, x_2 by $\sqrt{y_1 y_2 y_3 y_4}$, x_3 by $\sqrt{y_1 y_2 y_3 / y_4}$ and x_4 by $y_1 y_2$.

From Lemma (2.8) it follows that

$$F'(\underset{\sim}{y}; a, b) = \text{FUN } (F(\underset{\sim}{x}; a, b)) \tag{5.8}$$

is symmetric in y_1, y_2, y_3. Now we can handle (5.2).

$$\begin{aligned} &C.T. \ \frac{p_2(\underset{\sim}{x})}{x_1 x_3 x_4 - x_2} \ F(\underset{\sim}{x}; a, b) \\ =&C.T. \ \frac{((12) p_2(\underset{\sim}{x}))}{x_2 x_3 x_4 - x_1} \ F(\underset{\sim}{x}; a, b) \qquad \text{(by Lemma (2.21))} \\ =&C.T. \left(\frac{x_1^{-1} (12) p_2(\underset{\sim}{x})}{x^{\alpha_2 - \alpha_1} - 1} \right) F(\underset{\sim}{x}; a, b) \\ =&C.T. \ \frac{q(\underset{\sim}{y})}{y_2 - y_1} \ F'(\underset{\sim}{y}; a, b) \qquad \text{(by applying FUN)} \end{aligned} \tag{5.9}$$

where $q(\underset{\sim}{y})$ is the Laurent polynomial

$$q(\underset{\sim}{y}) = y_1 \ \text{FUN } (x_1^{-1} (12) p_2(\underset{\sim}{x})). \tag{5.10}$$

Hence,

$$\begin{aligned} &C.T. \ \frac{p_2(\underset{\sim}{x})}{x_1 x_3 x_4 - x_2} \ F(\underset{\sim}{x}; a, b) \\ =&\frac{1}{2} \ C.T. \left(\frac{q(\underset{\sim}{y}) - (12) q(\underset{\sim}{y})}{y_2 - y_1} \right) F'(\underset{\sim}{y}; a, b) \\ =&\frac{1}{2} \ C.T. \ \text{UNFUN } \left(\frac{q(\underset{\sim}{y}) - (12) q(\underset{\sim}{y})}{y_2 - y_1} \right) F'(\underset{\sim}{x}; a, b) \end{aligned} \tag{5.11}$$

and UNFUN $\left(\dfrac{q(\underset{\sim}{y})-(12)q(\underset{\sim}{y})}{y_2-y_1}\right)$ is a Laurent polynomial in x_i $(1 \le i \le 4)$ as required.

We note that $\dfrac{p_1(\underset{\sim}{x})-(13)p_1(\underset{\sim}{x})}{x_1-x_3}$ and similarly UNFUN $\left(\dfrac{q(\underset{\sim}{y})-(12)q(\underset{\sim}{y})}{y_2-y_1}\right)$ can be easily computed by observing that if $p_1(\underset{\sim}{x}) = \sum_{\underset{\sim}{a}} c_{\underset{\sim}{a}}\, \underset{\sim}{x}^{\underset{\sim}{a}} = \sum_{\underset{\sim}{a}} c_{\underset{\sim}{a}} x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}$

then

(5.12)

$$\frac{p_1(\underset{\sim}{x})-(13)p_1(\underset{\sim}{x})}{x_1-x_3} = \sum_{\underset{\sim}{a}} \text{sgn } (a_1-a_3) c_{\underset{\sim}{a}} x_2^{a_2}x_4^{a_4} \sum_{\ell=0}^{|a_1-a_3|-1} x_1^{\min(a_1,a_3)+\ell} x_3^{\max(a_1,a_3)-\ell-1}.$$

6. STEP 4: Obtaining equations in the an(i).

In this section we describe the final step needed in converting the constant term equation, (3.7), into an equation involving the $an(i)$. An examination of STEPS 1-4 will then yield an algorithm whose input is a 4-tuple $\underset{\sim}{k} = (k_1,k_2,k_3,k_4) \in \mathbb{N}^4$ and whose output is an equation involving the $an(i)$, or an error message.

