RELATIONS BETWEEN THE RANKS AND CRANKS OF PARTITIONS

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Dedicated to the memory of Robert A. Rankin

Abstract. New identities and congruences involving the ranks and cranks of partitions are proved. The proof depends on a new partial differential equation connecting their generating functions.

1. Introduction

Dyson [D1], [D3, p.52] defined the rank of a partition as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m, then

$$\sum_m N(m,n) = p(n), \tag{1.1}$$

the number of partitions of n; and

$$N(m,n) = N(-m,n), \tag{1.2}$$

using the classical conjugacy of partitions.

Andrews and Garvan [A-G] defined the crank of a partition. It is the largest part if the partition contains no ones, and is otherwise the number of parts larger than the number of ones minus the number of ones. Let $M(m,n)$ denote the number of partitions of n with crank m, then

$$\sum_m M(m,n) = p(n), \tag{1.3}$$

the number of partitions of n; and

$$M(m,n) = M(-m,n), \tag{1.4}$$

for $n > 1$. A direct combinatorial proof of (1.4) was found recently by Berkovich and Garvan [B-G].

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We now state the generating function $R(z,q)$ for the rank. We have

$$R(z,q) = \sum_{n\ge 0}\sum_{m} N(m,n) z^m q^n \tag{1.5}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}, \tag{1.6}$$

and

$$\sum_{n\ge 0} N(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n}{2}(3n-1)+|m|n}(1-q^n). \tag{1.7}$$

Here we are using the notation

$$\begin{aligned}(a)_n &= (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),\\ (q)_\infty &= \lim_{n\to\infty}(q)_n,\end{aligned} \tag{1.8}$$

where $|q|<1$.

Below we state the generating function $C(z,q)$ for the crank. If we amend the definition of $M(m,n)$ for $n=1$, then the generating function can be given as an infinite product. Accordingly, throughout this paper we assume

$$M(0,1) = -1,\ M(-1,1) = M(1,1) = 1, \text{and } M(m,1) = 0 \text{ otherwise}. \tag{1.9}$$

Then we have

$$C(z,q) = \sum_{n\ge 0}\sum_{m} M(m,n) z^m q^n \tag{1.10}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}, \tag{1.11}$$

and

$$\sum_{n\ge 0} M(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n}{2}(n-1)+|m|n}(1-q^n). \tag{1.12}$$

Equation (1.11) follows from [A-G, eq. (1.11)] and [A-G, Thm 1]. Equation (1.12) then follows from [G1, eq. (7.20)].

The main result of this paper is a fundamental partial differential equation (PDE) connecting the rank and crank generating functions. See Theorem 1.1 below. Let

$$R^*(z,q) := \frac{R(z,q)}{(1-z)}, \tag{1.13}$$

$$C^*(z,q) := \frac{C(z,q)}{(1-z)}. \tag{1.14}$$

Define the differential operators

$$\delta_z = z\frac{\partial}{\partial z}, \qquad \delta_q = q\frac{\partial}{\partial q}. \tag{1.15}$$

Theorem 1.1. *We have*

$$z(q)_\infty^2\,[C^*(z,q)]^3 = \left(3\delta_q + \frac{1}{2}\delta_z + \frac{1}{2}\delta_z^2\right) R^*(z,q), \tag{1.16}$$

and

$$z(q)_\infty^2\,[C(z,q)]^3 = \left(3(1-z)^2\delta_q + \frac{1}{2}(1-z)^2\delta_z^2 - \frac{1}{2}(z^2-1)\delta_z + z\right) R(z,q). \tag{1.17}$$

We prove equation (1.16) in Section 2. The Rank-Crank PDE (1.17) follows easily from (1.16) by using (1.13) and (1.14).

Let $N(k,t,n)$ denote the number of partitions of n with rank congruent to k modulo t. Then for $t=5$ or $t=7$

$$N(k,t,n) = \frac{1}{t}\,p(n), \qquad 0 \le k \le t-1; \tag{1.18}$$

for all n satisfying $24n \equiv 1 \pmod{t}$. These combinatorial results immediately imply Ramanujan's partition congruences

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.19}$$

$$p(7n+5) \equiv 0 \pmod{7}. \tag{1.20}$$

Let $M(k,t,n)$ denote the number of partitions of n with crank congruent to k modulo t. Then for $t=5$, $t=7$, or $t=11$

$$M(k,t,n) = \frac{1}{t}\,p(n), \qquad 0 \le k \le t-1; \tag{1.21}$$

for all n satisfying $24n \equiv 1 \pmod{t}$. These combinatorial results again imply Ramanujan's partition congruences mod 5 and mod 7, and in addition his congruence

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.22}$$

There are many more rank identities. For example,

$$N(1,5,5n+1) = N(2,5,5n+1), \tag{1.23}$$

and others for the moduli 5, 7, 8, 9, and 12. The results for 5 and 7 were all found by Dyson [D1], [D3, p.53] and proved by Atkin and Swinnerton-Dyer [A-SD]. The results for 8, 9, and 12, were found by Lewis [L1] and subsequently proved by Lewis and Santa-Gadea in a series of papers [SG1], [SG2], and [L-SG].

There are similar identities for the crank. For example,

$$M(0,8,2n+1) + M(1,8,2n+1) = M(3,8,2n+1) + M(4,8,2n+1), \tag{1.24}$$

and others for the moduli 5, 7, 8, 9, 10, and 11. These were proved in [G1], [G2], and [G3].

There are identities between the rank and the crank. For example,

$$M(4,9,3n) = N(4,9,3n), \tag{1.25}$$

and others for the moduli 5, 7, 8, and 9. These results were proved in [G2], [G3], [L1], [L3], [L-SG1], and [SG1].

In this paper we consider linear relations modulo a prime p. There are congruences for the rank to the moduli 11 and 13. For example,

(1.26)
$$2N(2,11,11n)+N(3,11,11n)+7N(4,11,11n)+N(5,11,11n)\equiv 0 \pmod{11}.$$

Results of this type are due to Atkin and Hussain [A-H] and O'Brien [OB].

It is a surprising fact that there is one analogous relation for the crank for *every* prime p. This was a mystery to us until we realized that these crank congruences follow from the identity

$$\sum_{k=1}^{n} k^2\, M(k,n) = n\, p(n), \tag{1.27}$$

due to Dyson [D2], who gave a combinatorial proof.

There is an extra linear congruence for the crank modulo p for $p = 41$, 53, 83, and 120667369.

For each prime $p > 13$ there are seven congruences involving both the rank and the crank modulo p. For example,

(1.28)
$$\begin{aligned}
&6N(0,29,29n+23)+17N(1,29,29n+23)+24N(2,29,29n+23)\\
&+18N(3,29,29n+23)+17N(4,29,29n+23)+14N(5,29,29n+23)\\
&+22N(6,29,29n+23)+24N(7,29,29n+23)+2N(9,29,29n+23)\\
&+15N(10,29,29n+23)+19N(11,29,29n+23)+18N(12,29,29n+23)\\
&+20N(13,29,29n+23)+16N(14,29,29n+23)\\
&\equiv 11M(0,29,29n+23)+17M(1,29,29n+23)+28M(2,29,29n+23)\\
&\quad+26M(4,29,29n+23)+6M(5,29,29n+23)+28M(8,29,29n+23)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\pmod{29}.
\end{aligned}$$

These congruences come from certain exact relations between the rank and the crank.

