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# EXTENSION OF THE PERRON–FROBENIUS THEOREM TO HOMOGENEOUS SYSTEMS*

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**Abstract.** This paper deals with a particular class of positive systems. The state components of a positive system are positive or zero for all positive times. These systems are often encountered in applied areas such as chemical engineering or biology. It is shown that for this particular class the first orthant contains an invariant ray in its interior. An invariant ray generalizes the concept of an eigenvector of linear systems to nonlinear homogeneous systems. Then sufficient conditions for uniqueness of this ray are given. The main result states that the vector field on an invariant ray determines the stability properties of the zero solution with respect to initial conditions in the first orthant. The asymptotic behavior of the solutions is examined. Finally, we compare our results to the Perron–Frobenius theorem, which gives a detailed picture of the dynamical behavior of positive linear systems.

**Key words.** positive systems, cooperative systems, homogeneous systems, monotone flows, global asymptotic stability

**AMS subject classifications.** 37C10, 37C65, 34D23, 34D05

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## 1. Introduction.

A dynamical system is said to be *positive* if it leaves the first orthant of $\mathbb{R}^n$ invariant for future times when initiated in this orthant. Examples of these systems abound in a variety of applied areas such as biology, chemistry, economics, and sociology [6], [14], [9]. In a biological system, for example, a state component will typically be the number of individuals of a certain species in a population of interacting species. State components in a chemical system are typically concentrations or amounts of chemical substances. Important issues arising in the study of positive systems are the boundedness of solutions, permanence or persistence, and the (asymptotic) stability of equilibrium points. In this paper the stability properties of the zero solution of a class of positive systems is considered. At first glance, it might be surprising that we are interested in the zero solution, because in most applications this solution is not very interesting. For example, it corresponds to death of all species in a biological context, or to washout of all chemicals in chemical engineering. In the context of positive systems, nontrivial equilibrium points are of much more interest. However, these equilibrium points arise in models where some type of control action is already present in the model, although implicitly.

As an illustration, consider the simplest example of the well-known predator-prey model proposed by Volterra (see [6]) to explain the observed oscillations in the biomass of prey species (denoted by $x$) and predator species (denoted by $y$):

$$\begin{aligned}\dot{x} &= -axy + bx,\\ \dot{y} &= cxy - dy,\end{aligned}$$

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where $a, b, c$, and $d$ are positive constants. The term $bx$ is the growth rate of the prey species and suggests the availability of a feeding source. The abundance of the food is represented by the parameter $b$, which can be interpreted as the implicit control action hinted at before. If there is no food ($b = 0$), then the only equilibrium point of the system is the trivial one. If there is food available ($b > 0$), then there is a nontrivial equilibrium point. It might not come as a surprise that the stability behavior of the zero solution of the simpler system with $b = 0$ determines to some extent the behavior of the system with $b > 0$. (Bifurcation theory might serve as a tool here.) This motivates the study of the zero solution for the case in which $b = 0$.

We provide another example of a chemical reactor at the end of section 5.

Therefore, the study of the stability behavior of the zero solution is not only interesting in its own right, but also important for control purposes.

Homogeneous systems on $\mathbb{R}^n$ (see, e.g., [3]) are a particular class of nonlinear systems. *Invariant rays*—when they exist—play an important role in the study of the stability behavior of the zero solution of a homogeneous system. An invariant ray is a particular curve that is invariant for the flow of the system. It can be interpreted as the generalization of the concept of the linear space spanned by an eigenvector of a real eigenvalue, in the context of *linear* systems, to the context of *homogeneous*—and thus generally nonlinear—systems. Suppose that a homogeneous system possesses a number of invariant rays. A necessary condition for global asymptotic stability (GAS) of the zero solution is that the vector field on all invariant rays points towards the origin. Although this condition is not sufficient in general for GAS, we introduce a particular class of homogeneous systems for which invariant rays determine the stability behavior.

Within the class of positive systems, cooperative and irreducible systems are well examined [5], [13]. In this paper a class of positive systems is introduced, characterized by *homogeneous* cooperative and irreducible vector fields. It is shown that these systems enjoy a fairly simple dynamical behavior. This may come as a surprise, since it is well known that the behavior of homogeneous systems is in principle as involved as the behavior of general nonlinear systems.

Next, to describe our results in some detail, we digress to discuss positive *linear* systems, known to model a number of important physical systems; see, e.g., [9]. A necessary and sufficient condition for a linear system $\dot{x} = Ax$ to be positive is that $A$ be a Metzler matrix (i.e., have nonnegative off-diagonal entries) or equivalently that the system be cooperative. The principal tool for the analysis of the (stability) behavior of a positive linear system is the Perron–Frobenius theorem. It is natural to ask whether it is possible to generalize this to classes of positive *nonlinear* systems. The purpose of this paper is to show that this is indeed the case for homogeneous cooperative and irreducible systems.

First of all it is shown that these systems *always* possess an invariant ray in the interior of the first orthant. If the order of the homogeneous vector field is equal to zero, then this ray is unique in the first orthant; if the order of the vector field is strictly greater than zero, then the ray is unique in the first orthant if the vector field on this ray does not point away from the origin. In both cases the stability behavior of the zero solution of the system is determined by the behavior of the system on this unique invariant ray.

Several invariant rays may exist if the order of the homogeneous vector field is strictly greater than zero and if the vector field on every invariant ray points away from the origin. In this case the stability behavior of the zero solution of the system is also determined by the behavior of the system on the invariant rays.

This paper is organized as follows. Basic definitions are given in section 2. In section 3, results on homogeneous systems are reviewed. A class of positive homogeneous systems is introduced in section 4, leading to a criterion for GAS of the zero solution with respect to the initial conditions in $\mathbb{R}^n_+$ (section 5). The asymptotic behavior of solutions of this class of systems is examined in section 6. The paper is concluded in section 7 with a discussion; in particular, the classical Perron–Frobenius theorem for linear differential equations (see [1] or [13]) is compared to the results of this paper.

## 2. Preliminaries.

Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}^n$ the set of $n$-tuples with all components belonging to $\mathbb{R}$. For $x \in \mathbb{R}^n$, $|x|$ is the Euclidean norm of $x$. $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{R}^+_0 := (0, +\infty)$, and $\mathbb{R}^n_+$ ($\mathrm{int}(\mathbb{R}^n_+)$) is the set of $n$-tuples with all components belonging to $\mathbb{R}^+$ ($\mathbb{R}^+_0$). Finally, $\mathrm{bd}(\mathbb{R}^n_+) := \mathbb{R}^n_+ \setminus \mathrm{int}(\mathbb{R}^n_+)$ is the boundary of $\mathbb{R}^n_+$.

Let $x, y \in \mathbb{R}^n_+$; then $x \leq y$ means $x_i \leq y_i$ $\forall i = 1, \ldots, n$. Furthermore, $x < y$ if and only if $x \leq y$ and $x \neq y$, while $x \ll y$ if and only if $x_i < y_i$ $\forall i = 1, \ldots, n$. For subsets $U$ and $V$ of $\mathbb{R}^n_+$, we denote $U \leq (<, \ll) V$ if $x \leq (<, \ll) y$ $\forall x \in U$ and $y \in V$.

For $x \in \mathbb{R}^n$, $\mathrm{diag}(x)$ stands for an $n \times n$ diagonal matrix, where the $i$th diagonal entry is equal to $x_i$, the $i$th component of the vector $x$. A real $n \times n$ matrix $A = (a_{ij})$ is *Metzler* if and only if its off-diagonal entries $a_{ij}$, $\forall i \neq j$, belong to $\mathbb{R}^+$.

A real $n \times n$ matrix $A = (a_{ij})$ is *reducible* if and only if the index set $N := \{1, 2, \ldots, n\}$ can be split into two sets $J$ and $K$, with $J \cup K = N$ and $J \cap K = \emptyset$ such that $a_{jk} = 0$ $\forall j \in J$ and $k \in K$.

It is clear that $A$ is reducible if and only if there exists a permutation matrix $P$ such that

$$PAP^T = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where $B$ and $D$ are square matrices.

The *standard basis* of the vector space $\mathbb{R}^n$ is given by $\{e_i | i \in N\}$, where the $i$th entry of $e_i$ is equal to 1, while the other entries are equal to 0. An $m$-dimensional *coordinate subspace* of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ with a basis $\{e_{k_1}, e_{k_2}, \ldots, e_{k_m}\}$, with $1 \leq k_1 < k_2 < \cdots < k_m$.

The matrix $A$ is reducible if and only if the linear operator, associated to the matrix $A$ and the standard basis, has an $m$-dimensional invariant coordinate subspace with $1 \leq m < n$.

When $A$ is not reducible, it is *irreducible*.

Consider the system

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ and $f(x)$ is a continuous vector field on $\mathbb{R}^n$, continuously differentiable (of class $C^1$) on $\mathbb{R}^n \setminus \{0\}$, and such that $f(0) = 0$. Later we give conditions such that the uniqueness of solutions for system (1) is guaranteed. The *forward solution* of system (1) with initial condition $x_0 \in \mathbb{R}^n$ at $t = 0$ is denoted as $x(t, x_0)$, $t \in \mathcal{I}_{x_0} := [0, T_{\max}(x_0))$, where $\mathcal{I}_{x_0}$ is the *maximal forward interval of existence*. A set $D \subset \mathbb{R}^n$ is *forward invariant* for system (1) if and only if $\forall x_0 \in D$, $x(t, x_0) \in D$ $\forall t \in \mathcal{I}_{x_0}$. System (1) is *positive* if and only if $\mathbb{R}^n_+$ is forward invariant.

Suppose that $D \subset \mathbb{R}^n$ is a forward invariant set for system (1). The flow of system (1) is *monotone in $D$* if and only if $\forall x_0$, $y_0 \in D$ with $x_0 \leq (<, \ll) y_0$ it holds that $x(t, x_0) \leq (<, \ll) x(t, y_0)$ $\forall t \in (\mathcal{I}_{x_0} \cap \mathcal{I}_{y_0})$.

The flow of system (1) is *strongly monotone in* $D$ if and only if it is monotone in $D$ and $\forall x_0,\, y_0 \in D$ with $x_0 < y_0$ it holds that $x(t,x_0) \ll x(t,y_0)\ \forall t \in (\mathcal{I}_{x_0} \cap \mathcal{I}_{y_0}) \setminus \{0\}$.