By STEP 3 we can write the constant term equation (3.7), in terms of constant terms of certain Laurent polynomials times $F(\underset{\sim}{x};a,b)$. In STEP 4 we would like to find such expressions in terms of the $an(i)$. To do this we use funch, defined in (2.17). Suppose we are given such an expression, say $p(\underset{\sim}{x})\ F(\underset{\sim}{x};a,b)$, where

$$p(\underset{\sim}{x}) = \sum_{\alpha\in L'} a_\alpha x^\alpha \qquad \text{(for some finite set } L'). \tag{6.1}$$

Then,

(6.2)

$$\begin{aligned} &C.T.\ p(\underset{\sim}{x})F(\underset{\sim}{x};a,b)\\ &\quad= C.T. \sum_{\alpha\in L'} a_a x^{\text{funch}(\alpha)}\ F(\underset{\sim}{x};a,b) && \text{(by Lemma (2.13), (2.17) and Lemma (2.20))}\\ &\quad= C.T. \sum_{\alpha\in S} a'_\alpha x^\alpha\ F(\underset{\sim}{x};a,b) && \text{(where } S \subseteq \text{funch } (L') \subseteq \overline{C} \text{ defined in (2.12))}\\ &\quad= \sum_{i=1}^{37} a'_{v(i)}\ an(i) && \text{(where } an(i) \text{ is defined in (3.6).} \end{aligned}$$

Note that here we are assuming $S \subseteq \{v(i)\}_{i=1}^{37}$ which may not necessarily be the case. However, for all values of the input $\underset{\sim}{k}$, that we use, this condition is satisfied. We leave it to the reader to write a subroutine that will do the reduction described in (6.2). This subroutine should check whether $S \subseteq \{v(i)\}_{i=1}^{37}$. If this condition is not satisfied the output of the subroutine should be some error message.

7. STEP 5: Generating the equations and completing the proof.

In the previous section we noted that STEPS 1-4 yield an algorithm whose input is a 4-tuple $\underset{\sim}{k} = (k_1,k_2,k_3,k_4) \in \mathbb{N}^4$ and whose output is an equation involving the $an(i)$. We have written a FORTRAN program that incorporates this algorithm. We leave it to the reader to use equation (4.3) and STEPS 3 and 4 to write such a program. We have found that the set of inputs $\underset{\sim}{k} = v(i)$ $(2 \leq i \leq 37)$, given in Appendix B, yield a system of 36 independent equations in the $an(i)$, as required. In fact a certain sequence of such inputs will yield a certain sequence of equations that can be solved easily using an algebra package like MAPLE. Our sequences have the following form:

INPUT	OUTPUT
$\underset{\sim}{k}_1 = v(2)$	$an(2) = \dfrac{-a}{(5a+6b+1)}\, an(1)$
$\underset{\sim}{k}_2 = v(3)$	$an(3) = \dfrac{-1}{(3a+5b+1)}\, \{b\ an(1) + 3a\ an(2)\}$
$\underset{\sim}{k}_3 = v(4)$	$an(4) = \dfrac{-1}{(3a+4b+1)}\, \{2(a+b)an(2) + a\ an(3)\}$
$\underset{\sim}{k}_4 = v(5)$	$an(5) = \dfrac{-1}{(5a+6b+2)}\, \{(a+2b)an(2) + 2a\ an(3) + 4(a+b)\ an(4)\}$
$\underset{\sim}{k}_5 = v(7)$	$an(7) = \dfrac{-1}{(5a+6b+2)}\, \{a\ an(1) + 2a\ an(2) + 6b\ an(3) + 4a\ an(4)\}$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
$\underset{\sim}{k}_{36} = v(37)$	$an(37) = \dfrac{-1}{(6a+8b+6)}\, \{a\ an(4) + \ 2a\ an(6) + \ldots\ldots + a\ an(36)\}$

We note that each input $v(i_0)$ produces an equation whose left hand side is $an(i_0)$. Each equation in the output is a linear equation in the $an(i)$ $(1 \leq i \leq 37)$ and the $an(i)$ that appear on the right hand side occur as left hand sides of equations that appear earlier in the output sequence. In other words, the system of equations is *triangular* in shape. Since this paper was first written we have found that this triangularity extends to all root systems. In fact, if the set of inputs has the following form:

$$\{\,\gamma : \gamma \prec \gamma_0\} \cap L \cap \overline{C},$$

then the corresponding system of equations is triangular with respect to any order that preserves the root order. Here $\prec$ is the usual root order, $\gamma_0 \in L \cap \overline{C}$, L is the root lattice and C is the fundamental chamber.

