Undecidability of the structure of the Solovay degrees of c.e. reals: extended abstract Full version in preparation.

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Abstract

We describe a method for showing that the elementary theory of the structure of the Solovay degrees of computably enumerable reals is undecidable, omitting the technical details involving the actual constructions.

1 Introduction

In this paper we work in Cantor space 2^ω, with basic clopen sets $[\sigma] = \{\sigma\alpha : \alpha \in 2^\omega\}$ having Lebesgue measure $\mu([\sigma]) = 2^{-|\sigma|}$. While this space is not homeomorphic to the real interval $(0,1)$, it is measure-theoretically isomorphic to it. We will identify a set A with its characteristic function χ_A, and hence with the real $0.\chi_A$. We write '$[s]$' after expressions to indicate that everything in the expression is taken with its value at stage s of the given construction.

We assume the basics of the theory of effective randomness, in particular the definitions of prefix-free Turing machine and prefix-free Kolmogorov complexity, which we denote by K. For definitions of these and related concepts, see for instance [6, 7, 11, 14].

Our basic objects of study will be the *computably enumerable* reals (which have also been called *left computable* and *left-c.e.*). The computably enumerable reals are those reals α such that the left cut $L(\alpha) = \{q \in \mathbb{Q} : q \leqslant \alpha\}$ forms a c.e. set of rationals. Equivalently, c.e. reals are those that are the limits of computable increasing sequences of rationals. These reals should not be confused with the *computable reals*, which are those that are the limits of a computable sequence of rationals for which the modulus of convergence is also a computable function. Nor should they be confused with the *strongly c.e. reals*,

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which are those that are of the form $0.\chi_A$ for some c.e. *set* A. Computably enumerable reals arise naturally as the measures of the domains of prefix-free Turing machines in the same way that in classical computability theory c.e. *sets* arise as the halting sets of Turing machines.

A classic example of a c.e. real that is not strongly c.e. is Chaitin's halting probability [5]:

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is a universal prefix-free machine. This real is famously *1-random*, in the sense that there is a constant c such that for all n,

$$K(\Omega \upharpoonright n) > n - c,$$

where $\Omega \upharpoonright n$ denotes the first n bits of Ω. As is well-known, this initial segment definition coincides with Martin-Löf's definition [15] of randomness as avoiding all "effectively presented" statistical tests.

The context of the current paper is a program to try to understand the relative initial segment complexity of (c.e.) reals as articulated in the papers [6, 8, 9, 10]. For instance, the above definition of Ω *seems* to depend upon the choice of the relevant universal machine U. Perhaps for a different machine $\widehat{U}$, the real $\Omega_{\widehat{U}}$ would behave rather differently than Ω_U.

The situation is akin to that for "the" halting set $\emptyset' = \{e : \varphi_e(e)\downarrow\}$, where φ_e is the eth partial computable function. Here we might well argue that the definition actually depends on the choice of universal machine enumerating the partial computable functions. Of course, by Myhill's Theorem (see [17]), we know that all versions of the halting problem are creative sets, and are all the same up to computable permutations of the natural numbers.

The first person to address this situation for Ω was Solovay [18, 19], who introduced an analytic version of m-reducibility appropriate for c.e. reals.

Definition 1 (Solovay [18, 19]). Let α and β be reals. We say β *dominates* α, or, alternatively, that α is *Solovay reducible to* β, and write $\alpha \leqslant_S \beta$, if there exist a partial computable function φ and a constant c such that for all rational $q < \beta$, we have $\varphi(q)\downarrow$ and

$$|\alpha - \varphi(q)| \leqslant c(\beta - q).$$

If A, $B \subseteq \mathbb{N}$, then $A \leqslant_S B$ if and only if $0.\chi_A \leqslant_S 0.\chi_B$.

Note that Solovay reducibility implies Turing reducibility. (See e.g. [8] for a proof.)

One reason that Solovay was interested in this reducibility is that if $\alpha \leqslant_S \beta$ then there is a constant c such that for all n,

$$K(\beta \upharpoonright n) > K(\alpha \upharpoonright n) - c.$$

(See e.g. [9] for a proof.) This fact makes Solovay reducibility a possible measure of relative randomness. Solovay defined a class of c.e. reals, the Ω-*like* c.e. reals, as being those that dominate Ω. Clearly, all Ω-like reals are 1-random. In the wonderful unpublished notes [18], Solovay proved a number of very interesting results about the initial segment complexity of Ω-like reals. He remarked: "It seems strange that we are able to prove so much about the behavior of Ω-like reals when, a priori, the definition of Ω is thoroughly model dependent...". Solovay's notes have not been published, save for a fragment in Solovay [19], but most of the material will appear in the forthcoming monograph of Downey and Hirschfeldt [7].

Solovay's intuition has been more recently confirmed by two groups of authors. First, Calude, Khoussainov, Hertling and Wang [4] used Kraft's inequality to show that if a c.e. real is Ω-like then it is the halting probability of some universal prefix-free machine. Thus it is a version of Ω. Second, Kučera and Slaman [13] proved that if a c.e. real is 1-random then it is Ω-like.

Thus we have the following very remarkable consequence. Fundamental work of Chaitin and Levin (see e.g. [11]) has shown that for all n there are strings σ of length n such that $K(\sigma) = n+K(n)+O(1)$. Thus it is possible for the generally-greater Kolmogorov complexity of Ω to oscillate upwards above $n + \log n$, and indeed we can show that infinitely often we have $K(\Omega \upharpoonright n) > n + \log n$. On the other hand, the complexity of a 1-random real can oscillate down towards n infinitely often. The Kučera-Slaman Theorem shows that all 1-random c.e. reals have the *same* initial segment behavior, and all oscillate downwards and upwards at the *same* n's. Thus the situation for halting probabilities is in this respect similar to that for versions of the halting set.

