Foundations Exam Two NAME:

Do any SIX problems.

1. Define a finite automaton for the language $\{a, b\}^*$ such that $L(M) = \{x : x \text{ has no occurrence of ``aa''}\}$.

2. Define a Turing machine M over the language $\{0, 1\}^*$ such that for any $w = a_0a_1 \ldots a_{k-1}$, $M(w) = a_0a_0a_1a_1 \ldots a_{k-1}a_{k-1}$.

3. Suppose that f is a (total) recursive function and let $g(x, y) = \sum_{i<y} f(x, i)$. Show carefully that g is a recursive function.

4. Let ϕ_n be the n'th formula in the language of arithmetic with one free variable x, using some sort of Godel numbering or coding.

(a) Explain why there is no single formula ϕ such that, for all natural numbers n and x, $\mathbb{N} \models \Phi(n, x)$ if and only if $\mathbb{N} \models \phi_n(x)$.

(b) Explain why there *is* a formula Ψ such that, for all natural numbers n and x, $\mathbb{N} \models \Psi(n, x)$ if and only if $PA \vdash \phi_n(x)$.

5. Let $f_0, f_1, \ldots$ be a list of all partial computable functions with one free variable, and let $g_0, g_1, \ldots$ be a list of all total computable functions with one free variable.

(a) Explain why there is no computable function G such that, for all i and n, $G(i, n) = g_i(n)$.

(b) Explain why there *is* a partial computable function F such that, for all i and n, $F(i, n) = f_i(n)$ – or both are undefined. (You may want to use the Turing machine model here.)

6. Use the Axioms of Peano Arithmetic to show that $(\forall x)(\forall y)x + y = y + x$. Prove any supporting lemmas. Hint: Induction.

7. Use the Axioms of Peano Arithmetic to show that $(\forall x)(\forall y)[x + y = 0 \rightarrow (x = 0 \ \& \ y = 0)]$

DUE Tuesday, April 28 5 PM.