For even $j \geq 2$, we define

$$N_j(n) = \sum_k k^j\, N(k,n), \tag{1.29}$$

$$M_j(n) = \sum_k k^j\, M(k,n). \tag{1.30}$$

The following is the simplest exact relation

$$N_4(n) = -(2n+\tfrac{2}{3})\, M_2(n) + \tfrac{8}{3}M_4(n) + (1-12n)\, N_2(n). \tag{1.31}$$

In fact, there are polynomials $P_k(n)$ of degree $k-1$ and $Q_{k,j}(n)$ of degree $k-j$ (for $1 \leq j \leq k$) such that

$$N_{2k}(n) = P_k(n)\, N_2(n) + \sum_{j=1}^{k} Q_{k,j}(n)\, M_{2j}(n), \tag{1.32}$$

for $k = 2$, 3, 4, and 5. For $k = 6$ there is no such relation. For $k = 7$ there is a similar relation but with an extra term $N_{12}(n)$. See Theorem 5.1 below. The proof of these exact relations depends on the Rank-Crank PDE (1.17).

We call the functions N_j and M_j (defined in (1.29), (1.30)) rank and crank moments, respectively. We define the following generating functions

$$R_j = R_j(q) = \sum_{n \geq 1} N_j(n) q^n, \tag{1.33}$$

$$C_j = C_j(q) = \sum_{n \geq 1} M_j(n) q^n, \tag{1.34}$$

for even j. We find that

$$\delta_z^j R(z,q)\big|_{z=1} = \begin{cases} R_j, & j \text{ even}, \\ 0, & j \text{ odd}, \end{cases} \tag{1.35}$$

using (1.2) and (1.5). Similarly we find that

$$\delta_z^j C(z,q)\big|_{z=1} = \begin{cases} C_j, & j \text{ even}, \\ 0, & j \text{ odd}. \end{cases} \tag{1.36}$$

In Section 2 we show how the Rank-Crank PDE (1.17) follows from a certain elliptic-function identity (2.3). In Section 3 we prove some results for Eisenstein series, modular forms and quasi-modular forms. In Section 4 we show how the crank moment functions can be written in terms of Eisenstein series, and we derive some results for the derivatives of crank moment functions. In Section 5 we show how the Rank-Crank PDE and certain results for the derivatives of Eisenstein series lead to exact relations between rank and crank moments. As a bonus we show that the 23rd power of the Dedekind eta-function can be written in terms of rank and crank moments. In Section 6 we consider congruence relations between rank and crank moments.

2. Proof of the rank-crank PDE

The rank-crank PDE follows easily from an identity in [A-SD]. Define

$$J(z,q) := \prod_{n=1}^{\infty}(1-z^{-1}q^n)(1-zq^{n-1}), \tag{2.1}$$

and

$$S(z,\zeta,q) := \sum_{n=-\infty}^{\infty}\frac{(-1)^n\zeta^n q^{3n(n+1)/2}}{1-zq^n}. \tag{2.2}$$

Then

$$\zeta^3 S(z\zeta,\zeta^3,q) + S(z\zeta^{-1},\zeta^{-3},q) - \zeta\frac{J(\zeta^2,q)}{J(\zeta,q)}S(z,1,q) = \frac{J(\zeta,q)J(\zeta^2,q)(q)_\infty^2}{J(\zeta z,q)J(z,q)J(z\zeta^{-1},q)}. \tag{2.3}$$

This identity is equation (5.1) in [A-SD, p.94] and was one of the key identities required to prove Dyson's results for the rank modulo 5 and 7. We let $g(\zeta)$ denote either side of (2.3) as a function of ζ. By considering the right side of (2.3) we see that $g(\zeta)$ has a double zero at $\zeta = 1$ and that

$$g''(1) = 4(q)_\infty^3\left[C^*(z,q)\right]^3, \tag{2.4}$$

where $C^*(z,q)$ is defined in (1.14). We let $h(\zeta)$ be the sum of the first two terms on the left side of (2.3); i.e.,

$$h(\zeta) = \zeta^3 S(z\zeta,\zeta^3,q) + S(z\zeta^{-1},\zeta^{-3},q). \tag{2.5}$$

We find that

$$\begin{aligned} h''(1) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{3n(n+1)/2}\left(\frac{6(3n^2+3n+1)}{1-zq^n} + \frac{4z(3n+2)q^n}{(1-zq^n)^2} + \frac{4z^2q^{2n}}{(1-zq^n)^3}\right) \\ &= (6+12\delta_q+6\delta_z+2\delta_z^2)S(z,1,q). \end{aligned} \tag{2.6}$$

We let $j(\zeta)$ be the third term on the right side of (2.3)

$$j(\zeta) = -\zeta\frac{J(\zeta^2,q)}{J(\zeta,q)}S(z,1,q). \tag{2.7}$$

We find that

$$j''(1) = -2\left(1 - 6\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}\right)S(z,1,q) = -2(1-6\Phi_1(q))S(z,1,q), \tag{2.8}$$

where

$$\Phi_1(q) = \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \tag{2.9}$$

The functions Φ_j are defined below in (3.1). We define

$$P(q)=\sum_{n=0}^{\infty}p(n)q^n=\frac{1}{(q)_\infty}. \tag{2.10}$$

By differentiating logarithmically with respect to q we obtain the well-known identity

$$\delta_q P(q)=\Phi_1(q)\,P(q), \tag{2.11}$$

or

$$\delta_q(q)_\infty=-\Phi_1(q)(q)_\infty. \tag{2.12}$$

The following identity is equation (7.10) in [G1]:

$$zS(z,1,q)=(q)_\infty(-1+R^*(z,q)). \tag{2.13}$$

We apply δ_q to both sides of (2.13) and use (2.12) to find that

$$z\delta_q S(z,1,q)=(q)_\infty\delta_q R^*(z,q)-z\Phi_1(q)S(z,1,q). \tag{2.14}$$

Similarly we find that

$$z\delta_z S(z,1,q)=(q)_\infty\delta_z R^*(z,q)-zS(z,1,q), \tag{2.15}$$

$$z\delta_z^2 S(z,1,q)=(q)_\infty\left(\delta_z^2-2\delta_z\right)R^*(z,q)+zS(z,1,q). \tag{2.16}$$

Now

$$g''(1)=h''(1)+j''(1). \tag{2.17}$$

Using (2.4), (2.6), (2.8), and (2.13)–(2.16) this equation becomes

$$z(q)_\infty^2\left[C^*(z,q)\right]^3=\left(3\delta_q+\frac{1}{2}\delta_z+\frac{1}{2}\delta_z^2\right)R^*(z,q). \tag{2.18}$$

From (1.13) and (1.14) we find that

$$\delta_z R^*(z,q)=\frac{\delta_z R(z,q)+zR^*(z,q)}{1-z}, \tag{2.19}$$

$$\delta_z^2 R^*(z,q)=\frac{\delta_z^2 R(z,q)+2z\delta_z R^*(z,q)+zR^*(z,q)}{1-z}, \tag{2.20}$$

$$\delta_q R^*(z,q)=\frac{\delta_q R(z,q)}{1-z}. \tag{2.21}$$

Using (2.19)–(2.21) we can write (2.18) in terms of $C(z,q)$ and $R(z,q)$:

$$z(q)_\infty^2\left[C(z,q)\right]^3=\left(3(1-z)^2\delta_q+\frac{1}{2}(1-z)^2\delta_z^2-\frac{1}{2}(z^2-1)\delta_z+z\right)R(z,q), \tag{2.22}$$

which is the rank-crank PDE.