This paper is part of the effort to understand the structure of the c.e. reals under Solovay reducibility. Note that for c.e. reals α and β, since there exist increasing computable functions of rationals $\alpha[s]$ and $\beta[s]$ such that $\lim_s \alpha[s] = \alpha$ and $\lim_s \beta[s] = \beta$, we have $\alpha \leqslant_{\mathrm{S}} \beta$ if and only if there exist a computable function φ and a constant c such that for all s,

$$\alpha - \alpha[\varphi(s)] < c \cdot (\beta - \beta[s]).$$

(This characterization was first given in [3]; see also [9] for a proof.) Notice that $\varphi(s) = s$ and $c = 1$ is a possibility here. Since we deal exclusively with c.e. reals in what follows, we generally use this equivalent characterization without explicit comment. One easy consequence of it that we will use below is that if for all s,

$$\alpha[s + 1] - \alpha[s] \leqslant \beta[s + 1] - \beta[s],$$

then $\alpha \leqslant_{\mathrm{S}} \beta$. Another useful variation is given in the following lemma.

Lemma 1.1. *Let $\alpha[s]$ and $\beta[s]$ be increasing sequences of rationals such that $\lim_s \alpha[s] = \alpha$ and $\lim_s \beta[s] = \beta$. Then $\alpha \leqslant_S \beta$ if and only if there exist a computable function φ, a constant c, and a nonincreasing computable function $\psi : \mathbb{N} \to \mathbb{Q}$ such that $\lim_s \psi(s) = 0$ and for all s,*

$$\alpha - \alpha[\varphi(s)] < c \cdot (\beta - \beta[s]) + \psi(s).$$

Proof. The "only if" direction follows from the previous characterization of Solovay reducibility. For the "if" direction, suppose that the displayed inequality holds for all s. Given s, find a $t > s$ such that $c \cdot (\beta[t] - \beta[s]) > \psi(t)$, which is possible since ψ goes to 0. Define $\theta(s) = \varphi(t)$. Then

$$\alpha - \alpha[\theta(s)] < c \cdot (\beta - \beta[t]) + \psi(t) = c \cdot (\beta - \beta[s]) - c \cdot (\beta[t] - \beta[s]) + \psi(t) < c \cdot (\beta - \beta[s]).$$

So $\alpha \leqslant_{\mathrm{S}} \beta$ by the previous characterization of Solovay reducibility. □

We call the equivalence classes of c.e. reals induced by the Solovay reducibility pre-ordering *c.e. Solovay degrees*. We denote the degree of α by $[\alpha]_{\mathrm{S}}$. Thus $[\alpha]_{\mathrm{S}} = \{\beta : \beta \equiv_{\mathrm{S}} \alpha\}$. The first paper to study this structure in detail was Downey, Hirschfeldt, and Nies [9], where it was shown that the Solovay degrees of c.e. reals form a dense distributive upper-semilattice, with join induced by ordinary arithmetical addition, that is, $[\alpha]_{\mathrm{S}} \vee [\beta]_{\mathrm{S}} = [\alpha + \beta]_{\mathrm{S}}$ (though some of these facts were previously known). It was also shown in [9] that while every nonrandom degree splits over all lesser ones, remarkably, $[\Omega]_{\mathrm{S}}$ is qualitatively different in that if α and β are c.e. reals and $\alpha + \beta \equiv_{\mathrm{S}} \Omega$, then at least one of α or β is 1-random. (According to Kučera, the latter result had been proved earlier by Demuth.)

The goal of this paper is to add to our global understanding of the structure of the c.e. Solovay degrees. We prove that the structure of the c.e. Solovay degrees has an undecidable first order theory. The proof is, of course, a priority argument, in this case one employing a $\mathbf{0}'''$ tree of strategies.

Of relevance to this paper is another measure of relative randomness.

Definition 2 (Downey, Hirschfeldt, and LaForte [8]). Let $A, B \subseteq \mathbb{N}$. We say A is *strongly weak-truth-table reducible (sw-reducible) to* B, and write $A \leqslant_{\mathrm{sw}} B$, if there exist a computable functional Γ and a constant c such that $A = \Gamma^B$ and $\gamma(x) \leqslant x + c$ for all x, where γ is the use function of Γ, which is independent of B. If α and β are reals with $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, then $\alpha \leqslant_{\mathrm{sw}} \beta$ if and only if $A \leqslant_{\mathrm{sw}} B$.

Again it is relatively easy to show that if $A \leqslant_{\mathrm{sw}} B$ then $K(B \upharpoonright n) > K(A \upharpoonright n) - O(1)$. For c.e. reals in general, sw-reducibility and Solovay reducibility are incomparable measures, but for strongly c.e. reals they coincide. For these results and others concerning the spectrum of measures of relative randomness see Downey, Hirschfeldt, and LaForte [8], the survey articles Downey [6] and Downey, Hirschfeldt, Nies, and Terwijn [11], or the forthcoming monograph Downey and Hirschfeldt [7].

Most of our notation is standard, and follows that of Soare [17]. A c.e. real can be identified with the characteristic function of a *nearly c.e. set*, that is, a set B with an approximation $B[s]$ such that for all s and x, if $B(x)[s+1] < B(x)[s]$ then there exists some $y < x$ such that $B(y)[s] < B(y)[s+1]$. In order to avoid possible confusion between c.e. *sets* and c.e. *reals*, in the following sections, we will talk only about c.e. sets and nearly c.e. sets. Given either c.e. or nearly c.e. sets Y and Z, we generally write $Y + Z$ and $Y - Z$ for the ordinary arithmetic sum and difference of the c.e. reals $0.\chi_Y$ and $0.\chi_Z$. This convention will never cause any confusion below, and simplifies our notation somewhat.

We show that the theory of the structure of the c.e. Solovay degrees is undecidable using the method of Nies [16], involving the notions of effectively dense boolean algebras and hereditarily undecidable theories. Suppose we have a structure $(\mathbb{N}, \preccurlyeq, \vee, \wedge)$ such that $\preccurlyeq$ is a Σ^0_k pre-ordering and $\vee$ and $\wedge$ are total computable binary functions, and let $\approx$ be the equivalence relation modulo which $\preccurlyeq$ becomes a partial order. (In other words, $m \approx n$ if and only if $m \preccurlyeq n$ and $n \preccurlyeq m$.) If the quotient structure $\mathcal{B} = (\mathbb{N}, \preccurlyeq, \vee, \wedge)/\approx$ is a boolean algebra, then we call $\mathcal{B}$ a Σ^0_k *boolean algebra*. (We also abuse terminology and call $\mathcal{B}$ a Σ^0_k boolean algebra if it is isomorphic to a Σ^0_k boolean algebra.) A boolean algebra $\mathcal{B}$ with least element 0 is *effectively dense* if there is a computable function F such that $0 \prec F(x) \prec x$ for all $x \neq 0$ in the domain of $\mathcal{B}$.