A point $p \in \mathbb{R}^n$ is an *omega limit point* of $x_0$ if there exists an increasing sequence of time instances $\{t_k\}$, with $t_k \to +\infty$ when $k \to +\infty$, such that $\lim_{t_k \to +\infty} x(t_k, x_0) = p$. The set of all omega limit points of $x_0$ is the *omega limit set* of $x_0$ and is denoted by $\omega(x_0)$. Notice that the *omega limit set* of $x_0$ may be the empty set, for instance if the solution starting in $x_0$ diverges. If $T_{\max(x_0)} = +\infty$, then the set $\mathcal{O}(x_0) := \{x(t,x_0) | t \in \mathbb{R}^+\}$ is the *forward orbit* of the forward solution $x(t,x_0)$. It follows from classical results on the theory of ordinary differential equations that if $\mathrm{cl}(\mathcal{O}(x_0))$, the closure of the forward orbit $\mathcal{O}(x_0)$, is compact, then $\omega(x_0)$ is nonempty and compact and $d(\omega(x_0), x(t,x_0)) \to 0$ when $t \to +\infty$ (where $d(A,z) := \inf_{y \in A} d(y,z)$ and $d(y,z)$ is the Euclidean distance between $y$ and $z$).

## 3. Homogeneous systems.

In this section we review the concept of a homogeneous system and discuss some of its properties. Many of these results are known, and no originality is claimed here. However, we have chosen to include the proofs to make the paper self-contained.

### 3.1. Definition and Euler's formula.

We first introduce the concept of a homogeneous vector field [12].

DEFINITION 3.1. *A vector field* $f(x)$, $x \in \mathbb{R}^n$, *is* homogeneous *of order* $\tau \in \mathbb{R}$ *with respect to the dilation map* $\delta^r_\lambda(x) : \mathbb{R}^n \to \mathbb{R}^n$ *with* $\delta^r_\lambda(x) = (\lambda^{r_1} x_1, \lambda^{r_2} x_2, \ldots, \lambda^{r_n} x_n)$, *where* $r := (r_1, r_2, \ldots, r_n)$ *is a fixed* $n$*-tuple*$(r_i \in \mathbb{R}^+_0 \forall i \in N)$, *and* $\forall \lambda \in \mathbb{R}^+_0$ *if and only if*

$$\forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}^+_0 \qquad f(\delta^r_\lambda(x)) = \lambda^\tau \delta^r_\lambda(f(x)). \tag{2}$$

To every $n$-tuple of positive real numbers, one can associate a dilation map $\delta^r_\lambda(x)$. When $r = (1, 1, \ldots, 1)$, then $\delta^r_\lambda(x)$ is the *standard dilation map*.

System (1) is *homogeneous* if $f(x)$ is homogeneous.

We introduce the following hypothesis:

$(\mathcal{H}1)$ $f(x)$ is a homogeneous vector field of order $\tau \in \mathbb{R}^+$ with respect to a dilation map $\delta^r_\lambda(x)$.

Notice that if $(\mathcal{H}1)$ holds, then $f(0) = 0$ (by continuity of $f$ on $\mathbb{R}^n$), and thus $x = 0$ is an equilibrium point of system (1). Since $f(x)$ is $C^1$ on $\mathbb{R}^n \setminus \{0\}$, solutions starting in $\mathbb{R}^n \setminus \{0\}$ exist and are unique. On the other hand, the vector field $f(x)$ is only continuous at $x = 0$. This implies that a solution starting in $x = 0$ exists (the zero solution satisfies the differential equation) but might not be unique. The additional hypothesis $(\mathcal{H}1)$ excludes the possibility that there are multiple solutions starting in $x = 0$ as proved in [10].

Let $U$ be an open subset of $\mathbb{R}^n$, and suppose that $f(x)$ is a homogeneous vector field of order $\tau$ with respect to the dilation map $\delta^r_\lambda(x)$ and of class $C^1$ on $\mathbb{R}^n$. For future reference we recall Euler's formula:

$$\forall x \in U \qquad \frac{\partial f}{\partial x}(x)\mathrm{diag}(r)x = \mathrm{diag}(r + \tau^*) f(x),$$

where $\tau^* := (\tau, \ldots, \tau)$. This formula is easily proved by first taking the derivative with respect to $\lambda$ on both sides of (2) and then evaluating the resulting equation for $\lambda = 1$.

### 3.2. Invariant rays.

For $x \in \mathbb{R}^n \setminus \{0\}$ and a fixed but arbitrary dilation map $\delta^r_\lambda(x)$, $R_x := \{\delta^r_\lambda(x) | \lambda \in \mathbb{R}^+_0\}$ is the *ray through* $x$.

The $\omega$ limit sets of points on a ray are related as follows.

LEMMA 3.2. *If system* (1) *satisfies* ($\mathcal{H}1$) *and if* $p \in \omega(x_0)$, *then* $\delta_\lambda^r(p) \in \omega(\delta_\lambda^r(x_0))$.

This follows immediately from the scaling property [7] of solutions of homogeneous differential equations. By the scaling property we mean the following: Suppose that $x(t, x_0)$, $t \in [0, T_{\max}(x_0))$, is a solution of system (1). Then $\forall \lambda \in \mathbb{R}_0^+$ the term $\delta_\lambda^r(x(\lambda^\tau t, x_0))$, $t \in [0, \frac{T_{\max}(x_0)}{\lambda^\tau})$, is also a solution of system (1).

LEMMA 3.3. *If system* (1) *satisfies* ($\mathcal{H}1$) *and if there exists a point* $\bar{x} \in \mathbb{R}^n \setminus \{0\}$ *such that*

$$f(\bar{x}) = \gamma_{\bar{x}} \mathrm{diag}(r) \bar{x} \tag{3}$$

*for some* $\gamma_{\bar{x}} \in \mathbb{R}$, *then the vector field* $f(x)$ *is tangent to* $R_{\bar{x}}$ *at each point of* $R_{\bar{x}}$.

*Proof.* Indeed, $\frac{d}{d\lambda}(\delta_\lambda^r(\bar{x}))|_{\lambda=1} = \mathrm{diag}(r)\bar{x}$. This and (3) imply that the vector field $f(x)$ is tangent to $R_{\bar{x}}$ at the point $\bar{x}$. Moreover, $\forall \lambda \in \mathbb{R}_0^+$

$$f(\delta_\lambda^r(\bar{x})) = (\gamma_{\bar{x}} \lambda^\tau) \mathrm{diag}(r) \delta_\lambda^r(\bar{x}), \tag{4}$$

and thus the vector field $f(x)$ is tangent to $R_{\bar{x}}$ in *every* point of $R_{\bar{x}}$, which proves the lemma. Notice that (4) implies that $\forall \lambda \in \mathbb{R}_0^+$

$$\gamma_{\delta_\lambda^r(\bar{x})} = \gamma_{\bar{x}} \lambda^\tau. \qquad \square \tag{5}$$

Suppose that there exists a point $\bar{x} \in \mathbb{R}^n \setminus \{0\}$ such that (3) holds. Then it follows from Lemma 3.3 that the forward (and backward) solution of system (1), starting in an arbitrary point of $R_{\bar{x}}$, stays on this ray for all future (and past) times for which this solution is defined. Such a ray is an *invariant ray* for system (1).

An invariant ray $R_y$ is *asymptotically stable*, *stable*, or *unstable* if and only if $\gamma_x < 0$, $\gamma_x \leq 0$, respectively, $\gamma_x > 0$ for some $x \in R_y$ and by (5) for any $x \in R_y$.

An easy calculation shows that solutions starting on an invariant ray $R_{\bar{x}}$ satisfy the set of decoupled differential equations

$$\dot{x}_k = \left( \gamma_z r_k z_k^{-\frac{\tau}{r_k}} \right) x_k^{1+\frac{\tau}{r_k}} \qquad \forall k \in N \tag{6}$$

for some $z \in R_{\bar{x}}$, and thus $f(z) = \gamma_z \mathrm{diag}(r) z$ for some $\gamma_z \in \mathbb{R}$. Notice that invariant rays do not always exist for homogeneous systems. The linear harmonic oscillator, for example, is homogeneous of order zero with respect to the standard dilation map but does not possess an invariant ray.

### 3.3. Projection of a homogeneous system.

We next introduce the concept of a homogeneous norm.

DEFINITION 3.4. *A* homogeneous norm *associated to the dilation map* $\delta_\lambda^r(x)$ *is a function* $\rho : \mathbb{R}^n \to \mathbb{R}^+$ *satisfying the following:*

1. $\rho(x)$ *is continuous on* $\mathbb{R}^n$ *and of class* $C^1$ *on* $\mathbb{R}^n \setminus \{0\}$.
2. $\rho(x) = 0$ *only if* $x = 0$.
3. $\forall x \in \mathbb{R}^n$ *and* $\forall \lambda \in \mathbb{R}_0^+$, $\rho(\delta_\lambda^r(x)) = \lambda \rho(x)$.

For example, the function

$$\left( \sum_{i=1}^n |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}, \tag{7}$$

with $p > \max_{i=1,\ldots,n}(r_i)$, is a homogeneous norm.

Pick an arbitrary homogeneous norm $\rho(x)$. Then the *$\rho$-homogeneous unit $(n-1)$-sphere* is defined as $S_\rho := \{x \in \mathbb{R}^n | \rho(x) = 1\}$.

LEMMA 3.5. *Suppose that system (1) satisfies ($\mathcal{H}1$). Consider the system*

$$\dot{x} = g(x) := \begin{cases} \frac{1}{(\rho(x))^\tau} f(x) & \text{for } x \in \mathbb{R}^n_+ \setminus \{0\}, \\ 0 & \text{for } x = 0. \end{cases} \tag{8}$$

*Then $g(x)$ is a continuous vector field on $\mathbb{R}^n$, of class $C^1$ on $\mathbb{R}^n \setminus \{0\}$, and homogeneous of order zero with respect to $\delta^r_\lambda(x)$.*

*Proof.* Indeed, it is clear that $g(x)$ is continuous on $\mathbb{R}^n \setminus \{0\}$, so it only has to be shown that $g(x)$ is continuous at $x = 0$. Pick a sequence $\{x_k\} \to 0$ when $k \to +\infty$. Associated to this sequence is the sequence $\{x'_k\} \subset S_\rho$ with

$$x_k = \delta^r_{\lambda_k}(x'_k) \tag{9}$$

for suitable $\lambda_k \in \mathbb{R}^+_0$ and such that $\{\lambda_k\} \to 0$ when $k \to +\infty$. Then

$$\begin{aligned} \lim_{k \to +\infty} \frac{1}{(\rho(x_k))^\tau} f(x_k) &= \lim_{\lambda_k \to 0} \frac{1}{(\rho(\delta^r_{\lambda_k}(x'_k)))^\tau} f(\delta^r_{\lambda_k}(x'_k)) \\ &= \lim_{\lambda_k \to 0} \frac{1}{\lambda_k^\tau (\rho(x'_k))^\tau} \lambda_k^\tau \delta^r_{\lambda_k}(f(x'_k)) \quad \text{by homogeneity of } f(x) \text{ and of } \rho(x) \\ &= \lim_{\lambda_k \to 0} \delta^r_{\lambda_k}(f(x'_k)) \qquad \text{because } \rho(x'_k) = 1 \text{ as } \{x'_k\} \subset S_\rho \\ &= 0. \end{aligned}$$

Since $f(x)$ and $\rho(x)$ are of class $C^1$ on $\mathbb{R}^n \setminus \{0\}$, and since $\rho(x) > 0$ $\forall x \in \mathbb{R}^n \setminus \{0\}$, it follows that $g(x)$ is of class $C^1$ on $\mathbb{R}^n \setminus \{0\}$. It is also easily verified that $g(x)$ is homogeneous of order *zero* with respect to $\delta^r_\lambda(x)$. ☐

Uniqueness of the solutions for system (8) is guaranteed. A proof of this assertion proceeds along the same lines of the proof of the uniqueness of solutions of system (1). In addition, system (8) is topologically equivalent to system (1). Indeed, the direction of both the vectors $f(x)$ and $g(x)$ is the same $\forall x \in \mathbb{R}^n$. This implies that the solutions of system (1) are transformed to solutions of system (8) by a change in time scale. In particular, a ray is invariant for system (8) if and only if it is invariant for system (1), and system (8) is positive if and only if system (1) is positive.