3. Eisenstein series, modular forms and derivatives

Following Ramanujan [Ram, p.163] we define

$$\Phi_j(q) = \sum_{n=1}^{\infty} \frac{n^j q^n}{1-q^n} = \sum_{m,n\geq 1} n^j q^{nm} = \sum_{n=1}^{\infty} \sigma_j(n) q^n, \tag{3.1}$$

for $j \geq 1$ odd and where $\sigma_j(n) = \sum_{d|n} d^j$. As usual we let $q = \exp(2\pi i\tau)$ where τ is in the complex upper half-plane $\mathcal{H}$ so that $|q| < 1$. For n even the Eisenstein series $E_n(\tau)$ is defined by

$$\begin{aligned} E_n(\tau) &= 1 - \frac{2n}{B_n} \sum_{k=1}^{\infty} \sigma_{n-1}(k) e^{2\pi i k\tau}, \\ &= 1 - \frac{2n}{B_n} \Phi_{n-1}(q), \end{aligned} \tag{3.2}$$

where the Bernoulli numbers B_n are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Ramanujan [R, p.140] considered in particular the Eisenstein series

$$\begin{aligned} E_2 &= 1 - 24\,\Phi_1, \\ E_4 &= 1 + 240\,\Phi_3, \\ E_6 &= 1 - 540\,\Phi_5. \end{aligned} \tag{3.3}$$

For even $n \geq 4$, E_n is a modular form of weight n for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ [Ran]. E_2 is not a modular form, but is transformed by the generators of Γ according to

$$\begin{aligned} E_2(\tau+1) &= E_2(\tau), \\ \tau^{-2} E_2(-1/\tau) &= E_2(\tau) + \frac{12}{2\pi i\tau}. \end{aligned} \tag{3.4}$$

See [K, p.113]. Ramanujan [Ram, p.165] found that

$$\begin{aligned} \delta_q(E_2) &= \frac{E_2^2 - E_4}{12}, \\ \delta_q(E_4) &= \frac{E_2 E_4 - E_6}{3}, \\ \delta_q(E_6) &= \frac{E_2 E_6 - E_4^2}{2}, \end{aligned} \tag{3.5}$$

where

$$\delta_q = q\,\frac{d}{dq}.$$

More generally, it is known that if f is a modular form of weight k then

$$12\delta_q(f) - kE_2 f$$

is a modular form of weight $(k+2)$. See [SD, p.19].

Following Serre [S, p.88], we let $\mathcal{M}_k$ denote the vector space of modular forms of weight $2k$. Then

$$\dim \mathcal{M}_k = \begin{cases} [k/6], & \text{if } k \equiv 1 \pmod{6}, \\ [k/6]+1, & \text{otherwise}, \end{cases} \tag{3.6}$$

and the set

$$\{E_4^a E_6^b \,:\, 2a+3b=k \text{ with } a \text{ and } b \text{ nonnegative integers}\} \tag{3.7}$$

forms a basis for $\mathcal{M}_k$. See Serre [S, p.88]. Thus any Eisenstein series E_{2n} (for $n \geq 2$) can be written in terms of E_4 and E_6. This can be done explicitly using well-known recurrences. See [Ap, pp.12-13] and [B, pp.331-332].

The graded algebra of modular forms is given by

$$\mathcal{M} = \sum_{k=0}^{\infty} \mathcal{M}_k = \mathbb{C}[E_4, E_6]. \tag{3.8}$$

See Serre [S,p.89]. We need to extend this algebra to include E_2. We say that f is a *quasi-modular form* if it is in the algebra generated by E_2 and $\mathcal{M}$. We extend the definition of weight by defining the weight of E_2 to be 2. Let n be a nonnegative integer. Then the space of quasi-modular forms of weight $\leq 2n$ is

$$\mathcal{E}_n = \left\{ \sum_{j=0}^{n} f_j E_2^j \,:\, f_j \in \sum_{k=0}^{n-j} \mathcal{M}_k \right\}, \tag{3.9}$$

which is clearly a vector space over $\mathbb{C}$. Below in Corollary 3.6 we give a basis for $\mathcal{E}_n$.

Quasi-modular forms were first studied systematically by Kaneko and Zagier [K-Z]. We need some independence results for modular forms and quasi-modular forms. The results for quasi-modular forms follow from [K-Z; Proposition 1(b), p.167]. We have included proofs for completeness since they are elementary and the details of the relevant proof in [K-Z] are omitted.

Lemma 3.1. *Let n be a nonnegative integer. Suppose that*

$$f_k \,:\, \mathcal{H} \longrightarrow \mathbb{C}, \ \text{where } f_k(\tau+1) = f_k(\tau), \qquad (0 \leq k \leq n) \tag{3.10}$$

for all $\tau \in \mathcal{H}$, and

$$\sum_{k=0}^{n} \tau^k f_k(\tau) = 0, \tag{3.11}$$

for all $\tau \in \mathcal{H}$. *Then*

$$f_0(\tau) = f_1(\tau) = \cdots = f_n(\tau) = 0, \tag{3.12}$$

for all $\tau \in \mathcal{H}$.

Proof. Suppose (3.10) and (3.11) hold. Let $\tau \in \mathcal{H}$ be arbitrary but fixed. Then

$$\sum_{k=0}^{n} (\tau + m)^k f_k(\tau) = 0, \tag{3.13}$$

for all integers m. Hence the polynomial

$$p(z) = \sum_{k=0}^{n} f_k(\tau) z^k$$

has infinitely many zeros. The result follows. $\square$

Corollary 3.2. *Non-zero modular forms of different weights are linearly independent over* $\mathbb{C}$.

Proof. Suppose that there are complex constants c_k such that

$$\sum_{k=0}^{n} c_k f_k(\tau) = 0, \tag{3.14}$$

for all $\tau \in \mathcal{H}$, where $f_k(\tau)$ is a modular form of weight k. We apply $\tau \to -1/\tau$ to obtain

$$\sum_{k=0}^{n} c_k \tau^k f_k(\tau) = 0,$$

for all $\tau \in \mathcal{H}$. The functions f_k satisfy (3.10) so by Lemma 3.1 we have

$$c_0 f_0(\tau) = c_1 f_1(\tau) = \cdots = c_n f_n(\tau) = 0,$$

and the result follows. $\square$

Proposition 3.3. *Let* n *be a nonnegative integer. Suppose that*

$$f(\tau) := \sum_{j=0}^{n} f_j(\tau) E_2^j(\tau) = 0, \tag{3.15}$$

for all $\tau \in \mathcal{H}$, *and* f *is a quasi-modular form of weight* $\leq 2n$, *so that each* f_j *is a sum of modular forms of weight* $\leq 2n - 2j$. *Then*

$$f_0 = f_1 = \cdots = f_n = 0.$$

Proof. Suppose (3.15) holds and suppose for $0 \leq j \leq n$

$$f_j = f_{j,0} + f_{j,4} + \cdots + f_{j,2n-2j},$$

where each $f_{j,k}$ is a modular form of weight k. Hence

$$\sum_{j=0}^{n}\left(f_{j,0}+\sum_{k=2}^{n-j} f_{j,2k}\right)E_2^j = 0. \tag{3.16}$$