A theory T in a first-order language L is *hereditarily undecidable* if every set $X \subseteq T$ containing the valid L-sentences is undecidable. This notion is useful to us because of the following *transfer principle* (see e.g. [1, 2]): if A is an L_1-structure with a hereditarily undecidable theory, and A can be interpreted with parameters in an L_2-structure B, then the theory of B is hereditarily undecidable.

A Solovay degree b is *complemented below* a Solovay degree a if there is a Solovay degree c such that $b \wedge c = a$ and $b \vee c = [0]_S$. In Nies [16] it is shown that the lattices of Σ^0_k ideals of effectively dense Σ^0_k boolean algebras have hereditarily undecidable theories. (Actually, Theorem 2.1 of [16] is proved only for the Σ^0_1 case, but it is clear how to relativize the proof for $k > 1$. See, for example, Nies and Downey [12, Theorem 2.1] for the case $k = 2$.) We use this result to show that the structure of the c.e. Solovay degrees is undecidable by finding a c.e. Solovay degree a such that the collection of c.e. Solovay degrees complemented below a (with join and meet) forms an effectively dense Σ^0_3 boolean algebra $\mathcal{B}(a)$ for which the lattice of Σ^0_3 ideals is definable in $\mathcal{B}(a)$.

2 The structure $\mathcal{B}(a)$

We produce the Solovay degree a mentioned above using two technical lemmas, Theorems 1 and 2 below, involving the following notion:

Definition 3. A c.e. set $A \subset \mathbb{N}$ is *somewhat sparse via f* if

1. $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing, computable function,
2. $A \subseteq \{\, f(k) : k \in \mathbb{N} \,\}$, and
3. $f(k) + k + 1 < f(k + 1)$ for all k.

This notion is a significant weakening of the notion of super-sparseness used in various results characterizing polynomial-time degrees of computable sets. A simple example of a somewhat sparse set is the set of positive squares $\{\, n^2 :$

$n > 0 \}$. In fact, for our main lemma, Theorem 2 below, we will choose a to be the Solovay degree of a subset of this set.

Let W_e denote the eth c.e. set, and let $B \sqcup C = A$ denote the fact that B and C are disjoint and their union is A. For a c.e. set A, let $\mathcal{B}(A)$ be the boolean algebra of c.e. subsets of A that have c.e. complements included in A, that is, $\{ W_e : \exists j \, (W_j \sqcup W_e = A) \}$, with union and intersection. If A is somewhat sparse, we can prove the following exact pair theorem characterizing the ideals of $\mathcal{B}(A)$ that are closed downward under sw-reducibility (or equivalently, since we are dealing with c.e. sets, Solovay reducibility).

Theorem 1. *Let A be somewhat sparse. Let $\mathcal{J}$ be a Σ^0_3 ideal of $\mathcal{B}(A)$. Then there exist c.e. sets X and Y such that $W_e \in \mathcal{J}$ if and only if $W_e \leqslant_{sw} X, Y$.*

We briefly describe the proof of this result below.

We can then produce a somewhat sparse set A with certain useful properties. For sets B_1 and B_2, we write $B_1 \wedge B_2 \equiv_{\mathrm{S}} 0$ to denote the fact that the infimum of $[B_1]_{\mathrm{S}}$ and $[B_2]_{\mathrm{S}}$ exists and is $[0]_{\mathrm{S}}$.

Theorem 2. *There exists a c.e., non-computable, somewhat sparse set A such that*

1. *for all c.e. splittings $A_1 \sqcup A_2 = A$, the infimum of the Turing degrees of A_1 and A_2 is $\mathbf{0}$; and*

2. *for all nearly c.e. sets B_1 and B_2 such that $B_1 + B_2 \equiv_S A$ and $B_1 \wedge B_2 \equiv_S 0$, there exist c.e. sets A_1 and A_2 with $A_1 \sqcup A_2 = A$, such that $B_1 \equiv_S A_1$ and $B_2 \equiv_S A_2$.*

We briefly describe the proof in Section 3 below. From now on, fix a set A as in Theorem 2 and let $a = [A]_{\mathrm{S}}$.

We will show that for all $B, C \in \mathcal{B}(A)$, we have $[B]_{\mathrm{S}} \wedge [C]_{\mathrm{S}} = [B \cap C]_{\mathrm{S}}$ and $[B]_{\mathrm{S}} \vee [C]_{\mathrm{S}} = [B \cup C]_{\mathrm{S}}$. Together with Theorem 2, this fact implies that the collection of c.e. Solovay degrees complemented below a (with join and meet) forms a boolean algebra $\mathcal{B}(a)$, which is equal to $\{ [B]_{\mathrm{S}} : B \in \mathcal{B}(A) \}$. We will also show that this boolean algebra is Σ^0_3 and effectively dense. Thus the theory of the lattice $\mathcal{I}(a)$ of Σ^0_3 ideals of $\mathcal{B}(a)$ is hereditarily undecidable. Since Theorem 1 will imply that there is an interpretation of the theory of $\mathcal{I}(a)$ in the theory of the Solovay degrees below a, this will show that the theory of the Solovay degrees of c.e. reals is undecidable, by the transfer principle mentioned in the introduction.

We now prove a series of lemmas showing that the set of c.e. splittings of A together with the relation of Solovay reducibility and the operations of intersection and union can be used to construct a Σ^0_3 boolean algebra. First we show that this algebra can be presented in a c.e. way, and that the density of the structure for sw-reducibility is effective. Since sw-reducibility is equivalent to Solovay reducibility on c.e. sets, this result will serve our purposes. Recall that the ordinary arithmetic sum gives the join in the Solovay degrees of c.e. reals.

Lemma 2.1. *There exist a computable enumeration* $\{ A_e : e \in \mathbb{N} \}$ *of c.e. sets and computable functions* f, g, *and* h *such that the following hold.*

(a) *For each* i *and* j *such that* $W_i \sqcup W_j = A$, *there exists an* e *such that* $W_i = A_e$.

