The Methods of Approximation and Lifting in Real Computation

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Computable Analysis

versus

Real Recursive Functions:

$\mathrm{TM}[\mathbb{R}] = \mathrm{FA}[\mathbb{R}]$

Bournez and Hainry: 2006, 2005

Real Recursive Functions

versus

Discrete Functions:

$\mathrm{DiscretePart}(\mathrm{FA}[\mathbb{R}]) = \mathrm{FA}[\mathbb{N}]$

Campagnolo, Moore, Costa 2002

2 Technologies:

Approximation: $\mathcal{A} \preceq^{\mathcal{E},[\omega]} \mathcal{B}$:

$\forall f \in \mathcal{A}\ \forall \alpha \in \mathcal{E}\ \exists f^* \in \mathcal{B}$ such that $f \preceq^{\alpha,[\omega]} f^*$

Lifting:

From TM[$\mathbb{N}$] = FA[$\mathbb{N}$] to
TM[$\mathbb{Q}$] = FA[$\mathbb{Q}$] to
TM[$\mathbb{R}$] = FA[$\mathbb{R}$]

$\mathcal{E}-$LIM:

Input: $\alpha(\overline{x};t) \in \mathcal{E}$ and $f(\overline{x};t)$

Returns: $F(\overline{x}) = \lim_{t\to\infty} f(\overline{x};t)$

(if limit exists and $F \preceq^{\alpha} f$)

Advantages of Lifting:

- Re-use classic results.
- Avoid Turing Machine Simulation.
- It is general.

Lemma:

$\mathcal{B}_1 \preceq^{\mathcal{E},[\omega]} \mathrm{FA}[\mathcal{B}_2;\mathcal{O}_2]$ and $\mathcal{O}_1 \preceq^{\mathcal{E},[\omega]} \mathrm{OP}[\mathcal{B}_2;\mathcal{O}_2]$

$\Rightarrow \mathrm{FA}[\mathcal{B}_1;\mathcal{O}_1] \preceq^{\mathcal{E},[\omega]} \mathrm{FA}[\mathcal{B}_2;\mathcal{O}_2]$

Approximation:

- Relate sets of functions by various notions of approximation.

- Derive equalities from approximations.

Advantages:

- Lends itself to natural inductive proofs on function algebras (avoid Turing Machine simulations).

- Facilitates general lemmas which can be re-used.

- Break apart tasks, using transitivity.

Future Work:

- Continue showing correspondences and developing the methods.

- Show (relative) separation results (Some work: Costa/Mycka).

  Ex: $\mathcal{F}_{\mathbb{R}} \not\approx \mathcal{G}_{\mathbb{R}} \Rightarrow \mathcal{F}_{\mathbb{N}} \neq \mathcal{G}_{\mathbb{N}}$