Applying $\tau \to -1/\tau$ we obtain

$$\sum_{j=0}^{n}\left(f_{j,0}+\sum_{k=2}^{n-j}\tau^{2k} f_{j,2k}\right)\left(\tau^2E_2+\alpha\tau\right)^j = 0, \tag{3.17}$$

where

$$\alpha = -\frac{6i}{\pi}, \tag{3.18}$$

by (3.4). We rewrite (3.17) as

$$\sum_{j=0}^{n}\left(f_{j,0}+\sum_{k=2}^{n-j}\tau^{2k} f_{j,2k}\right)\tau^j\sum_{\ell=0}^{j}\binom{j}{\ell}\tau^\ell E_2^\ell\alpha^{j-\ell} = 0,$$

whence

(3.19)

$$f_{0,0}+\alpha f_{1,0}\tau$$
$$+\sum_{m=2}^{2n}\left(\sum_{\substack{0\le j\le n\\0\le\ell\le j\\j+\ell=m}} f_{j,0}\binom{j}{\ell}E_2^\ell\alpha^{j-\ell}+\sum_{\substack{0\le j\le n\\2\le k\le n-j\\0\le\ell\le j\\j+\ell+2k=m}}\binom{j}{\ell}\alpha^{j-\ell}f_{j,2k}E_2^\ell\right)\tau^m = 0.$$

Since the $f_{j,k}$ and E_2 satisfy (3.10), by Lemma 3.1 we have

(3.20)

$$f_{0,0} = 0,$$
$$f_{1,0} = 0,$$
$$\sum_{\substack{0\le j\le n\\0\le\ell\le j\\j+\ell=m}} f_{j,0}\binom{j}{\ell}E_2^\ell\alpha^{j-\ell}+\sum_{\substack{0\le j\le n\\2\le k\le n-j\\0\le\ell\le j\\j+\ell+2k=m}}\binom{j}{\ell}\alpha^{j-\ell}f_{j,2k}E_2^\ell = 0, \quad 2\le m\le 2n.$$

There are $2n+1$ equations in (3.20). Each $f_{j,2k}$ occurs in some equation. We prove that

$$f_{j,2k} = 0, \tag{3.21}$$

for all j, k such that $f_{j,2k}$ occurs in the m-th equation of (3.20). We proceed by induction on m. The result holds for $m = 0$, and $m = 1$. Suppose it holds for $m < M$ where M is a fixed positive integer $\le 2n$. We consider the M-th equation. It is clear that if a term $f_{j,2k}E_2^\ell$ with $\ell \ge 1$ occurs in this equation, then $f_{j,2k}$ must have appeared in a previous equation, and so is zero by the induction hypothesis. The remaining terms in the M-th equation all have $\ell = 0$, and are true modular forms, and it is easy to see that they have different weights. Hence they are all zero by Corollary 3.2, and thus the result holds for $m = M$, and so for all m with $0 \le m \le 2n$ by induction. □

Corollary 3.4. *Let n be a nonnegative integer. The set*

$$\{E_2^aE_4^bE_6^c\,:\,a+2b+3c\le n \text{ with } a,\ b,\ c \text{ nonnegative integers}\} \tag{3.22}$$

is a basis for the space of quasi-modular forms of weight $\le 2n$.

In the next section we shall see that the crank moment functions C_a can be written in terms of the Φ_j but are not modular forms. Thus we need some results for the Φ_j. Using (3.3) and (3.5) we find that

$$\begin{aligned} \delta_q(\Phi_1) &= \frac{5}{6}\,\Phi_3 + \frac{1}{6}\,\Phi_1 - 2\,{\Phi_1}^2,\\ \delta_q(\Phi_3) &= \frac{7}{10}\,\Phi_5 + \frac{1}{3}\,\Phi_3 - \frac{1}{30}\,\Phi_1 - 8\,\Phi_1\Phi_3,\\ \delta_q(\Phi_5) &= \frac{1}{2}\,\Phi_5 + \frac{1}{42}\,\Phi_1 - 12\,\Phi_1\Phi_5 + \frac{10}{21}\,\Phi_3 + \frac{400}{7}\,{\Phi_3}^2. \end{aligned} \tag{3.23}$$

We define the weight ω of a monomial $\Phi_1^a\Phi_3^b\Phi_5^c$ as

$$\omega(\Phi_1^a\Phi_3^b\Phi_5^c) = 2a+4b+6c. \tag{3.24}$$

It is clear from (3.3) that the monomial is a sum of quasi-modular forms of different weights, the form $E_2^aE_4^bE_6^c$ having the highest weight ω. From (3.2), (3.3) and the fact the set in (3.7) forms a basis for $\mathcal{M}_n$, we see that for $n>1$ there are constants $\alpha_{b,c}$ such that

$$\Phi_{2n-1} = \sum_{0<2b+3c\le n} \alpha_{b,c}\Phi_3^b\Phi_5^c. \tag{3.25}$$

In the sum $2b+3c$ is positive since $\Phi_{2n-1}(q)=0$ for $q=0$. For example,

$$\Phi_{11} = \frac{1}{13}\left(63(\Phi_3 + 240\,{\Phi_3}^2 + 19200\,{\Phi_3}^3 + 200\,{\Phi_5}^2) - 50\Phi_5\right). \tag{3.26}$$

For $n\ge 1$ we let $\mathcal{V}_n$ be the $\mathbb{C}$-vector space spanned by monomials $\Phi_1^a\Phi_3^b\Phi_5^c$ with $a+2b+3c=n$. We define

$$\mathcal{W}_n = \sum_{k=1}^{n}\mathcal{V}_n; \tag{3.27}$$

i.e., $\mathcal{W}_n$ is the $\mathbb{C}$-vector space spanned by monomials $\Phi_1^a\Phi_3^b\Phi_5^c$ with $0<a+2b+3c\le n$. Using (3.23) and induction we can show that

$$\delta_q\left(\mathcal{W}_n\right) \subset \mathcal{W}_{n+1}, \tag{3.28}$$

so that

$$\delta_q^m\left(\mathcal{W}_n\right) \subset \mathcal{W}_{n+m}, \tag{3.29}$$

for $m\ge 0$.

Theorem 3.5. *Let n be a nonnegative integer. The set*

$$\{\Phi_1^a\Phi_3^b\Phi_5^c \,:\, 0 < a + 2b + 3c \le n \text{ with } a,\ b,\ c \text{ nonnegative integers}\} \tag{3.30}$$

is a basis for $\mathcal{W}_n$.

