Classification of Computable Approximations by Divergence Boundings

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Computable and Computably Approximable Reals

A real number x is computably approximable if $x = \lim x_s$ for a computable sequence (x_s) of rational numbers. (**CA**)

x is computable if (x_s) converges to x effectively in one of the following senses. (**EC**)

- $(\forall n)(|x - x_n| \leq 2^{-n})$
- $(\forall n)(\forall s \geq n)(|x_s - x_n| \leq 2^{-n})$
- $(\forall n)(\forall s, t \geq n)(|x_s - x_t| \leq 2^{-n})$
- $(\forall n)(\forall s \geq e(n)(|x - x_s| \leq 2^{-n})$ for computable function e
- $(\forall n)(\forall s, t \geq e(n) \left(|x - x_s| \leq \frac{1}{d(n)}\right)$ where e, d are computable and d is unbounded.

  (e is modulus function and d is the distance function)

There are exceptions for non-computable real numbers. How to measure the non-computability?

The First Measurement

- A sequence (x_s) converges h-bounded effectively if there are at most $h(n)$ non-overlapping index-pairs (s, t) such that $|x_s - x_t| > 2^{-n}$ for all n.
- A real number x is h-bounded computable if there is a computable sequence (x_s) of rationals which converges h-bounded effectively to x. (h-**BC**)
- A real number x is x C-bounded computable if it is h-bc for some $h \in C$. (C-**BC**)

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Proposition 1.

1. *x is rational $\iff$ x is h-bc and $\liminf f(n) < \infty$.*
2. *h is unbounded, monotone and computable $\Longrightarrow$ **EC** $\subsetneq h$-**BC***
3. *$(\exists c)(\forall n)(|f(n) - g(n)| \leq c) \Longrightarrow f$-**BC** $= g$-**BC***

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1. (Hierarchy) For any computable functions f and g we have

$$(\forall c)(\exists n)(|f(n) - g(n)| > c) \Longrightarrow f\text{-}\mathbf{BC} \neq g\text{-}\mathbf{BC}.$$

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$$(\exists \gamma > 1)(\forall c \in \mathbf{N})(\forall^\infty n)(f(\gamma n) + n + c < g(n))$$

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Remark: The classification is coarse. No Ershov-style hierarchy.

The Second Measurement

A real number x is h-Cauchy computable if there is a computable sequence (x_s) converging to x such that there are at most $h(n)$ non-overlapping index-pairs (s, t) satisfy

$$s, t \geq n \;\&\; 2^{-n} \geq |x_s - x_t| < 2^{-n+1}.$$

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Remark: There is an Ershov-style hierarchy. The classification is too sensitive to arithmetical operations.

The Third Measurement (More General Form)

A real number x is (f, e, d)-effectively computable if there is a computable sequence of rationals converging to x such that there are at most $f(n)$ non-overlapping index-pairs (s, t) satisfy

$$s, t \geq e(n) \ \& \ \left(|x_s - x_t| > \frac{1}{d(n)} \right).$$

f	— founding function;
e	— modulus function;
d	— distance function
(f, e, d)-**EC**	— class of all (f, e, d)-effectively computable real numbers.

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The properties we are interested in:

- Closure properties under arithmetical operations and computable functions; and
- possible nice hierarchy properties.

Reduction of (f, e, d)-Effective Computability

The distance function d and modulus function e should be computable, monotone and unbounded (cmu).

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Theorem 3.

1. A cmu modulus function e can be reduced to the identity function $id(n) := n$, i.e.,

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2. A cmu distance function d can be reduced to the identity function id, too, i.e.,

$$(f, d)\text{-}\mathbf{EC} = (f \circ d^{-1}, id)\text{-}\mathbf{EC}$$

   where $d^{-1}(n) := \min\{t \in \mathbf{N} : d(t) \geq n\}$ (upper inverse function of d).

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3. A cmu distance function d can also be reduced to the exponential function $ep(n) := 2^n$, i.e.,

$$(f, d)\text{-}\mathbf{EC} = (f \circ d^{-2}, ep)\text{-}\mathbf{EC}$$

   where $d^{-2}(n) := \min\{t \in \mathbf{N} : d(t) \geq 2^n\}$.

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f-Effectively Computable Real Numbers

A sequence (x_s) converges f-effectively if, for all n, there are at most $f(n)$ non-overlapping index-pairs (s,t) such that

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A real x is f-effectively computable if there is a computable sequence (x_s) which converges f-effectively to x.

f-**EC** is the class of all f-ec real numbers and C-**EC** $:= \bigcup_{f \in C} C$-**EC**.

ω-**EC** $:= C$-**EC** for the class of computable functions.

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Proposition 2.

- 0-**EC** $=$ **EC**;
- ω-**EC** $= \omega$-**BC** $=$ **DBC**;
- f-**EC** $\subseteq f$-**BC**.