(b) *For each* e, *there exists a* k *such that* $A_e \sqcup A_k = A$.

(c) *For all* i *and* j, *we have* $A_i \cup A_j = A_{f(i,j)}$ *and* $A_i \cap A_j = A_{g(i,j)}$.

(d) *For all* e, *if* A_e *is noncomputable, then* $\emptyset <_{sw} A_{h(e)} <_{sw} A_e$, *and hence* $\emptyset <_S A_{h(e)} <_S A_e$.

Proof. We first note that the collection of sets that are halves of c.e. splittings of A can be enumerated by the following procedure. Let $\langle W_e, V_e \rangle$ be an enumeration of all pairs of c.e. subsets of A. For each e, define a sequence of e-expansionary stages $s_0^e < s_1^e, \dots$ recursively by letting $s_0^e = 0$ and letting s_{k+1}^e be the least stage $s > s_k^e$ such that $\forall t \leqslant s\, (W_e \cap V_e = \emptyset)$ and $A[s_k^e] \subseteq (W_e \cup V_e)[s]$. Note that this sequence of stages could be finite. Let $B_e = \bigcup_k W_e[s_k^e]$. Then $\{ B_e : e \in \mathbb{N} \}$ is an indexing of all the c.e. sets that can be used to split A. This collection is closed under union and intersection, and under c.e. splittings.

We construct the sequence $\{ A_e : e \in \mathbb{N} \}$ from $\{ B_e : e \in \mathbb{N} \}$ by first constructing a natural computable enumeration of all closed terms of the first-order language $\mathcal{L} = \{\wedge, \vee, h_D\} \cup \{ b_e : e \in \mathbb{N} \}$, where $\wedge$ and $\vee$ are two-place function symbols, h_D is a one-place function symbol, and each b_e is a constant symbol. Let $\{ t_e : e \in \mathbb{N} \}$ be this sequence. The construction familiar from the Sacks splitting theorem (see [17]) is uniform in the enumeration of the set being split, so that given any noncomputable c.e. set B, we can uniformly find a c.e. set $B_D \subset B$ that is half of a splitting of B and such that $\emptyset <_{\mathrm{T}} B_D <_{\mathrm{T}} B$. Since $+$ is the join in the sw-degrees of c.e. sets and B_D is half of a c.e. splitting of B, it follows that $B_D \leqslant_{\mathrm{sw}} B$. Combining this fact with the fact that sw-reducibility implies Turing reducibility, we see that $\emptyset <_{\mathrm{sw}} B_D <_{\mathrm{sw}} B$. Also, the operations of union and intersection are of course similarly uniform for c.e. sets. So we can define A_e to be the c.e. set with the obvious interpretation given by t_e, interpreting the symbol $\wedge$ as $\cap$, the symbol $\vee$ as $\cup$, the symbol h_D as the operation of taking the first member of the splitting given by the uniform version of the Sacks splitting theorem, and each symbol b_e as B_e. (Note that we can apply the construction in the proof of the uniform version of the Sacks splitting theorem even if B is computable, to obtain some c.e. set; in this case, we do not care what properties this set has, only that it exists, so that h_D is total.) For example, if $t_e = h_D((b_2 \wedge b_1) \vee b_3)$, then A_e is the first half of the Sacks splitting of $(B_2 \cap B_1) \cup B_3$. Clearly the sequence $\{ A_e : e \in \mathbb{N} \}$ is as required, since we can effectively find the indices of $t_i \vee t_j$, of $t_i \wedge t_j$, and of $h_D(t_i)$ from i and j, giving us the required functions f, g, and h. □

Fix an enumeration $\{ A_e : e \in \mathbb{N} \}$ as in the lemma.

As indicated in the introduction, we simplify notation by writing X for both a c.e. set and the real $0.\chi_X$, which means we often write '$-$' for both set complementation and ordinary arithmetic subtraction. When the sets involved in the expression $B - C$ are c.e. and not merely nearly c.e., and $C \subseteq B$, these two operations amount to the same thing. Since we only use this notation in such cases, no confusion should result. For a subset B of A, we write $\overline{B}$ for the complement of B in A, that is, $A - B$. Notice that $\overline{A}_e$ is a c.e. subset of A for every e. We now show that intersections and unions give infima and suprema, respectively, for the Solovay degrees of the sets A_e.

Lemma 2.2. *For all $i, j \in \mathbb{N}$, we have $A_i \cup A_j \equiv_S A_i + A_j$.*

Proof. First note that if n enters $A_i \cup A_j$ at stage $s+1$, then it contributes exactly 2^{-n} to the value of $(A_i \cup A_j)[s+1] - (A_i \cup A_j)[s]$, thought of as a real, and contributes at least 2^{-n} to the value of $(A_i + A_j)[s+1] - (A_i + A_j)[s]$. Thus

$$(A_i \cup A_j)[s+1] - (A_i \cup A_j)[s] \leqslant (A_i + A_j)[s+1] - (A_i + A_j)[s]$$

for all s. It follows that $A_i \cup A_j \leqslant_{\mathrm{S}} A_i + A_j$.

Showing that $A_i + A_j \leqslant_{\mathrm{S}} A_i \cup A_j$ is a little more involved. We can assume the usual convention that, at each stage, exactly one number enters one of A_i or A_j. (Clearly we can assume without loss of generality that $A_i \cup A_j$ is infinite.) Given $s \in \mathbb{N}$, let $f(s)$ be the least stage t at which for every $z \in (A_i \cup A_j)[s]$, we have $z \in (A_i \cap A_j)[t]$, or $z \in (A_i \cap \overline{A}_j)[t]$, or $z \in (\overline{A_i} \cap A_j)[t]$. Note that f is a computable function. Suppose $t \geqslant f(s)$ and $(A_i + A_j)[t+1] - (A_i + A_j)[t] = 2^{-k(t)}$. Then either $k(t) \in A_i[t+1] - A_i[t]$ or $k(t) \in A_j[t+1] - A_j[t]$. So $k(t) \notin (A_i \cup A_j)[s]$ by our choice of $f(s)$. Each k that enters $A_i \cup A_j$ at some t after stage $f(s)$ can contribute at most $2 \cdot 2^{-k} = 2^{-k+1}$ to the value $A_i + A_j$, since it can enter A_i at most once and A_j at most once. Thus

$$\begin{aligned} A_i + A_j - (A_i + A_j)[f(s)] &= \sum_{j=f(s)+1}^{\infty} 2^{-k(j)} \\ &\leqslant \sum_{k \in A_i \cup A_j - (A_i \cup A_j)[s]} 2 \cdot 2^{-k} \\ &= 2 \cdot (A_i \cup A_j - (A_i \cup A_j)[s]). \end{aligned}$$