Proof. The set in (3.30) spans $\mathcal{W}_n$ by definition. We show that the set is linearly independent, by showing a slightly larger set is linearly independent. By equation (3.3) and Corollary 3.4 the set of monomials $\Phi_1^a\Phi_3^b\Phi_5^c$, where $0 \le a + 2b + 3c \le n$, spans $\mathcal{E}_n$ and the number of such monomials is equal to $\dim \mathcal{E}_n$. Hence, these monomials form a basis for $\mathcal{E}_n$ and are linearly independent. □

Corollary 3.6. *For $n \ge 1$*

$$\dim \mathcal{V}_n = \sum_{k=0}^{n} \dim \mathcal{M}_k, \tag{3.31}$$

$$\dim \mathcal{W}_n = n + \sum_{k=2}^{n} (n - k + 1) \dim \mathcal{M}_k. \tag{3.32}$$

Proof. The result follows from Theorem 3.5 and the fact that the set in (3.7) forms a basis for $\mathcal{M}_k$. □

We have the following table.

k	$\dim \mathcal{M}_k$	$\dim \mathcal{V}_k$	$\dim \mathcal{W}_k$
1	0	1	1
2	1	2	3
3	1	3	6
4	1	4	10
5	1	5	15
6	2	7	22
7	1	8	30
8	2	10	40
9	2	12	52
10	2	14	66

4. Crank moments

We now prove that the crank moment functions C_a for a even can be written in terms of P and the Φ_j. From (1.11) we have

$$\delta_z C(z,q) = L(z,q)C(z,q), \tag{4.1}$$

where

$$\begin{aligned} L(z,q) &= \sum_{n\geq 1}\left(\frac{zq^n}{1-zq^n} - \frac{z^{-1}q^n}{1-z^{-1}q^n}\right), \\ &= \sum_{m,n\geq 1}\left(z^m q^{mn} - z^{-m}q^{mn}\right), \end{aligned} \tag{4.2}$$

so that

$$\delta_z^j L(z,q) = \sum_{m,n\geq 1}\left(m^j z^m q^{mn} - (-m)^j z^{-m} q^{mn}\right), \tag{4.3}$$

and

$$\delta_z^j L(z,q)\big|_{z=1} = \begin{cases} 0, & j \text{ even}, \\ 2\Phi_j, & j \text{ odd}. \end{cases} \tag{4.4}$$

Assume a is even and apply δ_z^{a-1} to both sides of (4.1). Then

$$\delta_z^a C = \sum_j \binom{a-1}{j} \delta_z^j(L)\, \delta_z^{a-1-j}(C). \tag{4.5}$$

Setting $z=1$ and using (4.4) and (1.36) we obtain the following recurrence

$$C_a = 2\sum_{j=1}^{\frac{a}{2}-1}\binom{a-1}{2j-1}\Phi_{2j-1}\, C_{a-2j} + 2\Phi_{a-1}\, P. \tag{4.6}$$

We compute some examples.

$$\begin{aligned} C_2 &= 2\,P\,\Phi_1, \\ C_4 &= 2\,P\,\left(\Phi_3 + 6\,{\Phi_1}^2\right), \\ C_6 &= 2\,P\,\left(\Phi_5 + 30\,\Phi_3\Phi_1 + 60\,{\Phi_1}^3\right), \\ C_8 &= 2\,P\,\left(\Phi_7 + 56\,\Phi_5\Phi_1 + 840\,\Phi_3{\Phi_1}^2 + 840\,{\Phi_1}^4 + 70\,{\Phi_3}^2\right). \end{aligned} \tag{4.7}$$

Using induction and (4.6) we can show that for $n \geq 1$ there are integers $\alpha_{a_1,a_2,\ldots,a_n}$ such that

$$C_{2n} = 2\,P \sum_{a_1+2a_2+\cdots+na_n=n} \alpha_{a_1,a_2,\ldots,a_n}\,\Phi_1^{a_1}\,\Phi_3^{a_2}\cdots\Phi_{2n-1}^{a_n}. \tag{4.8}$$

By (2.11) and (4.7) we have

$$C_2 = 2\delta_q P, \tag{4.9}$$

or

$$M_2(n) = \sum_k k^2\, M(k,n) = 2np(n). \tag{4.10}$$

A combinatorial proof of (4.10) was found by Dyson [D2].

We need the following

Lemma 4.1. *For $m \geq 1$, there exists a $\Phi \in \mathcal{W}_m$ such that*

$$\delta_q^m(P) = P\,\Phi. \tag{4.11}$$

Proof. We proceed by induction on m. From (2.11)

$$\delta_q(P) = P\,\Phi_1, \tag{4.12}$$

and the result holds for $m = 1$. Suppose the result holds for $m = a$; i.e.,

$$\delta_q^a(P) = P\,\Phi, \tag{4.13}$$

for some $\Phi \in \mathcal{W}_a$. Then

$$\begin{aligned} \delta_q^{a+1}(P) &= P\,\delta_q(\Phi) + \delta_q(P)\,\Phi, \\ &= P(\delta_q(\Phi) + \Phi_1\,\Phi), \end{aligned}$$

and the result holds for $m = a+1$ since $\delta_q(\Phi) \in \mathcal{W}_{a+1}$ by (3.29). The result for general m follows by induction. ☐

We may calculate $\delta_q^a(P)$ by using (4.12) and the recurrence

$$\delta_q^a(P) = \sum_{j=0}^{a-1} \binom{a-1}{j} \delta_q^j(\Phi_1)\,\delta_q^{a-j-1}(P), \tag{4.14}$$

which is obtained by applying δ_q^{a-1} to both sides of (4.12). For example, we have

$$\begin{aligned} \delta_q^2(P) &= -\frac{1}{6} P\,\left(6\,{\Phi_1}^2 - 5\,\Phi_3 - \Phi_1\right), \\ \delta_q^3(P) &= \frac{1}{12} P\,\left(36\,{\Phi_1}^3 - 90\,\Phi_1\Phi_3 - 6\,{\Phi_1}^2 + 7\,\Phi_5 + 5\,\Phi_3\right). \end{aligned} \tag{4.15}$$

Theorem 4.2. *For $m \geq 0$ and $n \geq 1$, there exists $\Phi \in \mathcal{W}_{n+m}$ such that*

$$\delta_q^m(C_{2n}) = P\,\Phi, \tag{4.16}$$

where $\mathcal{W}_k$ is defined by (3.27).