Below we give our complete sequence of inputs:

$$\underset{\sim}{k}_1 = v(2), \quad \underset{\sim}{k}_2 = v(3), \quad \underset{\sim}{k}_3 = v(4), \quad \underset{\sim}{k}_4 = v(5), \quad \underset{\sim}{k}_5 = v(7),$$

$$\underset{\sim}{k}_6 = v(8), \quad \underset{\sim}{k}_7 = v(6), \quad \underset{\sim}{k}_8 = v(11), \quad \underset{\sim}{k}_9 = v(14), \quad \underset{\sim}{k}_{10} = v(9),$$

$$\underset{\sim}{k}_{11} = v(10), \quad \underset{\sim}{k}_{12} = v(12), \quad \underset{\sim}{k}_{13} = v(13), \quad \underset{\sim}{k}_{14} = v(15), \quad \underset{\sim}{k}_{15} = v(16),$$

$$\underset{\sim}{k}_{16} = v(17), \quad \underset{\sim}{k}_{17} = v(18), \quad \underset{\sim}{k}_{18} = v(19), \quad \underset{\sim}{k}_{19} = v(20), \quad \underset{\sim}{k}_{20} = v(21),$$

$$\underset{\sim}{k}_{21} = v(22), \quad \underset{\sim}{k}_{22} = v(23), \quad \underset{\sim}{k}_{23} = v(24), \quad \underset{\sim}{k}_{24} = v(25), \quad \underset{\sim}{k}_{25} = v(29),$$

$$\underset{\sim}{k}_{26} = v(26), \quad \underset{\sim}{k}_{27} = v(27), \quad \underset{\sim}{k}_{28} = v(28), \quad \underset{\sim}{k}_{29} = v(30), \quad \underset{\sim}{k}_{30} = v(31),$$

$$\underset{\sim}{k}_{31} = v(32), \quad \underset{\sim}{k}_{32} = v(33), \quad \underset{\sim}{k}_{33} = v(34), \quad \underset{\sim}{k}_{34} = v(35), \quad \underset{\sim}{k}_{35} = v(36),$$

$$\underset{\sim}{k}_{36} = v(37).$$

We can now complete the proof. From (3.4) and (3.5) we have

$$f'(a+1,b) = 192\ an(1) - 768\ an(2) + \ldots\ldots + 192\ an(37) \tag{7.1}$$

(a complete list of the coefficients is given in Appendix B)

$$= 18432\ \frac{(3a+2)(3a+1)(2a+1)(6a+6b+5)(4a+4b+3)}{(3a+4b+3)(3a+5b+3)(5a+6b+5)(2a+3b+1)(3a+5b+2)}$$

$$\cdot \frac{(2a+6b+1)(4a+2b+3)(4a+4b+1)(6a+6b+1)(2a+4b+1)}{(5a+6b+3)(5a+6b+4)(3a+4b+2)(5a+6b+1)(5a+6b+2)}$$

$$\cdot \frac{(4a+2b+1)(2a+2b+1)^2\ an(1)}{(3a+4b+1)(3a+5b+1)(2a+3b+2)} \quad \text{(via MAPLE)}$$

$$= \frac{f(a+1,b)}{f(a,b)}\ f'(a,b),$$

which is (3.2) as required.

8. Other results.

In this section we give other results that have to do with adding roots to the F_4 case of the Macdonald-Morris root system conjecture. Recently we [5] have found a new proof of the G_2 case of the Macdonald-Morris root system conjecture that is solely in terms of integrals. Our proof was motivated by some conjectures of Askey [2], that have to do with adding roots to the G_2 case, and is analogous to Aomoto's [1] proof of Selberg's integral. We have been able to extend our integral-type proof, mentioned above, to the F_4 case. This proof involves converting equations involving integrals into equations analogous to (4.3) given in STEP 3 and then the proof is completed by proceeding as in STEPS 4 and 5. The proof given in this paper is more straightforward and direct.