With system (8) we now associate a system defined on $S_\rho$ as follows. Consider the projection map $\pi : \mathbb{R}^n \setminus \{0\} \to S_\rho$ with $\pi(x) := \delta^r_{1/\rho(x)}(x)$. Notice that

$$\pi = \pi \circ \delta^r_\lambda \tag{10}$$

$\forall \lambda \in \mathbb{R}^+_0$. This means that the image of a ray under $\pi$ is a unique point of $S_\rho$. Geometrically, $\pi(x)$ is the intersection of the ray through $x$ and $S_\rho$.

Now pick a point $m \in \mathbb{R}^n \setminus \{0\}$ and consider $R_m$. It will now be shown that *for all points $m'$ of the ray $R_m$*, the tangent mapping of $\pi$ maps the vector $g(m')$ to the *same* vector at $\pi(m)$. More precisely, it will be shown that

$$T_m\pi(g(m)) = T_{m'}\pi(g(m')) \tag{11}$$

$\forall m' \in R_m$, where $T_m\pi$ is the derivative of $\pi$ at $m$.

For $m' \in R_m$ there exists by the definition of $R_m$ a $\tilde{\lambda} \in \mathbb{R}^+_0$ such that

$$m' = \delta^r_{\tilde{\lambda}}(m). \tag{12}$$

Then

$$\begin{aligned} T_m\pi(g(m)) &= T_m(\pi \circ \delta^r_{\tilde{\lambda}})(g(m)) && \text{by (10)} \\ &= T_{\delta^r_{\tilde{\lambda}}(m)}\pi \circ T_m\delta^r_{\tilde{\lambda}}(g(m)) && \text{by the chain rule} \\ &= T_{m'}\pi \circ T_m\delta^r_{\tilde{\lambda}}(g(m)) && \text{by (12)} \\ &= T_{m'}\pi \circ \delta^r_{\tilde{\lambda}}(g(m)) && \text{by linearity of the dilation map} \\ &= T_{m'}\pi(g(m')) && \text{by homogeneity of } g \text{ of order zero and (12).} \end{aligned}$$

Since the preimage of a point $y \in S_\rho$, $\pi^{-1}(y)$ is equal to the ray through $y$, $R_y$, and by (11), it follows that $\forall y \in S_\rho$ a unique tangent vector $h(y) \in T_y S_\rho$ can be defined as follows:

$$h(y) = T_m\pi(g(m)) \qquad \forall m \in R_y. \tag{13}$$

It remains to be shown that $h$ defines a vector field of class $C^1$ on $S_\rho$. This is done by showing that for every $C^\infty$ function $\tilde{f} : S_\rho \to \mathbb{R}$ the function $h(\tilde{f}) : S_\rho \to \mathbb{R}$ is of class $C^1$ on $S_\rho$.

For all $m \in \mathbb{R}^n \setminus \{0\}$ it holds that

$$\begin{aligned} h(\pi(m))(\tilde{f}) &= T_m\pi(g(m))(\tilde{f}) && \text{by (13)} \\ &= g(m)(\tilde{f} \circ \pi) && \text{by definition of the tangent mapping.} \end{aligned}$$

This implies that

$$h(\tilde{f}) \circ \pi|_m = g(\tilde{f} \circ \pi)|_m \tag{14}$$

$\forall m \in \mathbb{R}^n \setminus \{0\}$, and thus $h(\tilde{f}) \circ \pi = g(\tilde{f} \circ \pi)$. Denoting the canonical injection by $j : S_\rho \to \mathbb{R}^n \setminus \{0\}$, we obtain that

$$h(\tilde{f}) \circ \pi \circ j = g(\tilde{f} \circ \pi) \circ j. \tag{15}$$

Since $\pi \circ j$ is the identity mapping on $S_\rho$, it follows that $h(\tilde{f}) = g(\tilde{f} \circ \pi) \circ j$, which is clearly of class $C^1$ on $S_\rho$.

Thus the following system can be considered:

$$\dot{y} = h(y), \tag{16}$$

where $y \in S_\rho$ and $h$ is the vector field of class $C^1$ defined above.

We conclude this section with the following claim.

PROPOSITION 3.6. *Suppose that $y_0 \in S_\rho$ is an equilibrium point for system* (16). *Then $R_{y_0}$ is an invariant ray for system* (8).

*Proof.* Since $h(y_0) = 0$, it follows by (13) that

$$T_m\pi(g(m)) = 0 \qquad \forall m \in R_{y_0}. \tag{17}$$

By the chain rule, we have that the tangent mapping of the mapping $j \circ \pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ equals

$$T_m(j \circ \pi) = T_{\pi(m)} j \circ T_m\pi. \tag{18}$$

We obtain from (17) and (18) that $\forall m \in R_{y_0}$

$$(T_m(j \circ \pi))(g(m)) = 0. \tag{19}$$

In local coordinates we have that $j \circ \pi(x) = (\frac{1}{\rho^{r_1}(x)} x_1, \dots, \frac{1}{\rho^{r_n}(x)} x_n)$, and this implies that

$$g(m) = \frac{1}{\rho(x)} \left( \sum_{i=1}^{n} \frac{\partial \rho}{\partial x_i}(m) g_i(m) \right) \operatorname{diag}(r) m \tag{20}$$

$\forall m \in R_{y_0}$, and therefore $R_{y_0}$ is an invariant ray for system (8). ∎

## 4. A class of positive homogeneous systems.

We call on the concept of a cooperative vector field, which has been widely studied [5], [13].

DEFINITION 4.1. *A vector field $f(x)$, $x \in \mathbb{R}^n$, is* cooperative *in $W \subset \mathbb{R}^n$ if the Jacobian matrix $\frac{\partial f}{\partial x}$ is Metzler $\forall x \in W$.*

System (1) is called *cooperative* if the following hypothesis holds.

($\mathcal{H}2$) $f(x)$ is cooperative in $\mathbb{R}^n_+ \setminus \{0\}$.

THEOREM 4.2. *If system* (1) *satisfies* ($\mathcal{H}1$) *and* ($\mathcal{H}2$)*, then $\mathbb{R}^n_+$ is a forward invariant set for system* (1).

*Proof.* Since $\tau \geq 0$ and $f(x)$ is cooperative in $\mathbb{R}^n_+ \setminus \{0\}$, it follows from Euler's formula that $f_i(x) \geq 0 \ \forall x \in \mathbb{R}^n_+ \setminus \{0\} : x_i = 0$. Also $f(0) = 0$ and the uniqueness of solutions for system (1) are guaranteed, as has been shown in the previous section. Then forward solutions cannot leave $\mathbb{R}^n_+$, proving Theorem 4.2. ∎

For future reference we apply Kamke's theorem to obtain the following.

PROPOSITION 4.3. *If system* (1) *satisfies* ($\mathcal{H}1$) *and* ($\mathcal{H}2$)*, then the flow of system* (1) *is monotone in $\mathbb{R}^n_+$.*

*Proof.* This follows from Kamke's theorem [13, Remark 1.4, p. 34] if the following conditions hold:

1. $f(x)$ is of *type K* on $\operatorname{int}(\mathbb{R}^n_+)$, i.e., $\forall x, y \in \operatorname{int}(\mathbb{R}^n_+)$ with $x \leq y$ it holds that $f_i(x) \leq f_i(y) \ \forall i : x_i = y_i$. It is easily checked that $f(x)$ is of type K on $\operatorname{int}(\mathbb{R}^n_+)$ since ($\mathcal{H}2$) holds (see [13, Remark 1.1, p. 33]).
2. $\mathbb{R}^n_+$ is a forward invariant set for system (1). This follows from Theorem 4.2.
3. $\forall x, y \in \mathbb{R}^n_+$ with $x < y$ there exist sequences $\{x_n\}, \{y_n\} \subset \operatorname{int}(\mathbb{R}^n_+)$ such that $x_n < y_n \ \forall n$ and $x_n \to x$, $y_n \to y$ when $n \to +\infty$. It is easily checked that this condition holds.
4. $f(x)$ is continuously differentiable on some neighborhood of $\mathbb{R}^n_+$. This condition is not necessarily satisfied here. Indeed, it may happen that $\frac{\partial f}{\partial x}$ is not defined at $x = 0$. A closer look at the proof of Kamke's theorem as stated in [13] indicates that only the continuity of solutions with respect to initial conditions is needed. Since we have shown that with ($\mathcal{H}1$) the solutions of system (1) are unique, it follows that solutions of system (1) are continuous with respect to initial conditions [4, Theorem 2.1, p. 94].

This concludes the proof. ∎

THEOREM 4.4. *If system* (1) *satisfies* ($\mathcal{H}1$) *and* ($\mathcal{H}2$)*, then there exists at least one invariant ray in $\mathbb{R}^n_+$ for system* (1).

*Proof.* Since, by Theorem 4.2, $\mathbb{R}^n_+$ is a forward invariant set for system (1), $\mathbb{R}^n_+$ is also a forward invariant set for system (8). It follows that the set $S_{\rho,+} := \mathbb{R}^n_+ \cap S_\rho$ is a forward invariant set for system (16). If not, there exists a forward solution $y(t, y_0)$, $y_0 \in S_{\rho,+}$ and $t \in \mathcal{I}_{y_0}$, for system (16) and a $T \in \mathcal{I}_{y_0}$ such that $y(T, y_0) \in S_\rho \setminus S_{\rho,+}$. But then the forward solution of system (8) starting in $y_0$ at $t = 0$, $z(t, y_0)$, is such that $z(T, y_0) \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, and thus we obtain a contradiction.