Proof. Let $n \geq 1$. We proceed by induction on m. The result is true for $m = 0$ using (4.8) and (3.25). The remainder of the proof is analogous to that of Lemma 4.1. ☐

We give some examples.

$$\begin{aligned} \delta_q(C_2) &= -\frac{1}{3} P\,\left(6\,{\Phi_1}^2 - 5\,\Phi_3 - \Phi_1\right), \\ \delta_q^2(C_2) &= \frac{1}{6} P\,\left(36\,{\Phi_1}^3 - 90\,\Phi_1\Phi_3 - 6\,{\Phi_1}^2 + 7\,\Phi_5 + 5\,\Phi_3\right), \\ \delta_q(C_4) &= -\frac{1}{15} P\,\left(-90\,\Phi_1\Phi_3 - 21\,\Phi_5 - 10\,\Phi_3 + \Phi_1 + 540\,{\Phi_1}^3 - 60\,{\Phi_1}^2\right). \end{aligned} \tag{4.17}$$

5. Relations between rank and crank moments

Let a be even. After applying δ_z^a to both sides of the rank-crank PDE (2.22), setting $z=1$ and using (4.9) we find that

$$\begin{aligned}
&\sum_{i=0}^{a/2-1}\binom{a}{2i}\sum_{\substack{\alpha+\beta+\gamma=a-2i\\ \alpha,\,\beta,\,\gamma\text{ even }\geq 0}}\binom{a-2i}{\alpha,\beta,\gamma}C_\alpha\,C_\beta\,C_\gamma\,P^{-2}-3\left(2^{a-1}-1\right)C_2\\
&=\frac{1}{2}(a-1)(a-2)R_a+6\sum_{i=1}^{a/2-1}\binom{a}{2i}\left(2^{2i-1}-1\right)\delta_q(R_{a-2i})\\
&\quad+\sum_{i=1}^{a/2-1}\left[\binom{a}{2i+2}\left(2^{2i+1}-1\right)-2^{2i}\binom{a}{2i+1}+\binom{a}{2i}\right]R_{a-2i}.
\end{aligned}\tag{5.1}$$

For $a=2$ we obtain $0=0$. For $a=4$ we obtain

$$C_4+6\,\frac{{C_2}^2}{P}-C_2=R_4-\;R_2+12\,\delta_q(R_2).\tag{5.2}$$

Using (4.7), (4.17) and (5.2) we find that

$$-\frac{2}{3}\,C_2-2\,\delta_q(C_2)+\frac{8}{3}\,C_4=R_4-R_2+12\,\delta_q(R_2),\tag{5.3}$$

or

$$N_4(n)=\frac{2}{3}\,(-3\,n-1)\;M_2(n)+\frac{8}{3}\,M_4(n)+(-12\,n+1)\;N_2(n),\tag{5.4}$$

for $n\geq 0$.

Similarly, for 2, 3, 4, and 5, there are polynomials $P_k(n)$ of degree $k-1$ and $Q_{k,j}(n)$ of degree $k-j$ (for $1\leq j\leq k$) such that

$$N_{2k}(n)=P_k(n)\,N_2(n)+\sum_{j=1}^{k}Q_{k,j}(n)\,M_{2j}(n),\tag{5.5}$$

for $n\geq 0$. For $k=6$ there is no such relation. For $k=7$ there is a similar relation but with an extra term $N_{12}(n)$. These relations are given in the following

Theorem 5.1. *For $n\geq 0$ we have*

$$N_4(n)=\frac{2}{3}\,(-3\,n-1)\;M_2(n)+\frac{8}{3}\,M_4(n)+(-12\,n+1)\;N_2(n),\tag{5.6}$$

$$\begin{aligned}
N_6(n)&=\frac{2}{33}\,\left(324\,n^2+69\,n-10\right)\;M_2(n)+\frac{20}{33}\,(-45\,n+4)\;M_4(n)\\
&\quad+\frac{18}{11}\,M_6(n)+\left(108\,n^2-24\,n+1\right)\;N_2(n),
\end{aligned}\tag{5.7}$$

(5.8)

$$\begin{aligned}N_8(n) &= \frac{2}{913}\left(-72972\,n^3 - 1728\,n^2 + 5667\,n - 289\right)\,M_2(n)\\ &\quad + \frac{280}{913}\left(732\,n^2 - 195\,n + 8\right)\,M_4(n) + \frac{84}{913}\,(-196\,n + 15)\,M_6(n)\\ &\quad + \frac{1248}{913}\,M_8(n) + \left(-864\,n^3 + 360\,n^2 - 36\,n + 1\right)\,N_2(n),\end{aligned}$$

(5.9)

$$\begin{aligned}&N_{10}(n)\\ &= \frac{2}{5951847}\left(3588144480\,n^4 - 805458600\,n^3 - 398007108\,n^2 + 56257647\,n - 1794592\right)\,M_2(n)\\ &+ \frac{140}{5951847}\left(-72270360\,n^3 + 36826920\,n^2 - 3625245\,n + 104002\right)\,M_4(n)\\ &+ \frac{210}{1983949}\left(1421544\,n^2 - 380744\,n + 13519\right)\,M_6(n)\\ &+ \frac{120}{1983949}\,(-282435\,n + 18796)\,M_8(n) + \frac{2724}{2173}\,M_{10}(n)\\ &+ \left(6480\,n^4 - 4320\,n^3 + 756\,n^2 - 48\,n + 1\right)\,N_2(n)\end{aligned}$$

(5.10)

$$\begin{aligned}N_{14}(n) = {} & \frac{1}{4505033323132497}\,(-6559188470167503542400\,n^6\\ & + 584104439765983424400\,n^5 - 88193910587689930464\,n^4\\ & - 5125598568960631736 4\,n^3 + 12889219681488512844\,n^2\\ & -1033571808069319887\,n + 23432656561492057)\,M_2(n)\\ & + \frac{364}{4505033323132497}\,(2544016408481081520\,n^5\\ & - 2986029950270749200\,n^4 + 1233083592931144500\,n^3\\ & - 185464100558325420\,n^2 + 12124758510318780\,n\\ & -229618708346923)\,M_4(n)\\ & + \frac{728}{500559258125833}\,(-12932704022040180\,n^4\\ & + 11781511098477120\,n^3 - 3661921161131415\,n^2 + 234233352768436\,n\\ & -7334109150929)\,M_6(n) +\\ & \frac{364}{500559258125833}\,(3327634333443960\,n^3\\ & - 2184561928177200\,n^2 + 464283118670595\,n\\ & -12774042869566)\,M_8(n)\\ & + \frac{2002}{6030834435251}\,(-758615153688\,n^2 + 404700708960\,n\\ & -24122003839)\,M_{10}(n)\\ & + \frac{25388554464}{2775349487}\,(n-1)\,M_{12}(n)\end{aligned}$$

$$+\frac{139497552}{120667369}\,M_{14}(n)$$
$$+\frac{1}{138}\left(-107775360\,n^{6}+143700480\,n^{5}\right.$$
$$\left.-70752528\,n^{4}+14978304\,n^{3}-1456488\,n^{2}+64320\,n-1045\right)\,N_{2}(n)$$
$$+\frac{91}{138}\,(-36\,n+13)\,N_{12}(n)$$

Proof. We first show that relations such as (5.6)–(5.10) must exist. For $k \geq 1$ we define

$$\begin{aligned}T_k &= (2k-1)(k-1)R_{2k} + 6\sum_{i=1}^{k-1}\binom{2k}{2i}\left(2^{2i-1}-1\right)\delta_q(R_{2k-2i})\\ &\quad+\sum_{i=1}^{k-1}\left[\binom{2k}{2i+2}\left(2^{2i+1}-1\right)-2^{2i}\binom{2k}{2i+1}+\binom{2k}{2i}\right]R_{2k-2i};\end{aligned}\tag{5.11}$$

i.e., T_k is the function on the right side of (5.1) with $a = 2k$. By considering the left side of (5.1) and using Theorem 4.2 we have

$$T_k \in P\,\mathcal{W}_k;\tag{5.12}$$

i.e., there is a $\Phi \in \mathcal{W}_k$ such that

$$T_k = P\,\Phi.\tag{5.13}$$

For $k \geq 1$ we consider the set of functions

$$\mathcal{C}_k = \{\delta_q^m(C_{2j}) \,:\, 1 \leq j \leq k,\, j+m \leq k\} \subset P\,\mathcal{W}_k.\tag{5.14}$$

