Computable Riesz Representation for the Dual of $C[0;1]$

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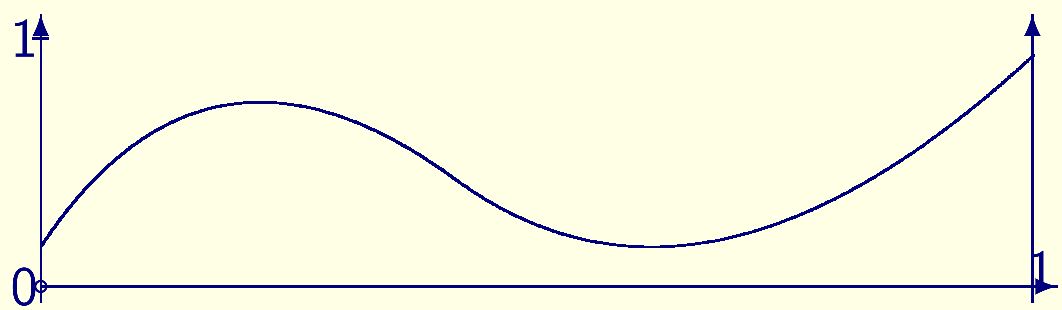
Riesz representation theorem:

For every continuous linear operator $F : C[0;1] \to \mathbb{R}$ there is a function $g : [0;1] \to \mathbb{R}$ of bounded variation such that

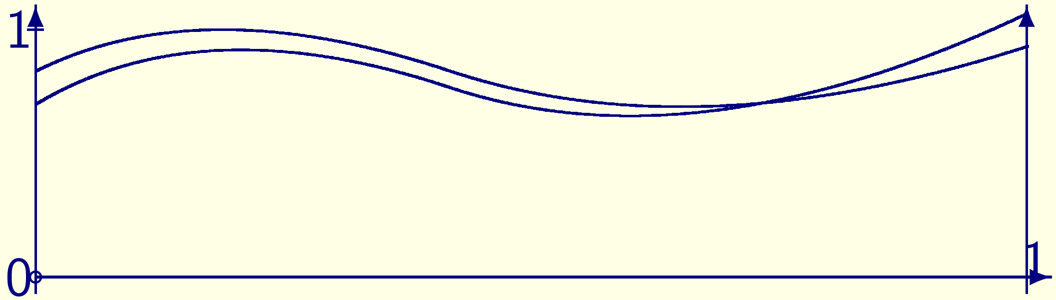
$$F(h) = \int h\,dg \quad (h \in C[0;1])$$

and

$$V(g) = \|F\|\,.$$



Operator $F : C[0;1] \to \mathbb{R}$ **:**



$C[0;1]$: the set of continuous functions $h : [0;1] \to \mathbb{R}$

norm of h: $\|h\| = \max_x |h(x)|$

modulus of continuity $m : \mathbb{N} \to \mathbb{N}$ **:**

$$|x - y| < 2^{-m(k)} \Longrightarrow |h(x) - h(y)| < 2^{-k}$$

F **is linear:** $F(g+h) = F(g) + F(h), \quad F(ah) = aF(h)$

F **is continuous:** $\|h_i - h\| \to 0 \implies \|F(h_i) - F(h)\| \to 0$

norm of F: $\|F\| = \sup_{\|h\| \leq 1} F(h)$

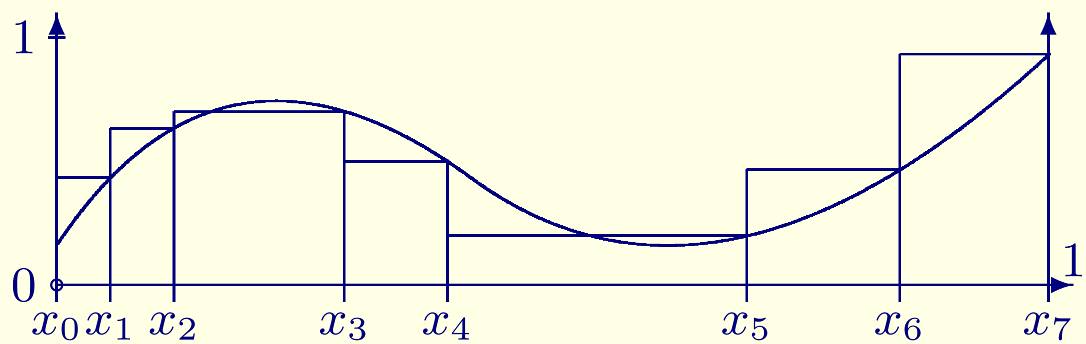
Stieltjes integral $\int h\,dg$ for $h \in C[0;1]$ and g of bounded varaition:

partition Z : $0 = x_0 < x_1 < \ldots < x_n = 1$

precision of Z : $|Z| = \max_{i=1}^{n}(x_i - x_{i-1})$

Riemann sum: $S(h, Z) = \sum_{i=1}^{n} h(x_i)(x_i - x_{i-1})$

Riemann integral: $\int h\,dx = \lim_{|Z|\to 0} S(h, Z)$



Riemann sum: $S(h, Z) = \sum_{i=1}^{n} h(x_i)(x_i - x_{i-1})$

For $g : [0; 1] \to \mathbb{R}$

Stieltjes sum: $S(g, h, Z) = \sum_{i=1}^{n} h(x_i)(g(x_i) - g(x_{i-1}))$

Stieltjes integral: $\int h\, dg = \lim_{|Z|\to 0} S(g, h, Z)$

$\int h\, dg$ exists, **if** $V(g) < \infty$ (g has bounded variation)

$$V(g) = \sup_Z \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

Theorem: (easy proof)
For every function $g : [0;1] \to \mathbb{R}$ of bounded variation $V(g)$ the operator

$$F : C[0;1] \to \mathbb{R}\,, \quad F(h) = \int h\,dg$$

is linear and continuous.

Riesz Representation Theorem:
For every linear and continuous operator $F : C[0;1] \to \mathbb{R}$, there is a function $g : [0;1] \to \mathbb{R}$ of bounded variation such that

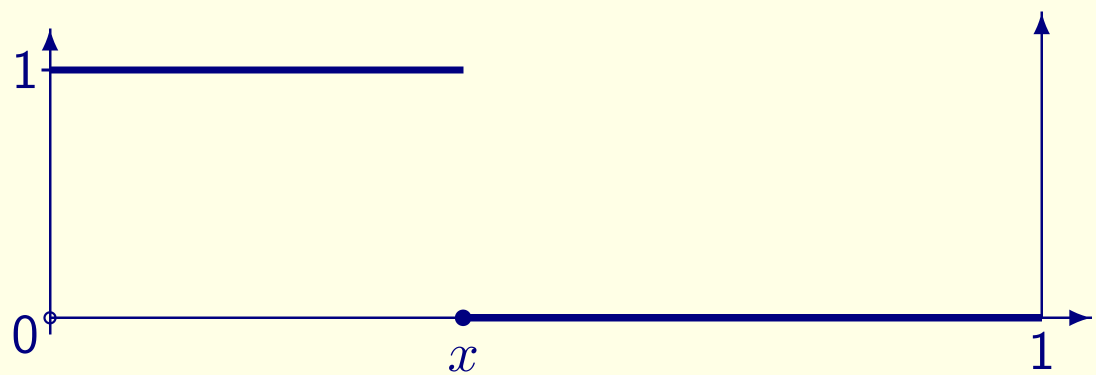
$$F(h) = \int h\,dg\,.$$

A Classical Proof of Riesz Representation Theorem:

By the Hahn-Banach Theorem $F : C[0;1] \to \mathbb{R}$ has a linear continuous extension $\overline{F} : B[0;1] \to \mathbb{R}$ to the set $B[0;1]$ of bounded real functions with the same norm. Define

$$g(x) := \overline{F}(\varphi_x)\,,$$

where φ_x is the following step function:



Computable version:

From F "compute" g and from g "compute" F such that

$$F(h) = \int h\, dg.$$

Representation approach (TTE) to Computable Analysis:

Machines operate on "concrete" infinite sequnces $p \in \Sigma^\omega$ of symbols, which are used as "names" of the "abstract" elements of the set M.

representation ("naming system") $\delta : \Sigma^\omega \to M$.