It follows that $A_i + A_j \leqslant_{\mathrm{S}} A_i \cup A_j$. □

Since $\overline{A_i \cup A_j} = \overline{A}_i \cap \overline{A}_j$, and, by Theorem 2, every Solovay degree in $\mathcal{B}(a)$ contains a c.e. set that is half of a splitting of A, the previous lemma shows that $\mathcal{B}(a)$ is closed under suprema. We now show that infima also exist in $\mathcal{B}(a)$. One direction is almost immediate.

Lemma 2.3. *For all $i, k \in \mathbb{N}$, we have $A_i \cap A_k \leqslant_S A_k$.*

Proof. At stage s, we search for the least $t > s$ such that $A_k[s] \subset A_i[t] \cup \overline{A_i}[t]$, and set $f(s) = t$. If $k \in A_i \cap A_k - (A_i \cap A_k)[f(s)]$, then $k \notin A_k[s]$, so $k \in A_k - A_k[s]$. Hence $A_i \cap A_k - (A_i \cap A_k)[f(s)] \leqslant A_k - A_k[s]$ for all s, and so $A_i \cap A_k \leqslant_{\mathrm{S}} A_k$. □

For the other direction, we first prove a useful lemma.

Lemma 2.4. *For all $i, k \in \mathbb{N}$, we have $A_i \leqslant_S A_k$ if and only if $A_i \cap \overline{A_k}$ is computable.*

Proof. First, suppose $A_i \leqslant_{\mathrm{S}} A_k$. By Lemma 2.3, $A_i \cap \overline{A_k} \leqslant_{\mathrm{S}} \overline{A_k}$ and $A_i \cap \overline{A_k} \leqslant_{\mathrm{S}} A_i \leqslant_{\mathrm{S}} A_k$. By Theorem 2, the infimum of A_k and $\overline{A_k}$ in the Turing degrees is $\mathbf{0}$, so $A_i \cap \overline{A_k}$ is computable.

Now, suppose that $A_i \cap \overline{A_k}$ is computable. Then $A_i \cap \overline{A_k} \leqslant_{\mathrm{S}} A_k$. By Lemma 2.3, also $A_i \cap A_k \leqslant_{\mathrm{S}} A_k$. Since $A_i = (A_i \cap \overline{A_k}) + (A_i \cap A_k)$, it follows that $A_i \leqslant_{\mathrm{S}} A_k$. □

Lemma 2.5. *For all $i, j, k \in \mathbb{N}$, if $A_j \leqslant_S A_i$ and $A_j \leqslant_S A_k$, then $A_j \leqslant_S A_i \cap A_k$.*

Proof. By Lemma 2.4, $A_j \cap \overline{A_i}$ and $A_j \cap \overline{A_k}$ are both computable. But then $A_j \cap \overline{A_i \cap A_k}$ is computable. Since $A_i \cap A_k = A_l$ for some l, Lemma 2.4 implies that $A_j \leqslant_{\mathrm{S}} A_i \cap A_k$. □

The relation of Solovay reducibility on c.e. reals is certainly Σ^0_3, since $X \leqslant_{\mathrm{S}} Y$ if and only if

$$\exists e, c \forall s, t\, \exists u\, (\varphi_e(s)[u]\!\downarrow \wedge (X_t - X[\varphi_e(s)]) < c \cdot (Y_t - Y[s])).$$

Thus, we have shown that $\mathcal{B}(a)$ is an effectively dense Σ^0_3 boolean algebra. We can now easily prove our main result.

Theorem 3. *The structure of the Solovay degrees of c.e. reals is undecidable.*

Proof. By Theorem 2 and the lemmas above, we can represent $\mathcal{B}(a)$ by $\{\, [A_e]_{\mathrm{S}} : e \in \mathbb{N} \,\}$, with the operations of union and intersection on these sets giving joins and meets on their Solovay degrees. (Hence $\mathcal{B}(a)$ actually is a boolean algebra.) Furthermore, this algebra is Σ^0_3, as explained above, and is effectively dense, by Lemma 2.1.(d), since $\leqslant_{\mathrm{sw}}$ and $\leqslant_{\mathrm{S}}$ coincide on c.e. sets. Let $\mathcal{I}(a)$ be the lattice of Σ^0_3 ideals of $\mathcal{B}(a)$. By Theorem 1, there is a (two-dimensional) interpretation of the theory of $\mathcal{I}(a)$ in the theory of the Solovay degrees below a, since the sets involved can all be taken to be c.e., so that Solovay reducibility and sw-reducibility coincide on them. Since the theory of $\mathcal{I}(a)$ is hereditarily undecidable, this fact implies that the theory of the structure of the Solovay degrees of c.e. reals is undecidable, by the transfer principle mentioned in the introduction. □

The proof of the first of our main lemmas involves an infinite-injury priority argument, which we now describe.

Theorem 1. *Let A be somewhat sparse. Let $\mathcal{J}$ be a Σ^0_3 ideal of $\mathcal{B}(A)$. Then there exist c.e. sets X and Y such that $W_e \in \mathcal{J}$ if and only if $W_e \leqslant_{sw} X, Y$.*

Proof. To simplify our description, we will consider all c.e. sets to be subsets of A. This convention just amounts to abusing standard notation by indexing only the c.e. subsets of A: in other words, W_e means for us what would ordinarily be written $W_e \cap A$. Let $\mathcal{J} = \{\, W_e : \exists p \forall s\, R(e, p, s) \,\}$, where R is a Σ^0_1 relation.

When W and V are c.e. sets, the sw-degree of the union of W and V is not in general the degree of the join of their sw-degrees, even if such a join exists. (Joins in the sw-degrees of c.e. reals do not always exist; see [8].) Fortunately, it is easy to see that for elements W and V of $B(A)$, we have $W, V \leqslant_{\mathrm{sw}} W \cup V$, which is all that we require in our construction.