We consider sets of the form $S = T \cup -T$ where T is a subset of the short roots of F_4 (i.e: D_4). We call two such subsets S_1 and S_2 *equivalent* if there is a $w \in W(F_4)$ such that $S_2 = w(S_1)$. This defines an equivalence relation on such sets. By utilizing a FORTRAN program we have found all the equivalence classes:

$\frac{\lvert S\rvert}{2}$	# of equivalence classes	representative of each equivalence class
1	1	$\pm\{\beta_1\}$
2	2	$\pm\{\beta_1,\beta_2\},\pm\{\beta_1,\beta_4\}$
3	4	$\pm\{\beta_1,\beta_2,\beta_3\},\pm\{\beta_1,\beta_2,\beta_4\}$
		$\pm\{\beta_1,\beta_4,\beta_5\},\pm\{\beta_2,\beta_4,\beta_5\}$
4	6	$\pm\{\beta_1,\beta_2,\beta_3,\beta_4\},\pm\{\beta_1,\beta_2,\beta_4,\beta_5\}$
		$\pm\{\beta_2,\beta_3,\beta_4,\beta_5\},\pm\{\beta_1,\beta_2,\beta_5,\beta_6\}$
		$\pm\{\beta_1,\beta_3,\beta_5,\beta_6\},\pm\{\beta_4,\beta_8,\beta_9,\beta_{10}\}$
5	7	$\pm\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\},\pm\{\beta_1,\beta_2,\beta_3,\beta_5,\beta_6\}$
		$\pm\{\beta_1,\beta_2,\beta_4,\beta_5,\beta_6\},\pm\{\beta_1,\beta_3,\beta_4,\beta_5,\beta_6\}$
		$\pm\{\beta_3,\beta_4,\beta_5,\beta_6,\beta_8\},\pm\{\beta_1,\beta_2,\beta_3,\beta_7,\beta_8,\}$
		$\pm\{\beta_1,\beta_4,\beta_8,\beta_9,\beta_{10}\}$
6	9	$\pm\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\},\pm\{\beta_1,\beta_2,\beta_3,\beta_5,\beta_6,\beta_7\}$
		$\pm\{\beta_1,\beta_2,\beta_4,\beta_5,\beta_6,\beta_8\},\pm\{\beta_1,\beta_3,\beta_4,\beta_5,\beta_6,\beta_8\}$
		$\pm\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_7,\beta_8\},\pm\{\beta_2,\beta_3,\beta_4,\beta_5,\beta_7,\beta_8\}$
		$\pm\{\beta_2,\beta_3,\beta_6,\beta_7,\beta_8,\beta_9\},\pm\{\beta_1,\beta_2,\beta_4,\beta_8,\beta_9,\beta_{10}\}$
		$\pm\{\beta_1,\beta_4,\beta_5,\beta_8,\beta_9,\beta_{10}\}$

Here the β_k are given in Appendix A. The results for $6 < \frac{|S|}{2} < 12$ follow easily from the results in the table by taking complements. We denote

$$[x^\alpha] = (1 - x^\alpha)(1 - x^{-\alpha}). \tag{8.1}$$

We have calculated

$$C.T. \prod_{\alpha \in T} [x^\alpha]\ F(\underset{\sim}{x}; a, b) \tag{8.2}$$

for all possible subsets T of the short roots of F_4 such that $T \cap -T = \phi$. By Lemma (2.21) it is enough to consider only those T where $S = T \cup -T$ is a representative of an equivalence class. The same FORTRAN program, that calculated $f'(a+1,b)$ in terms of the $an(i)$ (see (3.4), (3.5)), was used to calculate (8.2) in terms of the $an(i)$ and hence as a product of a rational function of a and b, and $f(a,b)$. To our surprise many of these rational functions factored completely into linear functions. In fact, for each $1 \le k \le 11$ there exists at least one T with $|T| = k$ such that the expression (8.2) factors completely into linear factors. Most of these linear factors seem to *glue* onto $f(a,b)$ to become factorials. Askey [2] observed a similar phenomenon in G_2. We do not see how to generalize these results to all root systems. At the very least our results seem to indicate that an easier non-computer proof of F_4 may be possible. The results that only involve linear factors are given below.

For $1 \le i_j \le 12$ we define

$$[i_1, i_2, \ldots, i_k] = \prod_{j=1}^{k} [\beta_{i_j}]. \tag{8.3}$$

(8.4)
$$C.T.[1]F = 2\frac{(6a+6b+1)}{(5a+6b+1)} f(a,b),$$

(8.5)
$$C.T.[1,2]F = 4\frac{(4a+4b+1)(6a+6b+1)}{(3a+5b+1)(5a+6b+1)} f(a,b),$$

(8.6)
$$C.T.[1,4]F = 2\frac{(4a+4b+1)(6a+6b+1)(5a+10b+2)}{(3a+4b+1)(3a+5b+1)(5a+6b+1)} f(a,b),$$