Since $S_{\rho,+}$ is compact, system (16), restricted to $S_{\rho,+}$, is forward complete; i.e., forward solutions are defined $\forall t \in \mathbb{R}^+$. For each $t \in \mathbb{R}^+$ consider the mapping

$\psi_t : S_{\rho,+} \to S_{\rho,+}$ of system (16), mapping $y_0 \to \psi_t(y_0) := y(t, y_0)$. Since $S_{\rho,+}$ is compact and since it is a retract of the $(n-1)$-dimensional unit disk $D^{n-1} := \{x \in \mathbb{R}^{n-1} | \, |x| \leq 1\}$, it follows from a generalization of Brouwer's fixed point theorem (see, e.g., [2, p. 171]) that each (continuous) mapping $\psi_t$, $t \in \mathbb{R}^+$, has a fixed point $x_t^*$.

Pick an arbitrary $T^* \in \mathbb{R}_0^+$ and consider the sequence of times $\{\frac{T^*}{n}\}$ with $n \geq 1$ integer and $n \to +\infty$. Then $\forall n$ the map $\psi_{\frac{T^*}{n}}$ has at least one fixed point $x_n^*$. Since the elements of the sequence $\{x_n^*\}$, $n \to +\infty$, belong to the compact set $S_{\rho,+}$, there exists a convergent subsequence $\{x_{n_k}^*\}$ with $n_k \to +\infty$ when $k \to +\infty$. The limit of this subsequence is denoted as $x^*$. We will prove that $\psi_t(x^*) = x^*$ $\forall t \in \mathbb{R}_0^+$ and thus that $x^*$ is an equilibrium point for system (16).

Pick an arbitrary $t \in \mathbb{R}_0^+$. We can find a sequence of nonnegative integers $\{l_k\}$ and a sequence of real numbers $\{d_k\}$ with $0 \leq d_k < \frac{T^*}{n_k}$ and $d_k \to 0$ when $k \to +\infty$ and such that $t = l_k \frac{T^*}{n_k} + d_k$. It follows that

$$\begin{aligned}
\psi_t(x^*) &= \lim_{k\to+\infty} \psi_t(x_{n_k}^*) && \\
&= \lim_{k\to+\infty} \psi_{d_k}(\psi_{l_k \frac{T^*}{n_k}}(x_{n_k}^*)) && \text{because } t = l_k \tfrac{T^*}{n_k} + d_k \\
&= \lim_{k\to+\infty} \psi_{d_k}(x_{n_k}^*) && \text{because } x_{n_k}^* \text{ is a fixed point of } \psi_{l_k \frac{T^*}{n_k}} \\
&= \lim_{k\to+\infty} y(d_k, x_{n_k}^*) && \text{by definition of } \psi_{d_k} \\
&= x^*. &&
\end{aligned}$$

In the above, the third equality is valid since if $x_n^*$ is a fixed point of $\psi_{\frac{T^*}{n}}$, then $x^*$ is also a fixed point of $\psi_{k\frac{T^*}{n}}$ $\forall$ nonnegative integers $k$; the fifth equality is valid since $y(t, x_0)$ is continuous on $\mathbb{R}^+ \times S_{\rho,+}$. Since $t \in \mathbb{R}_0^+$ was arbitrary, we have proved that $\psi_t(x^*) = x^*$ $\forall t \in \mathbb{R}_0^+$, and thus $x^*$ is an equilibrium point of system (16).

Then by Proposition 3.6, $R_{x^*}$ is an invariant ray for system (8) and thus also for system (1). ☐

We introduce the following hypothesis:

($\mathcal{H}3$) For $x \in \text{int}(\mathbb{R}_+^n)$, $\frac{\partial f}{\partial x}$ is irreducible. For $x \in \text{bd}(\mathbb{R}_+^n) \setminus \{0\}$, either $\frac{\partial f}{\partial x}(x)$ is irreducible or $f_i(x) > 0$ $\forall i : x_i = 0$.

System (1) is called *irreducible* if ($\mathcal{H}3$) holds.

PROPOSITION 4.5. *If system* (1) *satisfies* ($\mathcal{H}1$), ($\mathcal{H}2$), *and* ($\mathcal{H}3$), *then the flow of system* (1) *is strongly monotone in* $\mathbb{R}_+^n$.

*Proof.* The flow of system (1) is monotone by Proposition 4.3.

It will now be shown that $\forall x_0, y_0 \in \mathbb{R}_+^n$ with $x_0 < y_0$ it holds that

$$x(t, x_0) \ll x(t, y_0) \qquad \forall t \in (\mathcal{I}_{x_0} \cap \mathcal{I}_{y_0}) \setminus \{0\}. \tag{21}$$

*Case* 1. $y_0 \in \text{int}(\mathbb{R}_+^n)$.

1. If $x_0 = 0$, then $x_0 \ll y_0$, and thus (21) follows from Proposition 4.3.
2. If $x_0 \neq 0$, then (21) follows from the generalized Kamke theorem in [13, Remark 1.1, p. 58].

*Case* 2. $y_0 \in \text{bd}(\mathbb{R}_+^n) \setminus \{0\}$. Notice that in this case $x_0$ belongs to $\text{bd}(\mathbb{R}_+^n)$ since $x_0 < y_0$. We distinguish two cases:

1. $x_0 = 0$. We distinguish two subcases:
    (a) $f_i(y_0) > 0$ $\forall i : (y_0)_i = 0$. It is clear that for $t$ small enough and strictly positive,

    $$0 = x(t, x_0) \ll x(t, y_0). \tag{22}$$

    Then it follows by Proposition 4.3 that (22) holds $\forall t \in \mathcal{I}_{y_0} \setminus \{0\}$, which proves (21).

(b) $\frac{\partial f}{\partial x}(y_0)$ is irreducible. It follows from Proposition 4.3 that $x(t,y_0) > 0$ $\forall t \in \mathcal{I}_{y_0} \setminus \{0\}$. Suppose that (21) does not hold; then there exists some $i \in N$ such that $x_i(t,y_0) = 0$ $\forall t \in [0,t']$, where $t'$ is some strictly positive real number.

On the other hand, since $\frac{\partial f}{\partial x}(y_0)$ is irreducible, it follows from Euler's formula that there exists some $j \in N$ such that $f_j(y_0) > 0$, implying that for small and strictly positive $t$, $x_j(t,y_0) > 0$.

If $j = i$, then we have reached a contradiction. If $j \neq i$, then we can pick $0 < t_1 < t'$ and consider $x(t_1,y_0) \in \mathrm{bd}(\mathbb{R}^n_+) \setminus \{0\}$.

If $\frac{\partial f}{\partial x}(x(t_1,y_0))$ is reducible, then it follows from $(\mathcal{H}3)$ that $f_k(x(t_1,y_0)) > 0$ $\forall k : x_k(t_1,y_0) = 0$ (and thus, in particular, for $k = i$), yielding a contradiction.

If $\frac{\partial f}{\partial x}(x(t_1,y_0))$ is irreducible, then there exists a $j' \in N$ with $j' \neq j$ such that $x_{j'}(t,y_0) > 0$ for small $t > t_1$. This argument is repeated and ends in a finite number of steps, leading to a contradiction.

2. $x_0 \neq 0$. It follows from Case 2.1 that both solution $x(t,x_0)$ and $y(t,y_0)$ belong to $\mathrm{int}(\mathbb{R}^n_+)$ for $t \in \mathcal{I}_{x_0} \setminus \{0\}$, respectively, $t \in \mathcal{I}_{y_0} \setminus \{0\}$.

   On the other hand, it follows from Proposition 4.3 that $x(t,x_0) < y(t,y_0)$ $\forall t \in (\mathcal{I}_{x_0} \cap \mathcal{I}_{y_0}) \setminus \{0\}$. Suppose that (21) does not hold; then there exists some $l \in N$ such that $x_l(t,x_0) = x_l(t,y_0)$ $\forall t \in [0,t'']$, where $t''$ is some strictly positive real number. This contradicts that the flow on $\mathrm{int}(\mathbb{R}^n_+)$ is strongly monotone, which follows from the generalized Kamke theorem in [13]. ☐

It follows from Theorem 4.4 that if system (1) satisfies $(\mathcal{H}1)$ and $(\mathcal{H}2)$, there exists at least one invariant ray in $\mathbb{R}^n_+$ for system (1). Adding hypothesis $(\mathcal{H}3)$ allows us to draw more conclusions regarding the location of these invariant rays in $\mathbb{R}^n_+$ and their possible uniqueness, by means of the strong monotonicity property of the flow of system (1) as expressed in Proposition 4.5.

THEOREM 4.6. *If system* (1) *satisfies* $(\mathcal{H}1)$, $(\mathcal{H}2)$, *and* $(\mathcal{H}3)$, *then the invariant rays for system* (1) *in* $\mathbb{R}^n_+$ *belong to* $\mathrm{int}(\mathbb{R}^n_+)$.

*If the order* $\tau$ *of the homogeneous vector field* $f(x)$ *is equal to zero, then there exists a* unique *invariant ray for system* (1) *in* $\mathrm{int}(\mathbb{R}^n_+)$.

*If the order* $\tau$ *of the homogeneous vector field* $f(x)$ *is greater than zero and if there exists a stable invariant ray for system* (1) *in* $\mathrm{int}(\mathbb{R}^n_+)$, *then this invariant ray is unique in* $\mathrm{int}(\mathbb{R}^n_+)$.

*If the order* $\tau$ *of the homogeneous vector field* $f(x)$ *is greater than zero and if there exists an unstable invariant ray for system* (1) *in* $\mathrm{int}(\mathbb{R}^n_+)$, *then this invariant ray is not necessarily unique in* $\mathrm{int}(\mathbb{R}^n_+)$. *There may be multiple invariant rays for system* (1) *in* $\mathrm{int}(\mathbb{R}^n_+)$, *all of them unstable.*

*Proof.* Let $R_{x^*} \subset \mathbb{R}^n_+$ be an invariant ray for system (1); then

$$f(x^*) = \gamma_{x^*} \mathrm{diag}(r) x^* \tag{23}$$

for some $\gamma_{x^*} \in \mathbb{R}$.

First it is shown that $R_{x^*} \subset \mathrm{int}(\mathbb{R}^n_+)$. Suppose not; then $R_{x^*} \subset \mathrm{bd}(\mathbb{R}^n_+)$. According to $(\mathcal{H}3)$, two cases can be distinguished: $\frac{\partial f}{\partial x}(x^*)$ is irreducible or $f_i(x^*) > 0$ $\forall i : x^*_i = 0$.