Needed: reasonable representations of

- the real numbers,
- the space $C'[0;1]$ of linear continuous operators $F : C[0;1] \to \mathbb{R}$,
- the set $\mathrm{BV}[0;1]$ of functions $g : [0;1] \to \mathbb{R}$ of bounded variation.

Representation $\boldsymbol{\rho}$ of $\mathbb{R}$: (Cauchy representation)
$\rho(p) = x$, iff p encodes a sequence of rational numbers converging to x rapidly.

Representation $\boldsymbol{\delta_C}$ of $\boldsymbol{C[0;1]}$: (Cauchy representation)
$\delta_C(p) = h$, iff p encodes a sequence of rational polygon functions converging to h rapidly.

Representation $[\boldsymbol{\delta_C \to \rho}]$ of $\boldsymbol{C'[0;1]}$: the (w.r.t. reducibility) largest (or "weakest") representation δ of $C'[0;1]$ such that evaluation $(F,h) \mapsto F(h)$ becomes (δ, δ_C, ρ)-computable.

Multi-Representation $\boldsymbol{\delta_{\mathrm{BV}} : \Sigma^\omega \rightrightarrows \mathbf{BV}[0;1]}$:

Observations:

- For defining the Stieltjes integral $\int h\,dg$ it suffices already to know the values $g(x)$ on a dense set $D \subseteq [0;1]$.
- A function $g \in \mathrm{BV}[0;1]$ cannot be defined by its values on D.

Therefore: **multi-representation**

Definition: $p \in \Sigma^\omega$ is a δ_{BV}-name of $g \in \mathrm{BV}[0;1]$, iff p encodes a sequence $(x_0, y_0), (x_1, y_1), \ldots \subseteq \mathrm{graph}(g)$ such that $\{x_0, x_1, \ldots\}$ is dense in $[0;1]$.

More precisely, $p = \langle p_0, q_0, p_1, q_1, \ldots \rangle$, $p_i, q_i \in \Sigma^\omega$ such that

$$\rho(p_0) = 0, \quad \rho(p_1) = 1,$$

$$\{\rho(p_i) \mid i \in \mathbb{N}\} \text{ is dense in } [0;1],$$

$$g\rho(p_i) = \rho(q_i) \text{ for } i \in \mathbb{N}.$$

Theorem: (computable Stieltjes integration) The operator

$$S : \mathrm{BV}[0;1] \times \mathbb{R} \to C'[0;1] \text{ is } (\delta_{\mathrm{BV}}, \rho, [\delta_C \to \rho])\text{-computable,}$$

$$(g, b) \mapsto F$$

where S maps every $g \in \mathrm{BV}[0;1]$ and every $b \in \mathbb{R}$ such that $V(g) < b$ to the linear continuous operator $F : C[0;1] \to \mathbb{R}$ such that

$$F(h) = \int h\,dg \qquad (h \in C[0,1])\,.$$

Lemma: If m is a modulus of uniform continuity of $h : [0;1] \to \mathbb{R}$ and Z is a partition of precision $2^{-m(k+1)}$, then

$$\left| \int h\,dg - S(g,h,Z) \right| \leq 2^{-k} V(g)$$

Stieltjes sum: $S(g,h,Z) = \sum_{i=1}^{n} h(x_i)(g(x_i) - g(x_{i-1}))$

Remarks: – A modulus of uniform continuity can be computed from h.
– It is not possible to compute an upper bound of $V(g)$ from g.

Theorem: (computable Riesz representation)

The operator

$$S' : C'[0;1] \times \mathbb{R} \rightrightarrows \mathrm{BV}[0;1] \text{ is } ([\delta_C \to \rho], \rho, \delta_{\mathrm{BV}})\text{-computable.}$$

$$(F,\ \|F\|) \ \mapsto\!\!\!\rightrightarrows\ g$$

where S' maps every linear continuous operator $F : C[0;1] \to \mathbb{R}$ **and** its norm to some function $g \in \mathrm{BV}[0;1]$ such that

$$F(h) = \int h\, dg \qquad (h \in C[0,1])$$

$$\text{and } \|F\| = V(g).$$

Remarks:

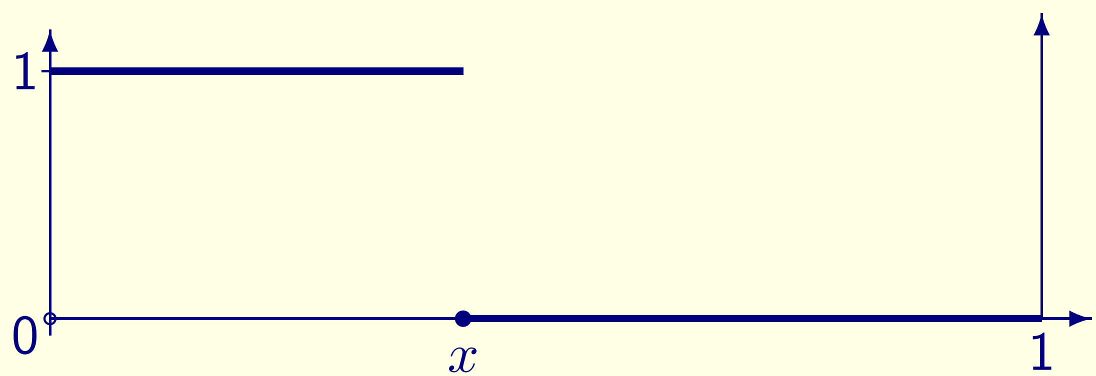
- An upper bound of $\|F\|$ can be computed from F, but
- it is not possible to compute $\|F\|$ from F.

Remember:

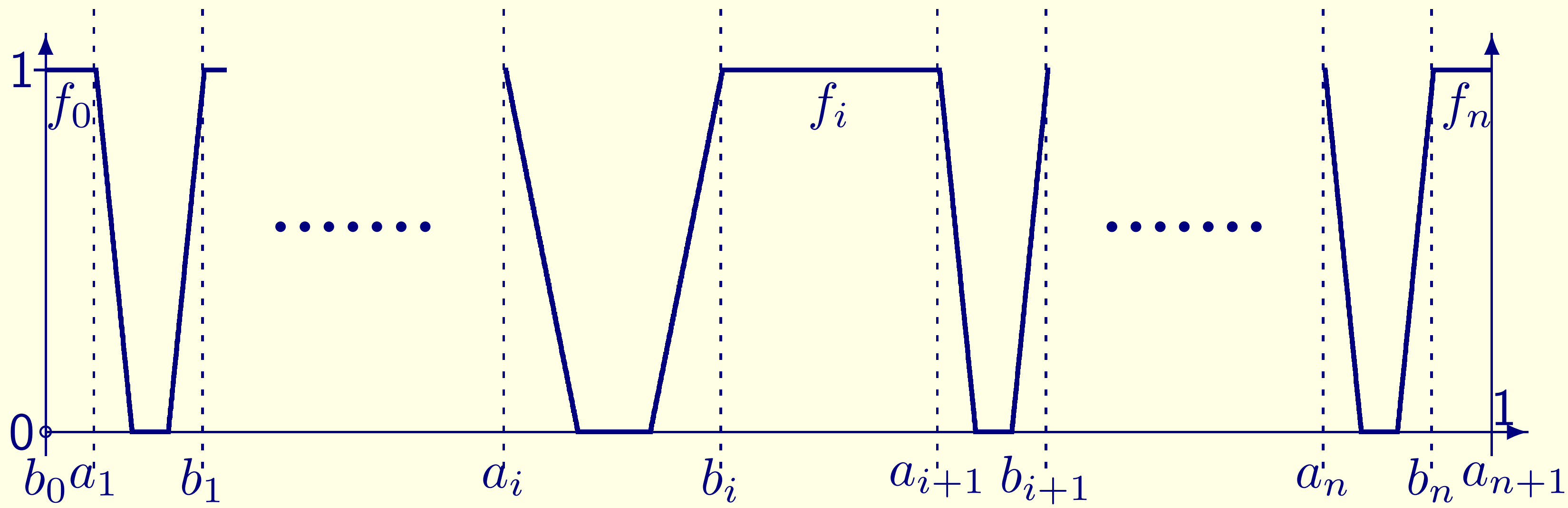
By the Hahn-Banach Theorem $F : C[0;1] \to \mathbb{R}$ has a linear continuous extension $\overline{F} : B[0;1] \to \mathbb{R}$ to the set $B[0;1]$ of bounded real functions with the same norm. Define

$$g(x) := \overline{F}(\varphi_x)\,,$$

where φ_x is the following step function:



Construction of g from F (idea) “approximate decomposition of $\mathbb{1}$”



Lemma: For each $\varepsilon > 0$ there are functions $f_0, f_1, \ldots, f_n$ such that

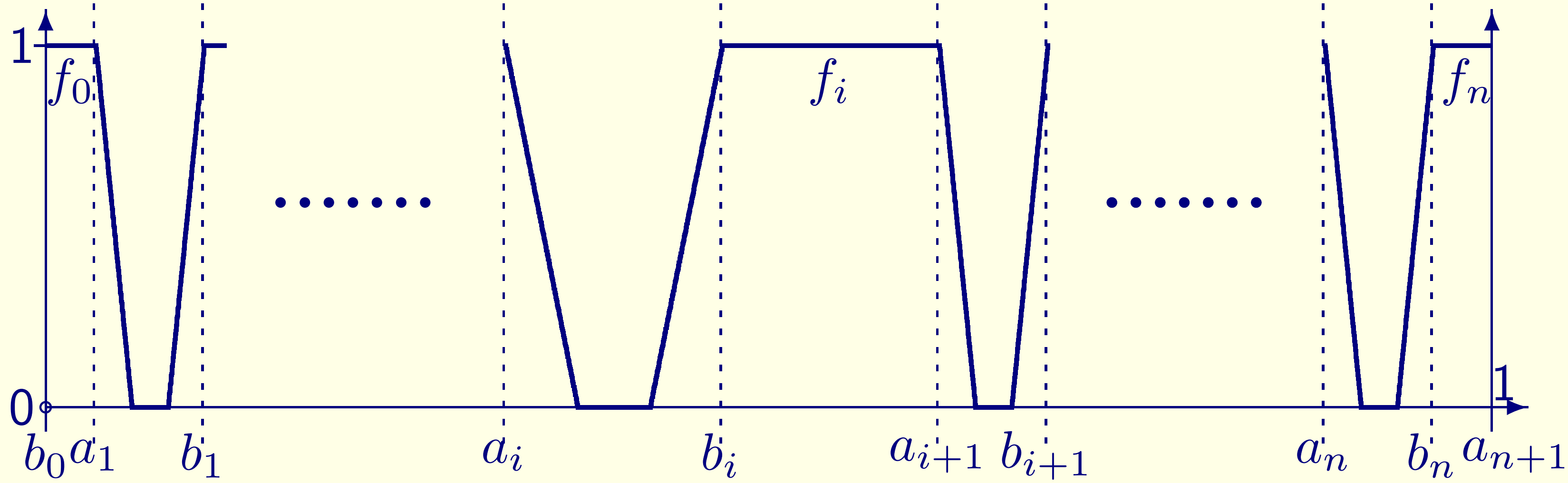
$$(\forall\, i)\ [b_i - a_i < \varepsilon \ \text{ and } \ a_i - b_{i-1} < \varepsilon], \quad (a_i, b_i \in \mathbb{Q})$$

$$\|F\| - \varepsilon < \sum_{i=0}^{n} |F(f_i)| \leq \|F\|.$$

(“the action of F in the gaps can be neglected”)

Add (x, y) to the countable subset of $\mathrm{graph}(g)$ where

$$x \approx (b_{i+1} - a_{i+1})/2 \ \text{ and } \ y \approx F(f_0 + f_1 + \ldots + f_i)$$



For ε functions $f_0, f_1, \ldots, f_n$ such that

$$(\forall\, i)\ [b_i - a_i < \varepsilon \text{ and } a_i - b_{i-1} < \varepsilon],\quad (a_i, b_i \in \mathbb{Q})$$

$$\|F\| - \varepsilon < \sum_{i=0}^{n} |F(f_i)| \leq \|F\|\,.$$

can be found by searching.

The norm $\|F\|$ must be known. $\|F\|$ cannot be computed from F.