Then

$$|\mathcal{C}_k| = \frac{k(k+1)}{2}.\tag{5.15}$$

For $1 \leq k \leq 5$ we observe from Corollary 3.6 that

$$\dim \mathcal{W}_k = \frac{k(k+1)}{2}.\tag{5.16}$$

Clearly,

$$\dim P\,\mathcal{W}_k = \dim \mathcal{W}_k,\tag{5.17}$$

for all k. Thus, for $1 \leq k \leq 5$ there is a linear relation between T_k and the functions in $\mathcal{C}_k$. For $k = 1$ the relation is $T_1 = 0$, which is trivial. For each $2 \leq k \leq 5$ we have used MAPLE to find this relation. This gives identities (5.6)–(5.9). Corollary 3.6 gives

$$\dim \mathcal{W}_6 = 22.\tag{5.18}$$

Since $|\mathcal{C}_6| = 21$ there is no reason to expect an analogous relation for $k = 6$, and in fact calculation shows that the functions in $\mathcal{C}_6$ together with T_6 are linearly independent. We note that

$$T_6,\ \delta_q(T_6),\ T_7 \in P\mathcal{W}_7. \tag{5.19}$$

Luckily we have

$$\dim \mathcal{W}_7 = 30, \quad \text{and} \quad \dim \mathcal{C}_7 = 28, \tag{5.20}$$

so there must exist a linear relation between the three functions in (5.19) and the functions in $\mathcal{C}_7$. We used MAPLE to find this relation and (5.10). $\square$

Although (5.5) does not hold for $k = 6$. There is a relation for $k = 6$ but involving an additional function. For an integer r define $p_r(n)$ by

$$\sum_{n\ge 0} p_r(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^r. \tag{5.21}$$

It is well-known that

$$\Delta(q) = \sum_{n\ge 1} \tau(n) q^n = \sum_{n\ge 1} p_{24}(n-1) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24} \tag{5.22}$$

is a modular form of weight 12. See [S,pp.95–97]. It follows that

$$P\Delta = \sum_{n\ge 1} p_{23}(n-1) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{23} \in P\mathcal{W}_6. \tag{5.23}$$

By (5.18) and the fact that $|\mathcal{C}_6| = 21$ we see that there must be a linear relation between $P\Delta$, T_{12} and the functions in $\mathcal{C}_6$. We used MAPLE to find this relation. It is given in the following

Theorem 5.2. *For $n \ge 1$ we have*

(5.24)

$$\begin{aligned}
&p_{23}(n-1) = \\
&\frac{1}{897196601564928}\big(-5791789754051878552\,n^5 + 3065207827654788955\,2\,n^4 \\
&\qquad +5952274737922797228\,n^3 - 2214892612179680256\,n^2 + 188772676333745691\,n \\
&\qquad -4410708034409819\big)\,M_2(n) \\
&+\frac{5}{224299150391232}\big(4089872889595634400\,n^4 - 3320629034843596140\,n^3 \\
&\qquad +593555423164294752\,n^2 - 40741028214970311\,n + 815166233039851\big)\,M_4(n) \\
&+\frac{13}{7120607948928}\big(-4612652508217680\,n^3 + 2500384365901740\,n^2 \\
&\qquad -190834728931028\,n + 5730847932535\big)\,M_6(n) \\
&+\frac{65}{24922127821248}\big(431597256867684\,n^2 - 112947999359631\,n + 3472477850182\big)\,M_8(n) \\
&+\frac{143}{600533200512}\big(-555655003092\,n + 33496841951\big)\,M_{10}(n) + \frac{16986177}{1919176}\,M_{12}(n) \\
&+\frac{24599722121}{3316336128}\big(-46656\,n^5 + 45360\,n^4 - 12096\,n^3 + 1296\,n^2 - 60\,n + 1\big)\,N_2(n) \\
&-\frac{24599722121}{3316336128}\,N_{12}(n).
\end{aligned}$$

6. Congruence relations between rank and crank moments

In the previous section we considered exact linear relations between rank and crank moments. In this section we consider the analogous problem modulo p. One way to find such relations is to examine the denominators of the rational numbers that occur in an exact relation. For example, if we multiply both sides of (5.7) by 11 and reduce modulo 11 we find

$$M_6(n) \equiv (2n+3)M_4(n) - (n+8)^2 M_2(n) \pmod{11}. \tag{6.1}$$

We could try the same sort of thing with (5.6). If we multiply both sides of (5.6) by 3 and reduce modulo 3 we obtain

$$M_4(n) \equiv M_2(n) \pmod{3}, \tag{6.2}$$

but this relation is trivial since $k^4 \equiv k^2 \pmod{3}$, for all integers k.

There is another way in which a congruence between rank and crank moments may arise. In (5.14) we defined the set $\mathcal{C}_k$. Let $k \geq 1$. The set $\mathcal{C}_k$ seems to be linearly independent over $\mathbb{Q}$. For small k and certain primes p the set $\mathcal{C}_k$ is linearly dependent over $\mathbb{Z}_p$. For example, consider the case $k = 3$.

$$\mathcal{C}_3 = \{C_2, \delta_q(C_2), \delta_q^2(C_2), C_4, \delta_q(C_4), C_6\}.$$

We want to find congruences between the elements of $\mathcal{C}_3$. We consider the 6×6 matrix A whose (i,j)-th entry is the coefficient of q^i in the q-expansion of the jth element of $\mathcal{C}_3$.

$$A = \begin{pmatrix} M_2(1) & M_2(1) & M_2(1) & M_4(1) & M_4(1) & M_6(1) \\ M_2(2) & 2\,M_2(2) & 4\,M_2(2) & M_4(2) & 2\,M_4(2) & M_6(2) \\ M_2(3) & 3\,M_2(3) & 9\,M_2(3) & M_4(3) & 3\,M_4(3) & M_6(3) \\ M_2(4) & 4\,M_2(4) & 16\,M_2(4) & M_4(4) & 4\,M_4(4) & M_6(4) \\ M_2(5) & 5\,M_2(5) & 25\,M_2(5) & M_4(5) & 5\,M_4(5) & M_6(5) \\ M_2(6) & 6\,M_2(6) & 36\,M_2(6) & M_4(6) & 6\,M_4(6) & M_6(6) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 8 & 16 & 32 & 32 & 64 & 128 \\ 18 & 54 & 162 & 162 & 486 & 1458 \\ 40 & 160 & 640 & 544 & 2176 & 8320 \\ 70 & 350 & 1750 & 1414 & 7070 & 32710 \\ 132 & 792 & 4752 & 3300 & 19800 & 103092 \end{pmatrix}$$

We find that

$$\det(A) = -110361968640 = -2^{17} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11.$$