*Case* 1. $\frac{\partial f}{\partial x}(x^*)$ is irreducible. From Euler's formula and (23),

$$(\mathrm{diag}(r + \tau^*))^{-1} \frac{\partial f}{\partial x}(x^*) \mathrm{diag}(r) x^* = \gamma_{x^*} \mathrm{diag}(r) x^*. \tag{24}$$

Since $r \in \operatorname{int}(\mathbb{R}^n_+)$, $\tau \in \mathbb{R}^n_+$, and $\frac{\partial f}{\partial x}(x^*)$ is Metzler and irreducible, $(\operatorname{diag}(r+\tau^*))^{-1}\frac{\partial f}{\partial x}(x^*)$ is also Metzler and irreducible. In addition, $\operatorname{diag}(r)x^* \in \operatorname{bd}(\mathbb{R}^n_+) \setminus \{0\}$. However, an irreducible Metzler matrix has no nonzero eigenvector belonging to $\operatorname{bd}(\mathbb{R}^n_+)$ [1]. Thus we obtain a contradiction.

*Case* 2. $f_i(x^*) > 0\ \forall i : x^*_i = 0$. Since $x^* \in \operatorname{bd}(\mathbb{R}^n_+)$ and with (23), there exists $i \in N$ such that $x^*_i = 0$ and $f_i(x^*) = 0$, yielding a contradiction.

Next it is shown that an invariant ray for system (1) is unique.

Suppose that there are two invariant rays $R_1, R_2 \subset \operatorname{int}(\mathbb{R}^n_+)$ $(R_1 \neq R_2)$ for system (1). Pick an arbitrary point $\bar{x} \in R_1$. There exist two points $p, q \in R_2$ such that $p < \bar{x} < q$ and $p_i = \bar{x}_i$, $q_j = \bar{x}_j$ for some $i \neq j$, $i, j \in N$. Indeed, pick an arbitrary $y \in R_2$. Since $R_2 \subset \operatorname{int}(\mathbb{R}^n_+)$, it follows that $y \gg 0$, and thus the following positive real numbers can be defined:

$$\lambda_1 = \min_{k \in N}\left(\left(\frac{\bar{x}_k}{y_k}\right)^{\frac{1}{r_k}}\right), \tag{25}$$

$$\lambda_2 = \max_{k \in N}\left(\left(\frac{\bar{x}_k}{y_k}\right)^{\frac{1}{r_k}}\right). \tag{26}$$

Since $R_1 \neq R_2$, it follows that $\lambda_1 \neq \lambda_2$. Then there exist $i, j \in N$ with $i \neq j$ such that $\lambda_1 = (\frac{\bar{x}_i}{y_i})^{\frac{1}{r_i}}$ and $\lambda_2 = (\frac{\bar{x}_j}{y_j})^{\frac{1}{r_j}}$. This implies that $p := \delta^r_{\lambda_1}(y)$ and $q := \delta^r_{\lambda_2}(y)$ satisfy the desired properties.

Solutions of system (1) starting on $R_1$, respectively on $R_2$, satisfy (6). Since $p$, $q \in R_2$, there exists a $\tilde{\lambda} \in \mathbb{R}^+_0$ such that $q = \delta^r_{\tilde{\lambda}}(p)$ and $\gamma_q = \tilde{\lambda}^\tau \gamma_p$ (see (5)). It follows from Proposition 4.5 that

$$x(t,p) \ll x(t,\bar{x}) \ll x(t,q), \tag{27}$$

where the first inequality holds $\forall t \in (\mathcal{I}_{\bar{x}} \cap \mathcal{I}_p) \setminus \{0\}$, and the second inequality $\forall t \in (\mathcal{I}_{\bar{x}} \cap \mathcal{I}_q) \setminus \{0\}$. We obtain from (6), (27) and since $p_i = \bar{x}_i$ and $\bar{x}_j = q_j$ that

$$\left(\gamma_p r_i p_i^{-\frac{\tau}{r_i}}\right) p_i^{1+\frac{\tau}{r_i}} < \left(\gamma_{\bar{x}} r_i \bar{x}_i^{-\frac{\tau}{r_i}}\right) \bar{x}_i^{1+\frac{\tau}{r_i}}, \tag{28}$$

$$\left(\gamma_{\bar{x}} r_j \bar{x}_j^{-\frac{\tau}{r_j}}\right) \bar{x}_j^{1+\frac{\tau}{r_j}} < \left(\gamma_q r_j q_j^{-\frac{\tau}{r_j}}\right) q_j^{1+\frac{\tau}{r_j}}, \tag{29}$$

or

$$\gamma_p < \gamma_{\bar{x}} < \gamma_q. \tag{30}$$

Two cases can be distinguished: $\tau = 0$ and $\tau > 0$.

*Case* 1. $\tau = 0$. If $\tau = 0$, then $\gamma_p = \tilde{\lambda}^0 \gamma_q = \gamma_q$, contradicting (30).

*Case* 2. $\tau > 0$. We introduce the sign function $\operatorname{sign} : \mathbb{R} \to \mathbb{R}$ as

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{for } x < 0, \\ \ \ 0 & \text{for } x = 0, \\ +1 & \text{for } x > 0. \end{cases} \tag{31}$$

From $\gamma_q = \tilde{\lambda}^\tau \gamma_p$ it follows that $\operatorname{sign}(\gamma_p) = \operatorname{sign}(\gamma_q)$.

*Case* 2(a). $\operatorname{sign}(\gamma_{\bar{x}}) \neq \operatorname{sign}(\gamma_p)$. Since $\operatorname{sign}(\gamma_p) = \operatorname{sign}(\gamma_q)$, it follows from (30) that $\operatorname{sign}(\gamma_{\bar{x}}) = \operatorname{sign}(\gamma_p) = \operatorname{sign}(\gamma_q)$, which is impossible because $\operatorname{sign}(\gamma_{\bar{x}}) \neq \operatorname{sign}(\gamma_p)$.

*Case* 2(b). $\operatorname{sign}(\gamma_{\bar{x}}) = \operatorname{sign}(\gamma_p)$. Two cases can be distinguished: $\operatorname{sign}(\gamma_{\bar{x}}) = 0$ and $\operatorname{sign}(\gamma_{\bar{x}}) = -1$.

1. $\mathrm{sign}(\gamma_{\bar{x}}) = \mathrm{sign}(\gamma_p) = 0$. This is impossible since, from (30), $\gamma_p < \gamma_{\bar{x}}$.
2. $\mathrm{sign}(\gamma_{\bar{x}}) = \mathrm{sign}(\gamma_p) = -1$. From (30),

$$\gamma_p < \gamma_q. \tag{32}$$

On the other hand, since $p < \bar{x} < q$ and $q = \delta^r_{\tilde{\lambda}}(p)$, it follows that $\tilde{\lambda} > 1$. Furthermore, $\gamma_q = \tilde{\lambda}^\tau \gamma_p$. But since $\mathrm{sign}(\gamma_p) = \mathrm{sign}(\gamma_q) = -1$, $\tau > 0$, and $\tilde{\lambda} > 1$, it follows that $\gamma_q \le \gamma_p$, contradicting (32). □

Notice that Theorem 4.6 does not exclude the possibility of several invariant rays in $\mathrm{int}(\mathbb{R}^n_+)$ for system (1). This can happen only if $\tau > 0$ and if all invariant rays are unstable. This situation can indeed occur; in particular, we will give an example of a planar cooperative irreducible and homogeneous system of order $\tau = 1$ possessing infinitely many unstable invariant rays in $\mathrm{int}(\mathbb{R}^n_+)$.

*Example.* Consider the following system:

$$\dot{x} = f_1(x) := (x_1 + x_2)x, \tag{33}$$

where $x := (x_1, x_2)^T \in \mathbb{R}^2$. This system is homogeneous of order $\tau = 1$ with respect to the standard dilation map. In addition, $f_1(x)$ is cooperative in $\mathbb{R}^2_+$, and $\frac{\partial f_1}{\partial x}$ is irreducible $\forall x \in \mathrm{int}(\mathbb{R}^2_+)$. Notice that ($\mathcal{H}3$) is not satisfied.

Next consider the following system:

$$\dot{x} = f_2(x) := \begin{pmatrix} x_1^2 + x_1x_2 + x_2^2 \\ x_1^2 + x_1x_2 + x_2^2 \end{pmatrix}. \tag{34}$$

This system is also homogeneous of order $\tau = 1$ with respect to the standard dilation map. In addition, $f_2(x)$ is cooperative in $\mathbb{R}^2_+$, and $\frac{\partial f_2}{\partial x}$ is irreducible $\forall x \in \mathbb{R}^2_+ \setminus \{0\}$. In particular, $\frac{\partial f_2}{\partial x}$ is irreducible when $x \in \mathrm{bd}(\mathbb{R}^2_+) \setminus \{0\}$.

Based on systems (33) and (34), we would like to construct a system with infinitely many unstable invariant rays in $\mathrm{int}(\mathbb{R}^2_+)$. Before doing so, partition $\mathbb{R}^2_+$ into five conic parts:

$$\mathbb{R}^2_+ := \bigcup_{i=1}^{5} C_i, \tag{35}$$

where $C_1 := \{x \in \mathbb{R}^2_+ | x_2 - \frac{1}{2}x_1 \ge 0, x_1 - \frac{1}{2}x_2 \ge 0\}$, $C_2 := \{x \in \mathbb{R}^2_+ | x_2 - \frac{1}{4}x_1 \le 0\}$, $C_3 := \{x \in \mathbb{R}^2_+ | x_2 - \frac{1}{2}x_1 < 0, x_2 - \frac{1}{4}x_1 > 0\}$, $C_4 := \{x \in \mathbb{R}^2_+ | x_1 - \frac{1}{4}x_2 \le 0\}$, and $C_5 := \{x \in \mathbb{R}^2_+ | x_1 - \frac{1}{4}x_2 > 0, x_1 - \frac{1}{2}x_2 < 0\}$.