(8.7)
$$C.T.[1,2,3]F = 8\frac{(4a+2b+1)(4a+4b+1)(6a+6b+1)}{(3a+4b+1)(3a+5b+1)(5a+6b+1)} f(a,b),$$

(8.8)
$$C.T.[1,4,5]F = 12\frac{(4a+4b+1)(4a+2b+1)(2a+6b+1)(6a+6b+1)}{(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)} f(a,b),$$

(8.9)
$$C.T.[1,2,4,5]F$$
$$= 8\frac{(2a+6b+1)(4a+2b+1)(4a+4b+1)(7a+8b+3)}{(2a+3b+1)(5a+6b+2)(3a+4b+1)(3a+5b+1)} \cdot \frac{(6a+6b+1)}{(5a+6b+1)} f(a,b),$$

(8.10)
$$C.T.[4,8,9,10]F$$
$$= 32\frac{(3a+1)(6a+6b+1)(4a+4b+1)(4a+2b+1)}{(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)} f(a,b),$$

(8.11)

$$C.T.[1,4,8,9,10]F = 48\frac{(3a+1)(2a+2b+1)(4a+2b+1)(6a+6b+1)}{(5a+6b+3)(2a+3b+1)(5a+6b+2)(3a+4b+1)} \cdot \frac{(4a+4b+1)(7a+12b+4)}{(3a+5b+1)(5a+6b+1)}f(a,b),$$

(8.12)

$$C.T.[1,2,4,5,6,8]F = 288\frac{(3a+1)(6a+6b+1)(2a+6b+1)(2a+2b+1)}{(5a+6b+3)(3a+5b+2)(2a+3b+1)(5a+6b+2)} \cdot \frac{(4a+4b+1)(2a+4b+1)(4a+2b+1)}{(3a+4b+1)(3a+5b+1)(5a+6b+1)}f(a,b),$$

(8.13)

$$C.T.[1,2,7,9,10,11,12]F = 384\frac{(3a+1)(2a+2b+1)(2a+4b+1)(2a+6b+1)(5a+5b+3)}{(3a+4b+2)(5a+6b+3)(3a+5b+2)(2a+3b+1)(5a+6b+2)} \cdot \frac{(4a+4b+1)(6a+6b+1)(4a+2b+1)}{(3a+4b+1)(3a+5b+1)(5a+6b+1)}f(a,b),$$

(8.14)

$$C.T.[1,2,3,5,6,7,11,12]F = 1152\frac{(3a+2)(3a+1)(4a+4b+3)(2a+2b+1)(4a+6b+3)}{(5a+6b+4)(3a+4b+2)(5a+6b+3)(3a+5b+2)(2a+3b+1)} \cdot \frac{(4a+4b+1)(4a+2b+1)(6a+6b+1)(4a+2b+3)}{(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)}f(a,b),$$

(8.15)

$$C.T.[1,6,7,8,9,10,11,12]F = 384\frac{(3a+1)(4a+2b+1)(2a+4b+1)(4a+4b+3)(2a+2b+1)}{(5a+6b+4)(3a+4b+2)(5a+6b+3)(3a+5b+2)(2a+3b+1)} \cdot \frac{(6a+6b+1)(2a+6b+1)(4a+4b+1)(13a+10b+7)}{(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)}f(a,b),$$

(8.16)

$$C.T.[2,4,7,8,9,10,11,12]F = 768\frac{(3a+1)(6a+6b+1)(2a+6b+1)(4a+2b+1)(4a+4b+3)}{(5a+6b+4)(3a+4b+2)(5a+6b+3)(3a+5b+2)(2a+3b+1)} \cdot \frac{(4a+4b+1)(2a+4b+1)(2a+2b+1)(7a+4b+4)}{(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)}f(a,b),$$

(8.17)

$$C.T.[1,3,6,7,8,9,10,11,12]F = 2304\frac{(3a+1)(2a+4b+1)(6a+6b+5)(4a+4b+3)(2a+6b+1)}{(5a+6b+4)(3a+5b+3)(3a+4b+2)(5a+6b+3)(3a+5b+2)} \cdot \frac{(4a+4b+1)(6a+6b+1)(4a+2b+1)(2a+2b+1)^2}{(2a+3b+1)(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)} \cdot f(a,b),$$