Finite Bounded Effective Computability

A real number x is bounded effectively computable if x if f-ec for a constant function f.

f-**EC** is denoted by k-**EC** if $f \equiv k$ and $*$-**EC** $:= \bigcup_{k \in \mathbf{N}} k$-**EC**.

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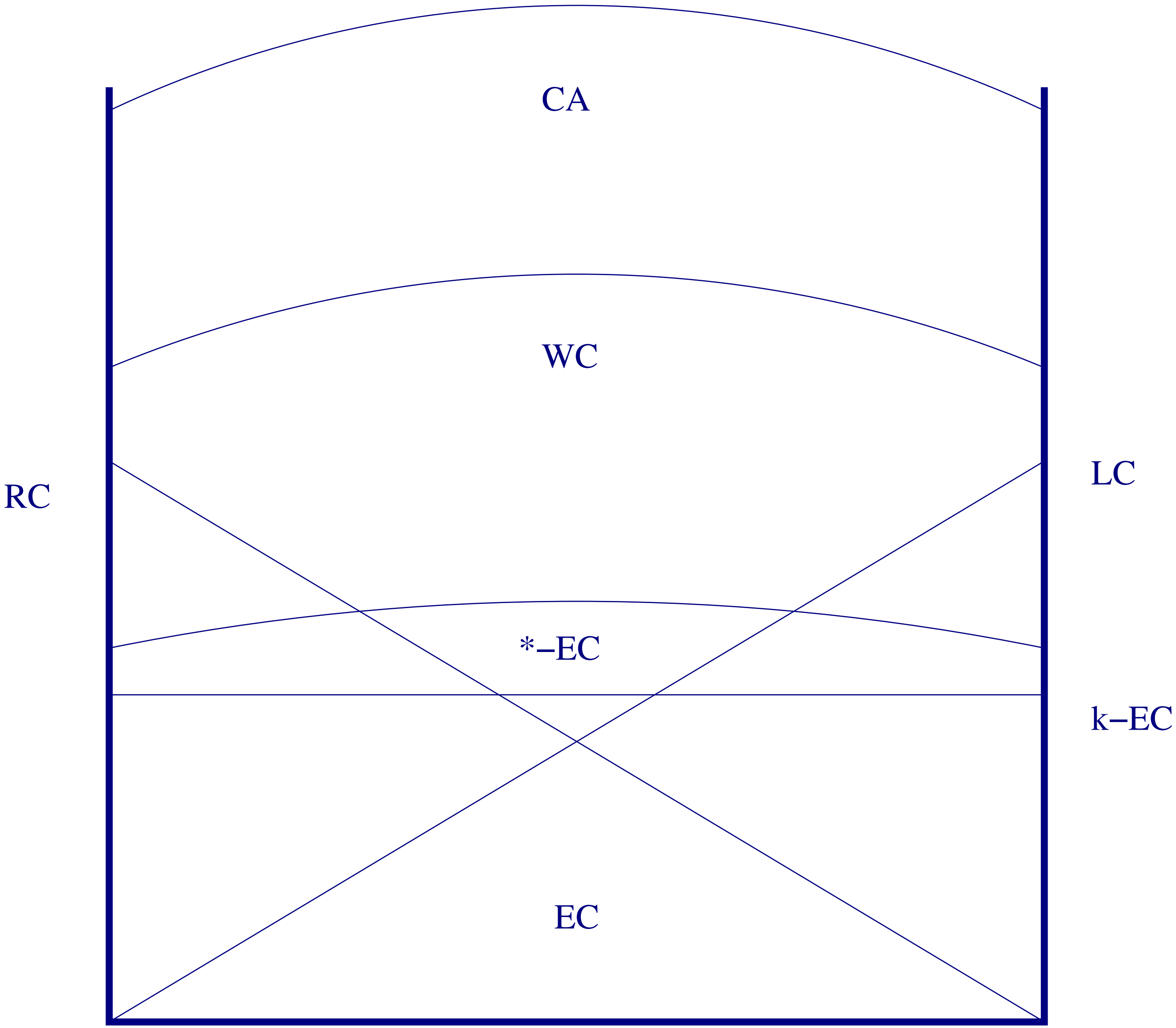
6. $*$-**EC** $\subsetneqq$ **WC**.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DBC $= \omega$-EC $:= \bigcup\{f$-EC : f is computable $\}$.

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1. (**Hierarchy of the classes**:) For any computable functions f, g we have

$$(\exists^{\infty} n)(f(n) < g(n)) \Longrightarrow g\text{-EC} \not\subseteq f\text{-EC};$$

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3. $\mathsf{SC} \subsetneqq \mathsf{WC} \subsetneqq o(2^n)\text{-}\mathsf{EC}$;

4. $\mathsf{SC} \not\subseteq o_e(2^n)\text{-}\mathsf{EC}$, where $o_e(2^n) := \{f \in o(2^n) : f \text{ is computable }\}$.