Now define the system

$$\dot{x} = f(x) := \begin{cases} f_1(x) & \text{for } x \in C_1, \\ f_2(x) & \text{for } x \in C_2, C_4, \\ f_1(x) + g_1(\theta)p(x) & \text{for } x \in C_3, \\ f_1(x) + g_2(\theta)p(x) & \text{for } x \in C_5, \end{cases} \tag{36}$$

where $\theta := \frac{x_2}{x_1}$, $p(x) := (x_2^2, x_1^2)^T$, and $g_1(\theta)$ ($g_2(\theta)$) is a continuously differentiable function defined on $[\theta_2, \theta_1] := [\frac{1}{4}, \frac{1}{2}]$ ($[\theta_3, \theta_4] := [2, 4]$) to be specified hereafter. The aim is to construct $g_1(\theta)$ and $g_2(\theta)$ such that $f(x)$ is continuously differentiable on $\mathbb{R}^2_+$, homogeneous of order $\tau = 1$ with respect to the standard dilation map, cooperative on $\mathbb{R}^2_+$, and such that $\frac{\partial f}{\partial x}$ is irreducible when $x \in \mathbb{R}^2_+ \setminus \{0\}$. First $g_1(\theta)$ is constructed. Notice that $f_1(x) + g_1(\theta)p(x)$ is homogeneous of order $\tau = 1$ with respect to the standard dilation map since $p(x)$ is homogeneous of order $\tau = 1$ with respect to the standard dilation map and since $g_1$ is constant along any ray in $C_3$. For $f(x)$ to be continuous on $\cup_{i=1}^3 C_i$, it suffices that the following hold: $g_1(\theta)$ is continuous on $[\theta_2, \theta_1]$, $g_1(\theta_1) = 0$, and $g_1(\theta_2) = 1$. Consider the Jacobian of $f_1(x) + g_1(\theta)p(x)$ on $C_3$:

$$\frac{\partial f_1}{\partial x} + g_1(\theta) \begin{pmatrix} 0 & 2x_2 \\ 2x_1 & 0 \end{pmatrix} + \frac{\partial g_1}{\partial \theta} \begin{pmatrix} -\frac{x_2^3}{x_1^2} & \frac{x_2^2}{x_1} \\ -x_2 & x_1 \end{pmatrix}. \tag{37}$$

For $f(x)$ to be continuously differentiable on $\cup_{i=1}^3 C_i$, it suffices that the following hold: $g_1(\theta)$ is continuously differentiable on $[\theta_2, \theta_1]$ and $\frac{\partial g_1}{\partial \theta}(\theta_1) = \frac{\partial g_1}{\partial \theta}(\theta_2) = 0$. Finally, $f(x)$ is cooperative in $\cup_{i=1}^3 C_i$ and irreducible in $(\cup_{i=1}^3 C_i) \setminus \{0\}$ if and only if the off-diagonal elements of the Jacobian (37) are strictly positive when $x \in C_3$. This is the case when the following two conditions are satisfied:

I. $x_1(1 + \theta(2g_1(\theta) + \theta \frac{\partial g_1}{\partial \theta})) > 0$ when $\theta \in [\theta_2, \theta_1]$ and $x \in C_3$.

II. $x_2 + 2g_1(\theta)x_1 - \frac{\partial g_1}{\partial \theta} x_2 > 0$ when $\theta \in [\theta_2, \theta_1]$ and $x \in C_3$.

Condition II is satisfied if $g_1(\theta)$ is chosen such that $g_1(\theta) \geq 0$ and $\frac{\partial g_1}{\partial \theta} \leq 0$ when $\theta \in [\theta_2, \theta_1]$. Summarizing, we are looking for $g_1(\theta) : [\theta_2, \theta_1] \to \mathbb{R}$ such that the following conditions are satisfied:

($\mathcal{C}1$) $g_1(\theta)$ is continuously differentiable in $[\theta_2, \theta_1]$.

($\mathcal{C}2$) $g_1(\theta) \geq 0$ in $[\theta_2, \theta_1]$ and $g_1(\theta_1) = 0$, $g_1(\theta_2) = 1$.

($\mathcal{C}3$) $\frac{\partial g_1}{\partial \theta} \leq 0$ in $[\theta_2, \theta_1]$ and $\frac{\partial g_1}{\partial \theta}(\theta_1) = \frac{\partial g_1}{\partial \theta}(\theta_2) = 0$.

($\mathcal{C}4$) $x_1(1 + \theta(2g_1(\theta) + \theta \frac{\partial g_1}{\partial \theta})) > 0$ when $\theta \in [\theta_2, \theta_1]$ and $x \in C_3$.

From this we propose that $g_1(\theta)$ is a third order polynomial in $\theta$:

$$g_1(\theta) = a \left( \frac{\theta^3}{3} - \frac{\theta_1 + \theta_2}{2} \theta^2 + \theta_1 \theta_2 \theta + c \right), \tag{38}$$

where $a$ and $c$ are real numbers that we will determine hereafter. It is clear that ($\mathcal{C}1$) and ($\mathcal{C}3$) are satisfied if $a > 0$. From ($\mathcal{C}2$) we find that $c = -\frac{1}{96}$ and $a = 384$, fixing the function $g_1(\theta)$. We have to check whether ($\mathcal{C}4$) holds. This amounts to the following question:

$$h(\theta) := 640\theta^4 - 576\theta^3 + 144\theta^2 - 8\theta + 1 > 0 \qquad \forall \theta \in [\theta_2, \theta_1]? \tag{39}$$

It is easily verified by means of Cardano's formula for finding the roots of a third order polynomial that $\frac{\partial h}{\partial \theta} = 2560(\theta - \theta')(\theta - \theta'')(\theta - \theta^*)$, where $\theta', \theta'' < \theta_2$ and $\theta^* \in (\theta_2, \theta_1)$. In addition, $\frac{\partial^2 h}{\partial \theta^2}(\theta^*) > 0$, and thus $h(\theta)$ reaches a global minimum in $[\theta_2, \theta_1]$. Finally, $h(\theta^*) > 0$, and this implies that $h(\theta) > 0 \ \forall \theta \in [\theta_2, \theta_1]$.

We are left with finding an appropriate $g_2(\theta) : [\theta_3, \theta_4] \to \mathbb{R}$. Because of the symmetry of both $f_1(x)$ and $f_2(x)$ and because $\frac{1}{\theta_3} = \theta_1$ and $\frac{1}{\theta_4} = \theta_2$, we can set $g_2(\theta) := g_1(\frac{1}{\theta}) \ \forall \theta \in [\theta_3, \theta_4]$.

In conclusion, system (36) is homogeneous of order $\tau = 1$ with respect to the dilation map, $f(x)$ is cooperative in $\mathbb{R}^2_+$, and $\frac{\partial f}{\partial x}$ is irreducible for $x \in \mathbb{R}^2_+ \setminus \{0\}$. There are an infinite number of unstable invariant rays for system (36) in $\text{int}(\mathbb{R}^2_+)$. Indeed, *every* ray in $C_1$ is invariant and unstable for system (36). $\square$

**5. Main result.** Suppose that system (1) satisfies ($\mathcal{H}1$), ($\mathcal{H}2$), and ($\mathcal{H}3$), and assume that initial conditions for system (1) belong to $\mathbb{R}^n_+$. From Theorem 4.6 it follows that there exists at least one invariant ray in $\mathbb{R}^n_+$ for system (1) and that this invariant ray belongs to $\text{int}(\mathbb{R}^n_+)$.

If $\tau = 0$, then there exists a unique invariant ray in $\text{int}(\mathbb{R}^n_+)$.

If $\tau > 0$ and if there exists a stable invariant ray in $\text{int}(\mathbb{R}^n_+)$, then this invariant ray is unique.

Consider the flow $\phi_t : \mathbb{R}^n_+ \to \mathbb{R}^n$ of system (1) mapping $x_0$ to $\phi_t(x_0) := x(t, x_0)$. $\forall x_0 \in \mathbb{R}^n_+$, $\phi_t(x_0)$ exists when $t \in \mathcal{I}_{x_0}$. In the following lemma, we provide sufficient conditions guaranteeing that $\phi_t$ is defined $\forall t \in \mathbb{R}^+$ when restricting initial conditions to $\mathbb{R}^n_+$.

LEMMA 5.1. *If system* (1) *satisfies* ($\mathcal{H}$1) *and if* $\tau = 0$, *then* $\phi_t : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *is defined* $\forall t \in \mathbb{R}^+$.

*If system* (1) *satisfies* ($\mathcal{H}$1), ($\mathcal{H}$2), *and* ($\mathcal{H}$3); *if* $\tau > 0$; *and if there exists a stable (and thus unique) invariant ray for system* (1) *in* $\text{int}(\mathbb{R}^n_+)$, *then* $\phi_t : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *is defined for all* $t \in \mathbb{R}^+$.

*Proof. Case* 1. $\tau = 0$. First it will be shown that every forward solution of system (1) remains in a compact set in finite time intervals. Therefore we consider the dynamics of a homogeneous norm:

$$\begin{aligned} \dot{\rho} &= \left(\frac{1}{\rho}\frac{\partial \rho}{\partial x} f\right)\rho \\ &= k(x)\rho. \end{aligned} \tag{40}$$

It is easily verified that

$$k(\delta^r_\lambda(x)) = k(x) \tag{41}$$

$\forall x \in \mathbb{R}^n \setminus \{0\}$ and $\lambda \in \mathbb{R}^+_0$. The function $k(x)$ takes a maximal value $M$ on the compact set $\{z \in \mathbb{R}^n | \rho(z) = 1\}$. In fact, by (41), $M$ is the maximum of $k(x)$ in $\mathbb{R}^n \setminus \{0\}$. Then $\dot{\rho} \le M\rho$, and thus

$$\rho(x(t, x_0)) \le e^{Mt}\rho(x_0). \tag{42}$$

This implies that $x(t, x_0)$, $t \in \mathcal{I}_{x_0}$, belongs to the compact set $K := \{z \in \mathbb{R}^n | \rho(z) \le e^{MT_{\max}(x_0)\rho(x_0)}\}$.

Now suppose that $\mathcal{I}_{x_0} = [0, T_{\max}(x_0))$, with $T_{\max}(x_0) < +\infty$. Pick a sequence $\{x_{t_n}\} \subset x(t, x_0)$ with $t_n \to T_{\max}(x_0)$ when $n \to +\infty$. Since $|f(x)|$ is continuous, it attains a maximum $M'$ on $K$. Then $\forall t_n$ and $t_m$

$$\begin{aligned} |x_{t_n} - x_{t_m}| &\le \int_{t_m}^{t_n} |f(x(t, x_0))| dt \\ &\le M'|t_n - t_m|. \end{aligned}$$

This implies that $|x_{t_n} - x_{t_m}| \to 0$ when $n$ and $m \to +\infty$. Therefore $\{x_{t_n}\}$ is a Cauchy sequence and thus converges.

Every sequence on $x(t, x_0)$ converges to the same point $p \in K$. Indeed, if this were not the case, then there would exist two Cauchy sequences on $x(t, x_0)$, converging to different points $p_1$ and $p_2 \in K$. Then there would exist a sequence on $x(t, x_0)$ with *two* limit points $p_1$ and $p_2$. This is impossible since, as we have shown, every sequence on $x(t, x_0)$ is a Cauchy sequence and therefore converges.