(8.18)

$$C.T.[3,5,6,7,8,9,10,11,12]F = 768\frac{(3a+1)(4a+2b+1)(2a+6b+1)(6a+6b+1)(6a+6b+5)}{(5a+6b+4)(3a+5b+3)(3a+4b+2)(5a+6b+3)(3a+5b+2)} \cdot \frac{(4a+4b+1)(4a+4b+3)(2a+4b+1)(2a+2b+1)(7a+4b+4)}{(2a+3b+1)(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)} \cdot f(a,b),$$

(8.19)

$$C.T.[4,5,6,7,8,9,10,11,12]F = 2304\frac{(3a+2)(3a+1)(6a+5+6b)(4a+4b+3)(6a+6b+1)}{(5a+6b+5)(5a+6b+4)(3a+4b+2)(5a+6b+3)(3a+5b+2)} \cdot \frac{(4a+4b+1)(2a+6b+1)(2a+2b+1)(4a+2b+3)(4a+2b+1)}{(2a+3b+1)(5a+6b+2)(3a+4b+1)(3a+5b+1)(5a+6b+1)} \cdot f(a,b),$$

(8.20)

$$C.T.[2,3,5,6,7,8,9,10,11,12]F = 2304\frac{(3a+1)(2a+4b+1)(6a+6b+5)(4a+2b+1)(2a+6b+1)}{(2a+3b+2)(5a+6b+4)(3a+5b+3)(3a+4b+2)(5a+6b+3)} \cdot \frac{(6a+6b+1)(4a+4b+1)(4a+2b+3)(4a+4b+3)(2a+2b+1)^2}{(3a+5b+2)(2a+3b+1)(5a+6b+2)(3a+4b+1)(3a+5b+1)} \cdot \frac{f(a,b)}{(5a+6b+1)},$$

(8.21)

$$C.T.[3,4,5,6,7,8,9,10,11,12]F = 4608\frac{(3a+2)(3a+1)(6a+6b+1)(4a+4b+1)(2a+6b+1)}{(5a+6b+5)(5a+6b+4)(3a+5b+3)(3a+4b+2)(5a+6b+3)} \cdot \frac{(4a+2b+1)(2a+4b+1)(6a+6b+5)(4a+2b+3)(2a+2b+1)}{(3a+5b+2)(2a+3b+1)(5a+6b+2)(3a+4b+1)(3a+5b+1)} \cdot \frac{(4a+4b+3)}{(5a+6b+1)}f(a,b),$$

(8.22)

$$
\begin{aligned}
&C.T.[1,2,3,4,5,6,7,8,9,10,11]F\\
&\quad= 3072\frac{(3a+2)(3a+1)(6a+6b+5)(2a+4b+1)(6a+6b+1)}{(3a+5b+1)(5a+6b+1)(3a+4b+2)(5a+6b+3)(3a+5b+2)}\\
&\quad\cdot\frac{(4a+4b+1)(4a+2b+1)(2a+6b+1)(4a+4b+3)(4a+2b+3)}{(2a+3b+1)(5a+6b+2)(3a+4b+1)(5a+6b+5)(5a+6b+4)}\\
&\quad\cdot\frac{(2a+2b+1)^2(7a+9b+7)}{(3a+5b+3)(2a+3b+2)(3a+4b+3)}f(a,b).
\end{aligned}
$$

It is a little unsettling that not all of the above results can be written as factorials. Since this paper was first written we have found nicer results. In fact, if we restrict to subsets of positive roots there is a chain of subsets in which each corresponding constant term formula can be written as a product of factorials. This chain seems to be related to the root order but we have been unable to generalize to other root systems.

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Note added in proof. Robert Gustafson (*A generalization of Selberg's beta integral*, preprint) has proved q-Macdonald-Morris for the affine root systems of types $S(C_\ell)$ and $S(C_\ell)^\vee$.

APPENDIX A

We write the roots of $\Phi^{(2)}(F_4)$ as $\mathbb{Z}$-linear combinations of α_i $(1 \leq i \leq 4)$ defined in (2.6).

