Random Continuous Functions

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Goals

Extend the notion of Martin-Löf random reals to

1. Random closed sets
2. Random continuous functions
3. Random structures.

ML Random Reals

A real $x \in 2^{\mathbb{N}}$ is Martin-Löf random if
for any effective sequence $S_1, S_2, \ldots$ of c.e. open sets with $\mu(S_n) \leq 2^{-n}$,

$$x \notin \cap_n S_n.$$

Closed Sets and Trees

For a finite string σ, let $I(\sigma)$ denote $\{x \in 2^{\mathbb{N}} : \sigma \prec x\}$.
We shall call $I(\sigma)$, the **interval** determined by σ.

A nonempty closed set P may be identified with a tree $T_P \subseteq \{0,1\}^*$ where

$$T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}.$$

Note that T_P has no dead ends, i.e., if $\sigma \in T_P$,

then either $\sigma^\frown 0 \in T_P$ or $\sigma^\frown 1 \in T_P$.

Closed Sets and Trees II

For an arbitrary tree $T \subseteq \{0,1\}^*$, let $[T]$ denote the set of infinite paths through T, that is,

$$x \in [T] \iff (\forall n) x\lceil n \in T.$$

$P \subseteq 2^{\mathbb{N}}$ is a closed set if and only if $P = [T]$ for some tree T.

P is a Π^0_1 class, or an effectively closed set, if $P = [T]$ for some computable tree T.

The complement of a Π^0_1 class is called a c.e. open set.

There is a natural effective enumeration $P_0, P_1, \ldots$ of the Π^0_1 classes and thus an enumeration of the c.e. open sets.

Random Closed Sets

Due to Broadhead, Cenzer, and Dashti

Given a closed set $Q \subseteq 2^{\mathbb{N}}$, let $T = T_Q$ be the tree without dead ends such that $Q = [T]$.
Let $\sigma_0, \sigma_1, \ldots$ enumerate the elements of T in order, first by length and then lexicographically.

We then define the code $x = x_Q = x_T$ by recursion such that for each n,
(i) $x(n) = 2$ if both $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ are in T,
(ii) $x(n) = 1$ if $\sigma_n{}^\frown 0 \notin T$ and $\sigma_n{}^\frown 1 \in T$, and
(iii) $x(n) = 0$ if $\sigma_n{}^\frown 0 \in T$ and $\sigma_n{}^\frown 1 \notin T$.

We then define a measure $\mu*$ on $\mathcal{C}$ by setting

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}) \tag{1}$$

for any $\mathcal{X} \subseteq \mathcal{C}$ where μ is the standard measure on $\{0, 1, 2\}^{\mathbb{N}}$.

Random Closed Sets II

Thus each of the following scenarios has equal probability $\frac{1}{3}$:

$\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$,

$\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \notin T$, and

$\sigma^\frown 0 \notin T$ and $\sigma^\frown 1 \in T$.

Definition. A closed set $Q \subseteq 2^{\mathbb{N}}$ is **(Martin-Löf) random** if x_Q is (Martin-Löf) random.

Random Closed Sets III

Theorem 1 *(BCD 2006) There is a* Π^0_2 *random closed set.*

(There are Δ^0_2 random reals)

Theorem 2 *(BCD 2006) If* Q *is a random closed set, then* Q *has no computable members.*

Theorem 3 *(BCD 2006) If* Q *is a random closed set, then* Q *has no isolated members.*

Theorem 4 *(BCD 2006) If* Q *is a random closed set, then* $\mu^*(Q) = 0$ *.*

Random Closed Sets IV

Theorem 5 *(BCD 2006) If Q is a Π^0_1 class with measure 0, then no subset of Q is a random closed set.*

Corollary 6 *No Π^0_1 class is a random closed set.*

Random Continuous Functions

A continuous function on $2^{\mathbb{N}}$ is a function with a closed graph.

Thus we might say that a function F is random if the graph $Gr(F) = \{x \oplus y : y = F(x)\}$ is a random closed set.

Problem: If $[T]$ is the graph of a function and $\sigma \in T$ has even length, then we must have $\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$.

Thus the family of closed sets which are the graphs of functions has measure 0 in the space of closed sets and hence a random closed set will not be the graph of a function.