Since every sequence on $x(t, x_0)$ converges to $p$, it follows that $\lim_{t \to T_{\max}(x_0)} x(t, x_0) = p$. The solution $x(t, x_0)$ can then be extended by concatenating it with the solution starting in $p$. This implies that the maximal forward interval of existence of the solution starting at $t = 0$ in $x_0$ contains $\mathcal{I}_{x_0}$ as a proper subset, contradicting the assumption that $\mathcal{I}_{x_0}$ is the maximal forward interval of existence. Thus $\forall x_0 \in \mathbb{R}^n_+$, $T_{\max}(x_0) = +\infty$, implying that $\phi_t$ is defined $\forall t \in \mathbb{R}^+$.

*Case* 2. $\tau > 0$. Let $R_{x^*}$ be the unique stable invariant ray for system (1) in $\operatorname{int}(\mathbb{R}^n_+)$. For each $x_0 \in \mathbb{R}^n_+$ we can find $y_0 \in R_{x^*}$ such that $x_0 < y_0$. Since $x(t, y_0)$ satisfies (6) with $z = x^*$ and $\gamma_{x^*} \leq 0$, we obtain that $\mathcal{I}_{y_0} = [0, +\infty)$. This implies that $x(t, x_0)$ belongs to the compact hypercube $C := \{z \in \mathbb{R}^n_+ | 0 \leq z \leq x(T_{\max}(x_0), y_0)\}$.

The rest of the proof follows the same lines as the proof of Case 1. The role of the compact set $K$ is now played by the compact set $C$. ☐

We are ready to state the main theorem.

THEOREM 5.2 (Main Theorem). *Assume that system* (1) *satisfies* ($\mathcal{H}1$), ($\mathcal{H}2$), *and* ($\mathcal{H}3$), *and assume that initial conditions for system* (1) *belong to* $\mathbb{R}^n_+$. *Then there exists at least one invariant ray* $R_{x^*} \subset \operatorname{int}(\mathbb{R}^n_+)$.

*If* $\tau = 0$, *then* $R_{x^*}$ *is unique in* $\operatorname{int}(\mathbb{R}^n_+)$. *If* $\tau > 0$ *and if* $R_{x^*}$ *is stable, then* $R_{x^*}$ *is unique. If* $\tau > 0$ *and if* $R_{x^*}$ *is unstable, then* $R_{x^*}$ *is not necessarily unique. If there are several invariant rays, all of them are unstable.*

*The zero solution of system* (1) *is*

- *unstable if and only if* $R_{x^*}$ *is unstable,*
- *stable if and only if* $R_{x^*}$ *is stable,*
- *globally asymptotically stable if and only if* $R_{x^*}$ *is asymptotically stable.*

*Proof. Sufficiency.*

1. $R_{x^*}$ is unstable. $\forall x_0 \in R_{x^*}$, the forward solution of system (1) starting at $t = 0$ in $x_0$ satisfies (6) with $z = x^*$. Since $R_{x^*}$ is unstable, $\gamma_{x^*} > 0$, and thus $x(t, x_0)$ diverges when $t \to T_{\max}(x_0)$. Then the zero solution of system (1) is unstable.
2. $R_{x^*}$ is stable. $\forall x_0 \in R_{x^*}$, the forward solution of system (1) starting at $t = 0$ in $x_0$ satisfies (6) with $z = x^*$. Since $R_{x^*}$ is stable, $\gamma_{x^*} \leq 0$, and thus $x(t, x_0) \leq x_0$ $\forall t \in \mathbb{R}^+$.

   It follows that $\forall x_0 \in R_{x^*}$ the hypercube $C_{x_0} := \{y_0 \in \mathbb{R}^n_+ | 0 \leq y_0 \leq x_0\}$ is a forward invariant set for system (1). Indeed, if $y_0 = x_0$, then $x(t, y_0) \in C_{x_0}$ $\forall t \in \mathbb{R}^+$, since $x(t, x_0) \leq x_0$ $\forall t \in \mathbb{R}^+$. If $y_0 < x_0$ with $y_0 \in C_{x_0}$, then from Proposition 4.5 and Lemma 5.1, $x(t, y_0) \ll x(t, x_0) \leq x_0$ $\forall t \in \mathbb{R}^+$, and thus $x(t, x_0) \in C_{x_0}$ $\forall t \in \mathbb{R}^+$.

   Since $x_0$ can be chosen arbitrarily close to the origin, the zero solution of system (1) is stable.
3. $R_{x^*}$ is asymptotically stable. $\forall x_0 \in R_{x^*}$, the forward solution of system (1) starting at $t = 0$ in $x_0$ satisfies (6) with $z = x^*$. Since $R_{x^*}$ is asymptotically stable, $\gamma_{x^*} < 0$, and thus $\lim_{t\to+\infty} x(t, x_0) = 0$. Also all forward solutions starting outside $R_{x^*}$ converge to the origin. Indeed, $\forall y_0 \notin R_{x^*}$ there exists $p \in R_{x^*}$ such that $y_0 < p$ since $R_{x^*} \subset \operatorname{int}(\mathbb{R}^n_+)$. Then it follows from Proposition 4.5 and Lemma 5.1 that $x(t, y_0) \ll x(t, p)$ $\forall t \in \mathbb{R}^+$. From $\lim_{t\to+\infty} x(t, p) = 0$ it follows that $\lim_{t\to+\infty} x(t, y_0) = 0$.

   It has been shown that all forward solutions converge to the origin. Stability of the zero solution follows from the proof of item 2 above. Thus the zero solution of system (1) is globally asymptotically stable.

*Necessity.*

1. System (1) is unstable. This follows from the contrapositive statement of the statement proved in item 2 of the sufficiency part of this theorem.
2. System (1) is stable. This follows from the contrapositive statement of the statement proved in item 1 of the sufficiency part of this theorem.

3. System (1) is asymptotically stable. Suppose that $R_{x^*}$ is not asymptotically stable. Then $\gamma_{x^*} \geq 0$, implying that no forward solution starting on $R_{x^*}$ converges to the origin and contradicting the assumption that system (1) is asymptotically stable. $\square$

*Example.* Consider a reversible chemical reaction at a given, constant temperature:

$$A \leftrightarrows B, \tag{43}$$

where $A$ and $B$ are chemical components. Denote the concentrations of $A$ and $B$, respectively, by $x_1$ and $x_2$. We assume that this reaction takes place in a *closed* chemical reactor and thus there is no exchange of material with the environment. The rate constant of reaction $A \to B$ is denoted by $k_1 \in \mathbb{R}^+$, and the rate constant of reaction $B \to A$ by $k_2 \in \mathbb{R}_+$. Supposing that the dynamics of both reactions is dictated by the *mass action principle* [11] and that both reactions are of second order, we obtain that the concentrations satisfy the following differential equations:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_1x_1^2 + k_2x_2^2 \\ k_1x_1^2 - k_2x_2^2 \end{pmatrix}. \tag{44}$$

System (44) is homogeneous of order $\tau = 1$ with respect to the standard dilation map, cooperative in $\mathbb{R}^2_+$, and $(\mathcal{H}3)$ holds. Therefore Theorem 5.2 can be applied. It is easily verified that $R_{x^*}$ with $x^* = (\sqrt{k_2}\ \sqrt{k_1})$ is the unique invariant ray in $\mathbb{R}^2_+$ that belongs to $\text{int}(\mathbb{R}^2_+)$ and that this ray is stable. We conclude that the zero solution of system (44) is also stable but not asymptotically stable.

One of the basic assumptions in model (44) is that the chemical reactor is closed, which is usually not satisfied. Indeed, in most models of chemical reactors there is exchange of chemicals with the environment, and a more realistic model would be (see [14])

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_1x_1^2 + k_2x_2^2 + (p_1(x_1, x_2) - q_1(x_1, x_2)) \\ k_1x_1^2 - k_2x_2^2 + (p_2(x_1, x_2) - q_2(x_1, x_2)) \end{pmatrix}, \tag{45}$$

where the functions $p_i \geq 0$, respectively $q_i \geq 0$, model the inflow, respectively outflow, of the chemicals. We show in [8] that the stability behavior of the trivial solution of the (simpler) system (44) plays an important role in determining the behavior of system (45) and that this can be extended to more general chemical reactors (in particular, to reactors containing more than two chemicals, in which several reactions take place).

**6. Asymptotic behavior.** Assume that system (1) satisfies $(\mathcal{H}1)$, $(\mathcal{H}2)$, and $(\mathcal{H}3)$, that the initial conditions of (1) belong to $\mathbb{R}^n_+$, and that $\tau = 0$. We recall from Theorem 4.6 that system (1) possesses a unique invariant ray $R_{x^*}$ in $\mathbb{R}^n_+$, which belongs to $\text{int}(\mathbb{R}^n_+)$, such that

$$f(x^*) = \gamma \text{diag}(r) x^* \tag{46}$$

for some $\gamma \in \mathbb{R}$. The sign of $\gamma$ then determines the stability properties of the zero solution of system (1). In this section the limiting behavior of solutions of systems satisfying these conditions will be described in more detail.

Introduce the following variable:

$$z(t, x_0) := (\text{diag}(e^{\gamma r t}))^{-1} \phi_t(x_0), \tag{47}$$

where $e^{\gamma r t} := (e^{\gamma r_1 t}, e^{\gamma r_2 t}, \ldots, e^{\gamma r_n t})$.

Since $f(x)$ is homogeneous of order $\tau = 0$ with respect to $\delta_\lambda^r(x)$, it is easily verified that $z(t, x_0)$ satisfies

$$\dot{z} = -\gamma \operatorname{diag}(r) z + f(z), \tag{48}$$

where $z \in \mathbb{R}_+^n$ and $z(0, x_0) = x_0$.

System (48) satisfies ($\mathcal{H}1$) with $\tau = 0$, ($\mathcal{H}2$), and ($\mathcal{H}3$), and therefore Theorem 4.6 can be applied to system (48). In particular, there exists a unique invariant ray for system (48) in $\operatorname{int}(\mathbb{R}_+^n)$. It is easy to see that this invariant ray is $R_{x^*}$, the unique invariant ray of system (1). For system (48) this ray consists of equilibrium points. It follows from Theorem 4.2 that system (48) is positive. Restricting initial conditions for system (48) to $\mathbb{R}_+^n$, it is possible to define the flow $\Phi_t : \mathbb{R}_+^n \to \mathbb{R}^n$ of system (48) (which is defined $\forall t \in \mathbb{R}^+$ by Lemma 5.1), mapping $\mathbb{R}_+^n$ *into* $\mathbb{R}_+^n$.