Since the determinant is nonzero the functions in $\mathcal{C}_3$ are linearly independent. If there is a linear congruence for the functions in $\mathcal{C}_3 \pmod{p}$, then p will divide this determinant. The occurrence of $p = 11$ is confirmed by (6.1). We suspect that there may be a relation modulo 7. We find that

$$(n+2)M_4(n) + (6n^2+4n+1)M_2(n) \equiv 0 \pmod{7}, \tag{6.3}$$

for all n. This is easily proved by writing the generating function of the left side of (6.3) in terms of P, Φ_1, Φ_3, Φ_5:

(6.4)

$$(\delta_q+2)C_4+(6\delta_q^2+4\delta_q+1)C_2 = -\frac{7}{15}P\,\left(180\,\Phi_1\Phi_3 - 30\,{\Phi_1}^2 - 18\,\Phi_5 - 35\,\Phi_3 - 7\,\Phi_1\right)$$

We have used both methods described above to obtain congruences between rank and crank moments. These are collected together in the following

Theorem 6.1. *For $n \geq 0$ we have*

$$(n+2)M_4(n) + (6n^2+4n+1)M_2(n) \equiv 0 \pmod{7}, \tag{6.5}$$

(6.6)
$$(n+5)^3\, M_4(n) \equiv \left(5\,n^4 + 10\,n^3 + 8\,n^2 + 8\,n + 9\right)\, M_2(n) \pmod{11},$$

(6.7)
$$M_6(n) \equiv 2(n+7)M_4(n) - (n+8)^2 M_2(n) \pmod{11},$$

(6.8)
$$M_8(n) \equiv 2\,(n+5)\left(n^2 + 5\,n + 10\right)\, M_2(n) + 6\left(n^2 + n + 1\right)\, M_4(n) \pmod{11},$$

(6.9)
$$\begin{aligned} M_{10}(n) \equiv{} & 4\,(n+7)\left(n^3 + 5\,n^2 + 29\,n + 32\right)\, M_2(n) \\ & + 39\,(n+7)(n+14)(n+39)\, M_4(n) + \left(6\,n^2 + 34\,n + 39\right)\, M_6(n) \\ & + 35\,(n+13)\, M_8(n) \pmod{41}, \end{aligned}$$

(6.10)
$$\begin{aligned} & (n+7)(n+25)(n+31)\left(n^2 + 28\,n + 6\right)\, N_2(n) + N_{12}(n) \\ & \equiv \left(4\,n^5 + 10\,n^4 + 6\,n^3 + 30\,n^2 + 31\,n + 33\right)\, M_2(n) \\ & \quad + \left(n^4 + 4\,n^3 + 42\,n^2 + 24\,n + 30\right)\, M_4(n) + 40\,(n+10)\left(n^2 + 32\,n + 7\right)\, M_6(n) \\ & \quad + 31\,(n+13)(n+41)\, M_8(n) + (n+11)\, M_{10}(n) + 22\, M_{12}(n) \pmod{43}, \end{aligned}$$

(6.11)
$$\begin{aligned} M_{10}(n) \equiv{} & 36\,(n+19)\left(n^3 + 50\,n^2 + 20\,n + 36\right)\, M_2(n) \\ & + \left(52\,n^3 + 28\,n^2 + 26\,n + 52\right)\, M_4(n) + \left(36\,n^2 + 11\,n + 32\right)\, M_6(n) \\ & + 47\,(n+17)\, M_8(n) \pmod{53}, \end{aligned}$$

(6.12)
$$\begin{aligned} M_8(n) \equiv{} & \left(10\,n^3 + 73\,n^2 + 40\,n + 82\right)\, M_2(n) + \left(72\,n^2 + 23\,n + 28\right)\, M_4(n) \\ & + 10\,(n+41)\, M_6(n) \pmod{83}, \end{aligned}$$

(6.13)
$$\begin{aligned} & N_{12}(n) - 367\,(n+332)(n+487)(n+664)\left(n^2 + 265\,n + 155\right)\, N_2(n) \\ & \equiv 352\,(n+247)(n+734)\left(n^3 + 147\,n^2 + 597\,n + 363\right)\, M_2(n) \\ & \quad + 88\,(n+530)(n+701)(n+709)(n+740)\, M_4(n) \\ & \quad + 577\,(n+114)(n+427)(n+682)\, M_6(n) + \left(295\,n^2 + 177\,n + 674\right)\, M_8(n) \\ & \quad + 271\,(n+336)\, M_{10}(n) + 654\, M_{12}(n) \pmod{797}, \end{aligned}$$

(6.14)
$$\begin{aligned}
M_{14}(n) \equiv \bigl(&44976165\,n^6 + 23584476\,n^5 + 19728425\,n^4 + 8711555\,n^3 + 36781660\,n^2 \\
&+70780973\,n + 108798274)\ M_2(n) \\
&+ 77429163\ (n + 4141548)\,(n + 113894720)\,\bigl(n^3 + 42853554\,n^2 + 28914352\,n \\
&\quad +100598975)\ M_4(n) \\
&+ \bigl(12571854\,n^4 + 82951807\,n^3 + 9501843\,n^2 + 38248242\,n + 118847240\bigr)\ M_6(n) \\
&+ 84218605\ (n + 53645347)\,\bigl(n^2 + 6688335\,n + 93582728\bigr)\ M_8(n) \\
&+ 73449678\ (n + 40889782)\,(n + 59666357)\ M_{10}(n) \\
&+ 89188917\ (n + 120667368)\ M_{12}(n) \\
&\pmod{120667369}.
\end{aligned}$$

Proof. We have already discussed (6.5) and (6.7). (6.12) follows from multiplying both sides of (5.8) by 83 and reducing modulo 83. (6.6) and (6.11) follow from multiplying both sides of (5.9) by 11 and 53 respectively. (6.14) follows from multiplying both sides of (5.10) by 120667369 and reducing modulo 120667369. The remaining relations follow by writing the generating functions for differences between the left and right sides in terms of P, Φ_1, Φ_3, and Φ_5, and checking that the coefficient of each monomial is divisible by the appropriate prime. $\square$

We may multiply both sides of (5.10) by 23 and then reduce modulo 23 to find a congruence for rank and crank moments. A stronger relation follows from reducing (5.24) modulo 23 and using

$$\sum_{n\ge1} p_{23}(n-1)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{23} \equiv q\prod_{n=1}^{\infty}(1-q^{23n}) \pmod{23}, \tag{6.15}$$

and Euler's pentagonal number theorem [An, p.11]

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}. \tag{6.16}$$

This gives

Theorem 6.2. *For $n \ge 0$ we have*

$$\begin{aligned}
&4\ \bigl(n^2+n+14\bigr)\bigl(n^3+n^2+15\bigr)\ M_2(n) + \bigl(10\,n^4+2\,n^3+8\,n^2+21\,n+22\bigr)\ M_4(n) \\
&+ 13\ (n+18)\bigl(n^2+21\,n+13\bigr)\ M_6(n) + 5\,n\,(n+6)\ M_8(n) + 15\ (n+19)\ M_{10}(n) \\
&+ M_{12}(n) + 12\ (n+10)\,(n+14)\,(n+19)\,(n+20)\,(n+21)\ N_2(n) + N_{12}(n) \\
&\equiv \begin{cases} (-1)^k \pmod{23}, & \textit{if } n = 23k(3k\pm1)/2\,+\,1, \\ 0 \pmod{23}, & \textit{otherwise.} \end{cases}
\end{aligned}$$

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