Representing Functions

A continuous function $F : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ may be represented by a function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ such that the following hold for all $\sigma \in \{0,1\}^*$.

(1) $|f(\sigma)| \leq |\sigma|$.

(2) $\sigma_1 \prec \sigma_2$ implies $f(\sigma_1) \preccurlyeq f(\sigma_2)$.

(3) For every n, there exists m such that for all $\sigma \in \{0,1\}^m$, $|f(\sigma)| \geq n$.

(4) For all $x \in 2^{\mathbb{N}}$, $F(x) = \bigcup_n f(x\lceil n)$.

We define the space $\mathcal{F}$ of representing functions $f : \{0,1\}^* \rightarrow \{0,1\}^*$ to be those which satisfy clauses (1) and (2) above.

Properties of Representing Functions

1) If f is a representing function of F, $f(\emptyset) = \emptyset$ by (1).

2) If $f((0)) = (i)$ where $i \in \{0, 1\}$, then for all $x \in I((0))$, $F(x)(0) = i$.

3) If $f((0)) = \emptyset$, then we shall take this to mean that there exist x_0 and x_1 in $I((0))$ such that $F(x_i)(0) = i$ for $i = 0, 1$.

4) $F(\sigma) \preccurlyeq \tau$ where τ is the longest string τ with $|\tau| \leq |\sigma|$ such that $\tau \prec F(x)$ whenever $\sigma \prec x$.

Coding Representing Functions

There is a one-to-one correspondence between $\mathcal{F}$ and $\{0,1,2\}^{\mathbb{N}}$ defined as follows.

Enumerate $\{0,1\}^*$ in order, first by length and then lexicographically, as $\sigma_0, \sigma_1, \ldots$. Thus $\sigma_0 = \emptyset$, $\sigma_1 = (0), \sigma_2 = (1), \sigma_3 = (00), \ldots$.

Then $r \in \{0,1,2\}^{\mathbb{N}}$ corresponds to the function $f_r : \{0,1\}^* \to \{0,1\}^*$ defined by declaring that $f_r(\emptyset) = \emptyset$ and that, for any σ_n with $|\sigma_n| \geq 1$,

$$f_r(\sigma_n) = \begin{cases} f_r(\sigma_k), & \text{if } r(n) = 2; \\ f_r(\sigma_k)^\frown i, & \text{if } r(n) = i < 2. \end{cases}$$

where k is such that $\sigma_n = \sigma_k{}^\frown j$ for some j.

Then every continuous function F has a representative f as described above, and, in fact, it has infinitely many representatives.

The measure μ^{**}

The measure μ^{**} on $\mathcal{F}$ is then induced by the standard probability measure on $\{0,1,2\}^{\mathbb{N}}$.

Definition. A **effectively random continuous function** on $2^{\mathbb{N}}$ is one which has a representation in $\mathcal{F}$ which is Martin-Löf random.

Remark: Fouche (2000) has a different approach to randomness for continuous functions connected with Brownian motion, first presented by Asarin and Prokovsky (1986).

Properties of Random Functions

Our first result will be to show that every random function represents a continuous function. To prove such a result, we need to prove the following lemma.

Lemma 7 *Let Σ be a finite set and let $Q \subseteq \Sigma^{\mathbb{N}}$ be a Π_1^0 class of measure 0. Then no element of Q is Martin-Löf random.*

Theorem 8 *The set of functions in $\mathcal{F}$ which represent a total continuous function has measure one. Hence every random function represents a continuous function.*

Theorem 9 *There exists a random continuous function which is Δ_2^0 computable.*

Properties of Random Functions II

Proposition 10 *If F is a random continuous function, then, for every $\sigma \in \{0,1\}^*$, the function F_σ is random continuous, where*

$$F_\sigma(x) = F(\sigma^\frown x).$$

Theorem 11 *If F is a random continuous function, then, for any computable real x, $F(x)$ is a random real.*

Remark: Fouche has shown that for his definiton of random continous function, any random continuous function F, $F(x)$ is not computable for any computable input x.

Theorem 12 *If F is a random continuous function, then the set of zeros of F is a random closed set.*

Properties of Random Functions III

Theorem 13 *If F is a random continuous function, then the image $F[2^{\mathbb{N}}]$ has no isolated elements.*

Theorem 14 *For any $\sigma \in \{0,1\}^*$, the probability that the image of a continuous function F meets $I(\sigma)$ is always $> \frac{3}{4}$.*

Corollary 15 *The image of a random continuous function need not be a random closed set.*

Random Structures

Idea Use random subsets of a universal structure.