Lemma 2.6. *Let A be a c.e. set. If W and V are elements of $B(A) = \{ W_e : \exists j\, (W_j \sqcup W_e = A) \}$, then $W, V \leqslant_{sw} W \cup V$.*

Proof. The proof is almost immediate, since the complements of V and W in A, and hence in $W \cup V$, are c.e. sets. Given $n \in \mathbb{N}$, let $g(n)$ be the least stage s at which for every $z \in (W \cup V) \restriction n$, one of the following holds: $z \in (W \cap V)[s]$, or $z \in (W \cap \overline{V})[s]$, or $z \in (\overline{W} \cap V)[s]$. Clearly $W \restriction n = W \restriction n[g(n)]$ and $V \restriction n = V \restriction n[g(n)]$. ☐

Let $V_{\langle e,p\rangle} = \bigcup \{ W_e[s] : \forall t < s\, R(e,p,s) \}$. To ensure that X and Y bound the ideal $\mathcal{J}$, we must satisfy two sequences of positive requirements:

$$\mathrm{P}^X_{e,p} : \quad \exists n\, \forall x > n\, (x \in V_{\langle e,p\rangle} \text{ if and only if } x + \langle e,p\rangle \in X)$$

and

$$\mathrm{P}^Y_{e,p} : \quad \exists n\, \forall x > n\, (x \in V_{\langle e,p\rangle} \text{ if and only if } x + \langle e,p\rangle \in Y).$$

These requirements clearly imply that

$$W_e \in \mathcal{J} \Longrightarrow W_e \leqslant_{\mathrm{sw}} X$$

and

$$W_e \in \mathcal{J} \Longrightarrow W_e \leqslant_{\mathrm{sw}} Y,$$

since if $W_e \in \mathcal{J}$, then there exists a p with $\forall s\, R(e,p,s)$, so that $W_e = V_{\langle e,p\rangle} \leqslant_{\mathrm{sw}} X, Y$.

To ensure exactness of the pair X, Y, we must satisfy the sequence of negative requirements

$$\mathrm{N}_e : \quad \Phi^X_e = \Phi^Y_e = h \text{ total } \Longrightarrow h \leqslant_{\mathrm{sw}} \bigcup_{\langle j,p\rangle < e} V_{\langle j,p\rangle},$$

where $\langle \Phi_e : e \in \mathbb{N}\rangle$ is a sequence consisting of all partial sw-reductions (i.e., all partial computable functionals such that, for any oracle, the use function is bounded by $x + O(1)$). Note that we list our sw-reductions so that their use functions are independent of the oracle; we can assume that the use function of Φ_e is exactly $n + c$ for some c. These requirements suffice because they imply that if a set W is sw-reducible to both X and Y, then W is also sw-reducible to a union of elements of $\mathcal{J}$ (since every $V_{\langle j,p\rangle}$ is either equal to W_j, in which case $W_j \in \mathcal{J}$, or is finite), and hence $W \in \mathcal{J}$. (The fact that we can use Φ_e twice in the statement of N_e, rather than having one requirement for each pair of reductions, follows by the usual Posner trick; see [17].)

Our construction will be similar to that of a minimal pair of Turing degrees (see [17]). The coding involved in our positive requirements of course prevents us from building an actual minimal pair, but we will see that it is compatible with our negative requirements. Some care will need to be taken because the positive requirements are infinitary. In order to arrange the priorities of the requirements, we employ the tree-of-strategies method pioneered by Lachlan. The positive requirements inflict only finite injury, so they need not appear on the tree of strategies, although it is conveneient to do so for the sake of uniformity. In our tree there are two possible outcomes for a strategy α working for a negative requirement N_e, depending on whether or not the partial functions Φ_e^X and Φ_e^Y produce the same total function. The infinitary outcome will be coded by 0 and the finitary one by 1. If Φ_e^X and Φ_e^Y do produce the same total function, then the restraint involved in ensuring that this function is computable from $\bigcup_{\langle j,p\rangle<e} V_{\langle j,p\rangle}$ will tend to infinity in the limit, but this restraint will be imposed only on positive strategies to the right of $\alpha^\frown 0$ on the tree of strategies. The idea is that the action of positive strategies above α can be accounted for by the $\bigcup_{\langle j,p\rangle<e} V_{\langle j,p\rangle}$ term in the statement of N_e, while positive strategies at or below $\alpha^\frown 0$ will not be able to destroy both Φ_e^X and Φ_e^Y computations between successive expansionary stages (that is, stages at which the length of agreement between Φ_e^X and Φ_e^Y increases), as usual in variations of the minimal pair construction. (Of course, if α is on the true path, the action of strategies to the left of α will be finite.) The technical details of the construction are relatively straightforward for an infinite injury construction of this type. □

3 The proof of Theorem 2

In this section we describe the proof of our main technical result, which involves a $\mathbf{0}'''$-priority argument. Because this is only an extended abstract, we omit the technical details and merely explain the ideas behind the strategies that are needed to construct A.

Theorem 2. *There exists a c.e., non-computable, somewhat sparse set A such that*

1. *for all c.e. splittings $A_1 \sqcup A_2 = A$, the infimum of the Turing degrees of A_1 and A_2 is $\mathbf{0}$; and*

2. *for all nearly c.e. sets B_1 and B_2 such that $B_1 + B_2 \equiv_S A$ and $B_1 \wedge B_2 \equiv_S 0$, there exist c.e. sets A_1 and A_2 with $A_1 \sqcup A_2 = A$, such that $B_1 \equiv_S A_1$ and $B_2 \equiv_S A_2$.*

Proof. We make A somewhat sparse by choosing all numbers enumerated into A to be from $\{ n^2 : n > 0 \}$. We must satisfy three types of requirements. The simplest are the requirements for noncomputability: for each $e \in \mathbb{N}$,

$$\mathrm{P}_e : \overline{A} \neq W_e.$$

To ensure that condition 1 on all c.e. splittings of A holds, we satisfy

$$\mathrm{N}_e : \ (U_e \sqcup V_e = A \text{ and } \Phi_e^{U_e} = \Phi_e^{V_e} = h \text{ total}) \Longrightarrow h \leqslant_{\mathrm{T}} \emptyset,$$

where $\langle U_e, V_e, \Phi_e \rangle$ is an enumeration of all triples consisting of pairs of c.e. sets together with a partial computable functional.