Bounding by Function Classes

C-**EC** $=$ $*$-**EC** is a field for the class C of constant functions.

Theorem 6.

If C is a function class which contains all constant functions and is closed under the addition and composition, then the class C-**EC** is a field.

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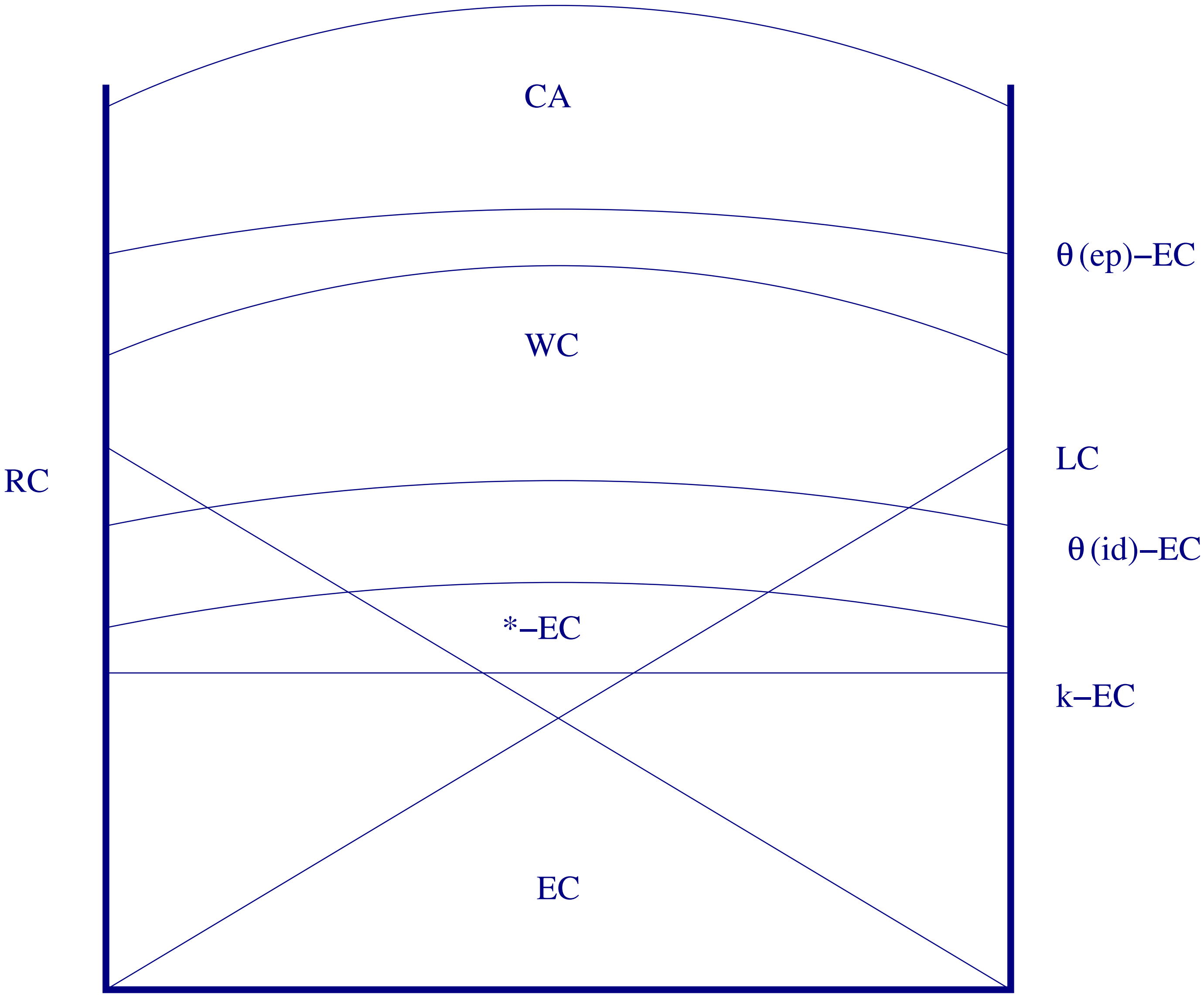
For any function f, let $\theta(f) := \{g : (\exists a, b, c)(\forall n)(g(n) \leq af(b + n) + c)\}$.

Corollary. *Let f, g be monotone functions.*

1. *The class $\theta(f)$-**EC** is a field;*

2. $f \in o(g) \Longrightarrow \theta(f)$-**EC** $\subsetneq \theta(g)$-**EC**.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Thank you