(Martin-Löf) Random Graphs. Start with $K_{\mathbb{N}}$ which is the complete graph on $\mathbb{N}$.

Let $e_0, e_1, \ldots$ be an effective list of all edges of $K_{\mathbb{N}}$.

Then given a real $r = r_0 r_1 \ldots \in \{0,1\}^{\mathbb{N}}$, we let K_r be the graph $(\mathbb{N}, E_r)$ where

$$e_i \in E_r \iff r_i = 1.$$

Definition We say that K_r is (Martin-Löf) random if and only if r is (Martin-Löf) random.

(Martin-Löf) Random Graphs

We say that K_r has property $P_{n,s}$ if for any subset $V = \{v_1, \ldots, v_n\}$ of vertices of K_r of size n and for any subset $S \subseteq V$ of size s,
there is a vertex $w \notin V$ such that for all $i = 1, \ldots, n$,

$$\{v_i, w\} \in E_r \iff v_i \in S.$$

Thus w witnesses that there is a vertex which is a neighbor of each vertex in S and is not a neighbor of each vertex in $V - S$.

Random Graphs

It is easy to see by a standard back and forth argument that if $K_1 = K_{r_1}$ and $K_2 = K_{r_2}$ have property $P_{n,s}$ for all $n \geq 1$ and $n \geq s \geq 0$, then K_1 and K_2 are isomorphic, i.e there is a bijection $F : \mathbb{N} \rightarrow \mathbb{N}$ such that
$\{i, j\} \in E_{r_1} \iff \{F(i), F(j)\} \in E_{r_2}$

Theorem 16 *If K_r is random graph, then K_r has property $P_{n,s}$ for all $n \geq 1$ and $n \geq s \geq 0$*

Corollary 17 *If K_{r_1} and K_{r_2} are two random graphs, then K_{r_1} and K_{r_2} are isomorphic.*

Classical Theory of Random Graphs

Consider the collection of all finite graphs on vertex set $[n] = \{1, \ldots, n\}$.

We can put a probability measure on this set by declaring the probability of any edge occuring in such a graph is p, $0 < p < 1$.

More formally for any graph $G = ([n], E)$ we declare that the probability of G is $p^{|E|}(1-p)^{\binom{n}{2}-|E|}$.

Glebskii, Kogan, Liagonkii and Talanov (1969) and, independently, Fagin (1976), proved the following theorem.

Theorem 18 *For any fixed $0 < p < 1$ and any first order sentence A in theory of graphs,*

$$lim_{n\to\infty} Pr(\{G = ([n], E) : G \models A\}) = 0 \text{ or } 1.$$

Classical Theory of Random Graphs II

Consider the first order theory of

$\{A : lim_{n\to\infty} Pr(\{G = ([n], E) : G \models A\}) = 1\}$.

It turn out that property Pm, s can be viewed as first order sentence so that $P_{m,s}$ is in this theory.

(Spencer's book, The Probabilistic Method)

Definition A graph G is said to have the **full level s extension property** if for any sets of vertices $v_1, \ldots, v_a$ and $u_1, \ldots, u_b$ where $a + b \leq s$, there is a vertex w such that for all $i \leq a$, $\{w, v_i\}$ is an edge in G and, for all $j \leq b$, $\{w, u_j\}$ is not an edge in G.

For any fixed $0 < p < 1$, the probability that the set of graphs $G = ([n], E)$ such that G has the full level extension property approaches 1 as n approaches ∞.

(Martin-Löf) Random Linear Orderings

Consider a computable dense linear ordering without end points over $L = (\mathbb{N}, \prec)$ with universe the natural numbers.

For any real $r = r_0 r_1 \cdots \in \{0,1\}^{\mathbb{N}}$, we let $L_r = (U_r, \prec)$ where $U_r = \{i : r_i = 1\}$.

Then we say that L_r is **(Martin-Löf) random** if and only if r is (Martin-Löf) random.

Theorem 19 *It L_r is random linear ordering, then L_r is dense linear ordering with out end points.*