We recall the following results (see [13, Theorem 2.3, p. 5, and Theorem 3.7, p. 8]).

Lemma 6.1. *If the forward flow of a system is strongly monotone, then a limit set cannot contain two limit points $x$ and $y$ with $x < y$.*

Lemma 6.2. *If the forward flow of a system is strongly monotone, if $z_1 < z_2$, and if $\omega(z_1)$ and $\omega(z_2)$ are nonempty, then $\omega(z_1) \leq \omega(z_2)$.*

It follows from Proposition 4.5 that the forward flow of system (48) is strongly monotone since system (48) satisfies ($\mathcal{H}1$) with $\tau = 0$, ($\mathcal{H}2$), and ($\mathcal{H}3$). This implies that Lemmas 6.1 and 6.2 apply to system (48).

Theorem 6.3. *If system* (1) *satisfies* ($\mathcal{H}1$) *with* $\tau = 0$, ($\mathcal{H}2$), *and* ($\mathcal{H}3$), *then* $\forall x_0 \in \mathbb{R}_+^n$ *there exists a* $p_{x_0} \in R_{x^*} \cup \{0\}$ *such that* $\lim_{t\to+\infty} (\operatorname{diag}(e^{\gamma r t}))^{-1} \phi_t(x_0) = p_{x_0}$.

*Proof.* The forward solutions of system (48) are bounded. Indeed, for each $x_0 \in \mathbb{R}_+^n$ there exists a $y \in R_{x^*}$ such that $x_0 < y$. From Proposition 4.5 it follows that $z(t, x_0) \ll z(t, y) \equiv y \ \forall t \in \mathbb{R}^+$, since $y$ is an equilibrium point of system (48). Thus all forward solutions of system (48) are bounded, implying that the omega limit set of every forward solution is nonempty.

The proof of the theorem proceeds in two steps:

1. The omega limit set of every forward solution of system (48) is a subset of $R_{x^*} \cup \{0\}$.
2. The omega limit set of every forward solution of system (48) consists of a single equilibrium point.

First a proof of item 1 is given. Suppose that there exists a $x_0 \in \mathbb{R}_+^n$ such that $\omega(x_0) \not\subset R_{x^*} \cup \{0\}$. Then there exist at least two points $p$ and $q \in \omega(x_0)$ with $p \neq q$ and $p \notin R_{x^*} \cup \{0\}$. Indeed, the existence of $p \in \omega(x_0)$ with $p \notin R_{x^*} \cup \{0\}$ follows immediately from the assumption that $\omega(x_0) \not\subset R_{x^*} \cup \{0\}$. Now suppose that $p$ is the only element in $\omega(x_0)$. Then $p$ is an equilibrium point of system (48) and thus belongs to $R_{x^*} \cup \{0\}$, since omega limit sets are (forward) invariant sets. This contradicts the fact that $p \notin R_{x^*} \cup \{0\}$. Therefore there exists a second element $q \in \omega(x_0)$ with $p \neq q$.

Three cases can occur: $p \leq q$, $q \leq p$, or $p$ and $q$ are not related by $\leq$.

*Case* 1. $p \leq q$. Now $\delta_\lambda^r(x_0) < x_0 \ \forall \lambda \in (0, 1)$, and using Lemmas 3.2 and 6.2, this implies in particular that $\forall \lambda \in (0, 1)$

$$\delta_\lambda^r(q) \leq p. \tag{49}$$

On the other hand, it follows from $p \leq q$ and $p \neq q$ that $p < q$. This means that there exist two subsets $J$ and $K$ of $N$, where $J$ is a subset of $N$ and $K$ can be empty such that $N = J \cup K$ and

$$\begin{aligned} p_j &< q_j \qquad \forall j \in J, \\ p_k &= q_k \qquad \forall k \in K. \end{aligned}$$

This implies that there exists a $\lambda^* \in (0,1)$ close to 1 such that

$$\begin{aligned} p_j &< (\delta^r_{\lambda^*}(q))_j \qquad \forall j \in J, \\ (\delta^r_{\lambda^*}(q))_k &\leq p_k \qquad \forall k \in K. \end{aligned}$$

This implies that $p < \delta^r_{\lambda^*}(q)$ or that $p$ and $\delta^r_{\lambda^*}(q)$ are not related by $\leq$, contradicting (49).

*Case* 2. $q \leq p$. If $q < p$, then a contradiction is obtained using an argument similar to that of Case 1.

*Case* 3. $p$ and $q$ are not related by $\leq$. We have $\delta^r_\lambda(x_0) < x_0$ $\forall \lambda \in (0,1)$, and using Lemmas 3.2 and 6.2, this implies in particular that $\forall \lambda \in (0,1)$

$$\delta^r_\lambda(p) \leq q. \tag{50}$$

On the other hand, since $p$ and $q$, $p \neq q$, are not related by $\leq$, there exist $i, j \in N$ with $i \neq j$ such that

$$\begin{aligned} p_i &< q_i, \\ p_j &> q_j. \end{aligned}$$

This implies that there exists a $\tilde{\lambda} \in (0,1)$ close to 1 such that

$$\begin{aligned} (\delta^r_{\tilde{\lambda}}(p))_i &< q_i, \\ (\delta^r_{\tilde{\lambda}}(p))_j &> q_j. \end{aligned}$$

This implies that $\delta^r_{\tilde{\lambda}}(p)$ and $q$ are not related by $\leq$, contradicting (50). This concludes the proof of item 1.

Next a proof of item 2 is given. Suppose that there exists a $x_0 \in \mathbb{R}^n_+$ such that $\omega(x_0) \subset R_{x^*} \cup \{0\}$ contains two points $p$ and $q$ with $p \neq q$. Since both $p$ and $q$ belong to $R_{x^*} \cup \{0\}$ and since $R_{x^*} \subset \text{int}(\mathbb{R}^n_+)$, we may assume that $p \ll q$. However, it follows from Lemma 6.1 that $p$ and $q$ cannot be related by $<$. Thus a contradiction is obtained, and this proves item 2. $\square$

## 7. Discussion of the results.

In this paper a particular class of *positive* homogeneous systems has been introduced for which the stability behavior with respect to initial conditions in $\mathbb{R}^n_+$ can be characterized by means of a simple criterion expressed in Theorem 5.2. This contrasts with the case of homogeneous systems on $\mathbb{R}^n$, where in general no criteria for (asymptotic) stability are available.

In the following we will review the Perron–Frobenius theorem. Although this theorem normally refers to *discrete*-time systems, we consider its linear continuous-time version.

Consider the linear system

$$\dot{x} = Ax, \tag{51}$$

where $A$ is an irreducible Metzler matrix and $x \in \mathbb{R}^n$. This system is cooperative and irreducible in $\mathbb{R}^n_+$ (in fact, also in $\mathbb{R}^n$) and homogeneous of order $\tau = 0$ with respect to the standard dilation map. Since $(Ax)_i \geq 0$ when $x_i = 0$, $\mathbb{R}^n_+$ is a forward invariant set for system (51). Thus system (51) is a positive system.

For this class of systems the Perron–Frobenius theorem states that there exists a unique eigenvector $z$ in $\mathbb{R}^n_+$ (up to multiplication with positive scalars) and such that $z \in \operatorname{int}(\mathbb{R}^n_+)$. Also, the eigenvalue $\gamma$ associated with $z$ is real and simple and has the property that if $\gamma'$ is an eigenvalue of $A$ and $\gamma' \neq \gamma$, then $\operatorname{Re}(\gamma') < \gamma$, where $\operatorname{Re}(\gamma')$ stands for the real part of $\gamma'$. The sign of $\gamma$ then determines the stability behavior of the zero solution of system (51): It is unstable if $\gamma > 0$, stable if $\gamma \leq 0$, and GAS if $\gamma < 0$. Furthermore, $\forall x_0 \in \mathbb{R}^n_+$ there exists a $c_{x_0}$ such that $\lim_{t\to+\infty} e^{At}x_0/e^{\gamma t} = c_{x_0} z$.

Theorems 4.6 and 5.2 generalize the Perron–Frobenius theorem for *linear* cooperative and irreducible systems to the class of *homogeneous* cooperative and irreducible systems and therefore to a nonlinear context. We distinguish two cases.

*Case* 1. $\tau = 0$. If the order of the homogeneous vector field equals zero, then according to Theorem 4.6 there exists a unique invariant ray in $\mathbb{R}^n_+$, and it belongs to $\operatorname{int}(\mathbb{R}^n_+)$. This ray plays the role of the unique eigenvector associated to the dominating eigenvalue featured in the Perron–Frobenius theorem for linear cooperative and irreducible systems.

According to Theorem 5.2, the stability behavior of the zero solution with respect to the initial conditions in $\mathbb{R}^n_+$ is completely determined by the flow on the unique invariant ray. This is reminiscent of the Perron–Frobenius theorem for linear cooperative and irreducible systems, where the sign of the eigenvalue associated to the unique eigenvector determines the stability behavior of the system. The only difference is that the stability behavior in the linear case holds with respect to initial conditions in $\mathbb{R}^n$ and not just in $\mathbb{R}^n_+$.

It follows from Theorem 6.3 that the properties of the asymptotic behavior of solutions of homogeneous order zero, systems which are cooperative and irreducible, are similar to those of solutions of system (51).

Therefore the conclusions of the Perron–Frobenius theorem for linear cooperative and irreducible systems remain valid for the class of homogeneous *order zero* cooperative and irreducible systems, provided one restricts initial conditions to $\mathbb{R}^n_+$.

*Case* 2. $\tau > 0$. If the order of the homogeneous vector field is strictly positive, then according to Theorem 4.4 there is at least one invariant ray in $\mathbb{R}^n_+$. It follows from Theorem 4.6 that every invariant ray in $\mathbb{R}^n_+$ belongs to $\operatorname{int}(\mathbb{R}^n_+)$.

1. An invariant ray is unique in $\mathbb{R}^n_+$ if it is stable or asymptotically stable. The stability behavior of the zero solution with respect to the initial conditions in $\mathbb{R}^n_+$ is completely determined by the flow on this unique invariant ray. Therefore the conclusions of the Perron–Frobenius theorem remain valid for the class of homogeneous systems of *positive order*, which are cooperative and irreducible, if the invariant ray is stable or asymptotically stable, provided one restricts initial conditions to $\mathbb{R}^n_+$.
2. If an invariant ray is unstable, then it is not necessarily unique. In case there are several invariant rays in $\mathbb{R}^n_+$, all of them belong to $\operatorname{int}(\mathbb{R}^n_+)$, and they are all unstable. The zero solution is then unstable. This is in contrast with the Perron–Frobenius theorem for linear cooperative and irreducible systems, where the eigenvector associated with the dominating eigenvalue is always unique.

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