The most complex requirements are those involving condition 2 on A. Say that $X \leqslant_{\mathrm{S}} Y$ *via* c, φ if $X - X[\varphi(s)] < c \cdot (Y - Y[s])$ for all s. Letting $\langle B_e, C_e, \varphi_e, \psi_e \rangle$ enumerate all pairs of nearly c.e. sets together with all pairs of partial computable functions, we must satisfy for each $c \in \mathbb{Q}$ and $e \in \mathbb{N}$ the requirement

$$\begin{aligned} \mathrm{R}_{\langle e,c \rangle} : \ &(B_e + C_e \leqslant_{\mathrm{S}} A \text{ via } c, \psi_e \text{ and } A \leqslant_{\mathrm{S}} B_e + C_e \text{ via } c, \varphi_e) \Longrightarrow \\ &\big(\exists \text{ c.e. } Q_e \, (Q_e \leqslant_{\mathrm{S}} B_e, C_e \ \wedge \ \forall i \, (\overline{Q_e} \neq W_i)) \ \vee \\ &\quad \exists \text{ c.e. } \widehat{B_e}, \widehat{C_e} \, (\widehat{B_e} \sqcup \widehat{C_e} = A \ \wedge \ \widehat{B_e} \equiv_{\mathrm{S}} B_e \ \wedge \ \widehat{C_e} \equiv_{\mathrm{S}} C_e)\big). \end{aligned}$$

The strategies for the first two classes of requirements are straightforward and familiar ones from the study of c.e. Turing degrees. For the requirements P_e, we pick some large $x \in \{ n^2 : n > 0 \}$ and wait for a stage s so that $x \in W_e[s]$, at which point we add x to $A[s+1]$. This action ensures that the complement of A is not equal to W_e.

For the requirements N_e, we use a slightly modified version of the strategy familiar from the standard construction of a minimal pair of c.e. degrees. To avoid introducing some essentially irrelevant details involved in checking whether $U_e \sqcup V_e$ splits A, we actually work with $U_e^* = U_e \cap A \cap (U_e \searrow V_e)$ and $V_e^* = V_e \cap A \cap (V_e \searrow U_e)$. (Recall that if X and Y are c.e. sets, then $X \searrow Y$ denotes the set of numbers enumerated into X before being enumerated into Y.) In what follows, we omit the *, just writing U_e and V_e for the restricted versions of these sets. Define the length of agreement for N_e at stage s by

$$l_e[s] = \max\{ x : \forall y < x \, (\Phi_e^{U_e}(y)[s]\downarrow = \Phi_e^{V_e}(y)[s]\downarrow) \}.$$

At each stage s we define a restraint $r[s]$ preventing lower-priority strategies from enumerating numbers into A. (In the full construction, this restraint function will be implicit in the way P-strategies choose their witnesses, so there will be no need to define it explicitly.) We recursively define a set of *expansionary* stages, with 0 being the first such, and let $r[0] = 0$. At each stage $s+1 > 0$, let s^- be the previous expansionary stage. If $A[s^- + 1] \nsubseteq (U_e \sqcup V_e)[s]$ or there is some $t \leqslant s$ such that $l_e[t] \geqslant l_e[s+1]$, then $r[s+1] = \max\{ \varphi_e^{U_e}(y)[s^-], \varphi_e^{V_e}(y)[s^-] : y < l_e[s^-] \}$. Otherwise, we declare $s+1$ to be expansionary and let $r[s+1] = 0$.

Since we intend to allow at most one number n to enter A below the restraint between expansionary stages, this procedure will ensure the satisfaction of the requirement, in much the same way as in the proof of Theorem 1: the restraint will not be allowed to drop again until n has entered either U_e or V_e, and both computations' values have been restored as signaled by the increase in the length of agreement.

3.1 Strategies for the splitting requirements

The description of our strategy for satisfying a requirement

$$\begin{aligned}\mathrm{R}:\ (B+C \leqslant_{\mathrm{S}} A \text{ via } c,\psi \text{ and } A \leqslant_{\mathrm{S}} B+C \text{ via } c,\varphi) \implies \\ \big(\exists \text{ c.e. } Q\,(Q \leqslant_{\mathrm{S}} B, C \,\wedge\, \forall i\,(\overline{Q} \neq W_i)) \vee \\ \exists \text{ c.e. } \widehat{B},\widehat{C}\,(\widehat{B}\sqcup\widehat{C} = A \,\wedge\, \widehat{B} \equiv_{\mathrm{S}} B \,\wedge\, \widehat{C} \equiv_{\mathrm{S}} C)\big)\end{aligned}$$

is much more involved.

Notice that we can assume that φ and ψ are strictly increasing functions, $\varphi(s) > s$ and $\psi(s) > s$ for all s, and $c \geqslant 1$. The strategy for satisfying R involves first approximating whether or not the condition

$$B+C \leqslant_{\mathrm{S}} A \text{ via } c,\psi \wedge A \leqslant_{\mathrm{S}} B+C \text{ via } c,\varphi$$

holds, by means of a length-of-correctness function that looks for the most recent stage below which the Solovay reductions appear to be correct. Let

$$\begin{aligned}l[s] = \mu t\big((B+C)[s] - (B+C)[\psi(t)] \geqslant c\cdot(A[s]-A[t]) \vee \\ A[s]-A[\varphi(t)] \geqslant c\cdot((B+C)[s]-(B+C)[t])\big).\end{aligned}$$

We can assume that $s > l[s]$ for all s. Notice that if the above condition does hold, then $\lim_s l[s] = \infty$.

As usual, we will call stages at which $l[s]$ appears to be approaching infinity as a limit *expansionary stages*. At such stages, we will attempt to construct a noncomputable c.e. set $Q \leqslant_{\mathrm{S}} B, C$ while satisfying the infinite sequence of subrequirements

$$\mathrm{S}_j:\ \overline{Q} \neq W_j.$$

(In the full construction, the jth subrequirement of $\mathrm{R}_{\langle e,c\rangle}$ will be called $\mathrm{S}_{\langle e,c,j\rangle}$.) We will arrange things so that the failure to satisfy any one of these subrequirements will allow us to construct c.e. sets $\widehat{B}$ and $\widehat{C}$ such that

$$\widehat{B}\sqcup\widehat{C} = A \,\wedge\, \widehat{B} \equiv_{\mathrm{S}} B \,\wedge\, \widehat{C} \equiv_{\mathrm{S}} C,$$

thus satisfying the full requirement R.