$\beta_1 = \alpha_1 = e_1 - e_2,$ $\beta_5 = \alpha_1 + \alpha_4 = e_1 - e_3,$ $\beta_8 = \alpha_1 + \alpha_2 + \alpha_4 = e_1 + e_4,$

$\beta_2 = \alpha_2 = e_3 + e_4,$ $\beta_6 = \alpha_2 + \alpha_4 = e_2 + e_4,$ $\beta_9 = \alpha_1 + \alpha_3 + \alpha_4 = e_1 - e_4,$

$\beta_3 = \alpha_3 = e_3 - e_4,$ $\beta_7 = \alpha_3 + \alpha_4 = e_2 - e_4,$ $\beta_{10} = \alpha_2 + \alpha_3 + \alpha_4 = e_2 + e_3,$

$\beta_4 = \alpha_4 = e_2 - e_3,$ $\beta_{11} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = e_1 + e_3,$ $\beta_{12} = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 = e_1 + e_2,$

$\alpha_1 + \alpha_2 = e_1 - e_2 + e_3 + e_4,$ $\alpha_1 - \alpha_2 = e_1 - e_2 - e_3 - e_4,$

$\alpha_1 + \alpha_3 = e_1 - e_2 + e_3 - e_4,$ $\alpha_1 - \alpha_3 = e_1 - e_2 - e_3 + e_4,$

$\alpha_2 + \alpha_3 = 2e_3,$ $\alpha_2 - \alpha_3 = 2e_4,$

$\alpha_1 + \alpha_2 + 2\alpha_4 = e_1 + e_2 - e_3 + e_4,$ $2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 = 2e_1,$

$\alpha_1 + \alpha_3 + 2\alpha_4 = e_1 + e_2 - e_3 - e_4,$ $\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = e_1 + e_2 + e_3 + e_4,$

$\alpha_2 + \alpha_3 + 2\alpha_4 = 2e_2,$ $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = e_1 + e_2 + e_3 - e_4.$

APPENDIX B

The vectors $v(i)$ $(1 \leq i \leq 37)$ that appear in (3.5) are listed below.

$v(1) = (0,0,0,0)$ $v(14) = (2,2,2,2)$ $v(27) = (5,3,3,1)$

$v(2) = (1,1,0,0)$ $v(15) = (4,2,2,0)$ $v(28) = (5,4,1,0)$

$v(3) = (1,1,1,1)$ $v(16) = (4,4,0,0)$ $v(29) = (4,4,2,2)$

$v(4) = (2,1,1,0)$ $v(17) = (3,3,2,2)$ $v(30) = (5,4,2,1)$

$v(5) = (2,2,1,1)$ $v(18) = (4,2,2,2)$ $v(31) = (6,2,2,2)$

$v(6) = (3,2,1,0)$ $v(19) = (4,3,2,1)$ $v(32) = (6,3,2,1)$

$v(7) = (2,2,0,0)$ $v(20) = (4,4,1,1)$ $v(33) = (6,3,3,0)$

$v(8) = (3,1,1,1)$ $v(21) = (5,2,2,1)$ $v(34) = (5,5,0,0)$

$v(9) = (3,2,2,1)$ $v(22) = (5,3,1,1)$ $v(35) = (5,5,1,1)$

$v(10) = (3,3,1,1)$ $v(23) = (5,3,2,0)$ $v(36) = (6,4,1,1)$

$v(11) = (3,3,0,0)$ $v(24) = (3,3,3,3)$ $v(37) = (6,4,2,0)$

$v(12) = (4,2,1,1)$ $v(25) = (4,3,3,2)$

$v(13) = (4,3,1,0)$ $v(26) = (5,3,2,2)$

The complete version of (3.5) is given below.

$$\sum_{\alpha\in S} a'_\alpha x^\alpha = 192x^{v(1)} - 768x^{v(2)} + 576x^{v(3)} + 960\ x^{v(4)} - 1152\ x^{v(5)}$$

$$+ 2688\ x^{v(6)} - 576\ x^{v(7)} - 1152\ x^{v(8)} - 576\ x^{v(10)} - 576\ x^{v(11)}$$

$$- 1152\ x^{v(12)} + 1152\ x^{v(13)} + 576\ x^{v(14)} - 192\ x^{v(15)} - 192\ x^{v(16)}$$

$$- 1152\ x^{v(17)} + 2304\ x^{v(18)} - 1152\ x^{v(21)} - 1152\ x^{v(22)} + 1152\ x^{v(23)}$$

$$+ 576\ x^{v(24)} - 1728\ x^{v(25)} + 2304\ x^{v(26)} - 1152\ x^{v(27)} + 384\ x^{v(28)}$$

$$+ 576\ x^{v(29)} - 1152\ x^{v(30)} - 576\ x^{v(31)} + 1152\ x^{v(32)} - 192\ x^{v(33)}$$

$$- 192\ x^{v(34)} + 576\ x^{v(35)} - 576\ x^{v(36)} + 192\ x^{v(37)}.$$

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