We will choose a sequence of witnesses targeted for $Q \cap W_j$, and restrain the set A so that we can exercise control over the approximations to B and C by using ψ. After each witness enters W_j, we will drop all restraint on A for exactly one stage, so that the positive requirements P_k will have a chance to be satisfied. After the length-of-correctness function rises enough for us to monitor what has occurred because of the dropping of the restraint, we will examine the effect on B and C. At this point we will either enumerate our witness into Q and satisfy the subrequirement, or, if this is impossible, split the total change in A since the witness was chosen between $\widehat{B}$ and $\widehat{C}$ in a way that records the changes in B and C, respectively.

As an aid to understanding, we will describe the procedure for this strategy in detail and indicate that it works. The strategy is complicated by the fact that we must divide the interval on which the permission is being sought into four pieces as more and more information about A, B, and C is provided by φ and ψ.

At each stage s at which the length of correctness increases, we will choose a witness x that we hope to enumerate into $Q \cap W_j$, and at the same time restrain A (thought of as a real) on 2^{-x}. Then we will wait for an expansionary stage $s_0 > \psi(s) > s$ such that $x \in W_j$. Let $s_1 > s_0$ be the least subsequent expansionary stage such that $l[s_1] > \varphi(s_0)$. At stage s_1 we drop the restraint on A, and then immediately reimpose this restraint at stage $s_1 + 1$. Finally, we end this attempt at permission by letting $s_2 > s_1 + 1$ be the least subsequent expansionary stage such that $l[s_2] > \psi(s_1 + 1)$. This stage is the one at which we hope to have finally gained permission to enumerate x into Q. If $B[s_2] - B[s_0] \geqslant 2^{-x-1}$ and $C[s_2] - C[s_0] \geqslant 2^{-x-1}$, then we can put x into Q, and the subrequirement S_j is thus permanently satisfied. Otherwise there are two possibilities:

1. If $C[s_2] - C[s_0] < 2^{-x-1}$, then we let $\widehat{B}[s_2] - \widehat{B}[s] = A[s_2] - A[\varphi(s_0)]$ and $\widehat{C}[s_2] - \widehat{C}[s] = A[\varphi(s_0)] - A[s]$.

2. If $B[s_2] - B[s_0] < 2^{-x-1}$ (and $C[s_2] - C[s_0] \geqslant 2^{-x-1}$), then we let $\widehat{C}[s_2] - \widehat{C}[s] = A[s_2] - A[\varphi(s_0)]$ and $\widehat{B}[s_2] - \widehat{B}[s] = A[\varphi(s_0)] - A[s]$.

In either case, we choose a new witness x' greater any number yet mentioned in the construction, restrain A on $2^{-x'}$, and repeat the entire cycle again, starting at s_2.

The point of this procedure is that we put the amount by which A changed during the stage at which it was unrestrained into the hatted set associated to the set that changed significantly, and the controlled part of A into the hatted version of the set that did not change enough to allow the enumeration of x into Q. If we are never able to enumerate a witness into Q, this procedure will give rise to an infinite sequence of pairs of stages at which we make the right decisions about which part of A to put into the hatted versions of each set.

We sum up the important facts about this sequence in the following definition. Call a computable sequence $s_2(-1) < s_0(0) < s_1(0) < s_2(0) < s_0(1) < \cdots$ *good for R* if there exists a computable sequence $x(0) < x(1) < \cdots$ and c.e. sets $\widehat{B}$ and $\widehat{C}$ such that for every $j \geqslant 0$,

(1) $s_0(j)$, $s_1(j)$, and $s_2(j)$ are expansionary stages;

(2) $s_0(j) > l[s_0(j)] > \psi(s_2(j-1))$;

(3) $s_1(j) > l[s_1(j)] > \varphi(s_0(j))$;

(4) $s_2(j) > l[s_2(j)] > \psi(s_1(j) + 1)$;

(5) $A[s_2(j)] - A[s_1(j) + 1] < 2^{-x(j)}$;

(6) $A[s_1(j)] - A[s_2(j-1)] < 2^{-x(j)}$;

(7) $C[s_2(j)] - C[s_0(j)] < 2^{-x(j)-1}$ if and only if $\widehat{B}[s_2(j)] - \widehat{B}[s_2(j-1)] = A[s_2(j)] - A[\varphi(s_0(j))]$ and $\widehat{C}[s_2(j)] - \widehat{C}[s_2(j-1)] = A[\varphi(s_0(j))] - A[s_2(j-1)]$; and

(8) $B[s_2(j)] - B[s_0(j)] < 2^{-x(j)-1}$ and $C[s_2(j)] - C[s_0(j)] \geqslant 2^{-x(j)-1}$ if and only if $\widehat{C}[s_2(j)] - \widehat{C}[s_2(j-1)] = A[s_2(j)] - A[\varphi(s_0(j))]$ and $\widehat{B}[s_2(j)] - \widehat{B}[s_2(j-1)] = A[\varphi(s_0(j))] - A[s_2(j-1)]$.

The last two properties ensure that if we fail to make Q noncomputable, then the "hatted" sets we build are *sw*-equivalent to the original sets B and C, thus satisfying the requirement. More precisely, using a proof involving a long sequence of sums and Lemma 1.1 above, we can show the following:

Lemma 3.1. *If there is a sequence that is good for R, then there is a c.e. splitting* $A = \widehat{B} \sqcup \widehat{C}$ *such that* $B \equiv_S \widehat{B}$ *and* $C \equiv_S \widehat{C}$.

We can show that our strategy will give rise to a good sequence if we infinitely often ask for and fail to receive permission to enumerate into $Q \cap W_j$.

Going further into the details entails giving a $\mathbf{0}'''$-priority construction and proving a sequence of technical lemmas. Since this is merely intended to be an extended abstract, we omit these further details. □

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