Medvedev degrees of generalized r.e. separating classes

Douglas Cenzer and Peter G. Hinman

Turing: for $f, g \in {}^\omega\omega$,

$$f \leq_T g \quad \iff \quad \exists\Phi\,[f = \Phi(g)]$$

Mučnik: for $P, Q \subseteq {}^\omega\omega$,

$$\begin{aligned} P \leq_w Q \quad &\iff \quad (\forall g \in Q)(\exists f \in P) f \leq_T g \\ &\iff \quad (\forall g \in Q)\exists\Phi\,[\Phi(g) \in P] \end{aligned}$$

Medvedev: for $P, Q \subseteq {}^\omega\omega$,

$$P \leq_M Q \quad \iff \quad \exists\Phi\,[\Phi : Q \to P]$$

$$P \equiv_M Q \quad \Longleftrightarrow \quad P \leq_M Q \ \wedge \ Q \leq_M P$$

$$\mathsf{dg}_M(P) = \{\, Q : P \equiv_M Q \,\}$$

$$\mathsf{dg}_M(P) \leq \mathsf{dg}_M(Q) \quad \Longleftrightarrow \quad P \leq_M Q$$

$$\begin{aligned} \mathsf{dg}_M(P) \ \Cup \ \mathsf{dg}_M(Q) &= \mathsf{dg}_M(P \ \Cup \ Q) \\ &= \mathsf{dg}_M(\{\, f \oplus g : f \in P,\ g \in Q \,\}) \end{aligned}$$

$$\begin{aligned} \mathsf{dg}_M(P) \ \Cap \ \mathsf{dg}_M(Q) &= \mathsf{dg}_M(P \ \Cap \ Q) \\ &= \mathsf{dg}_M(\{\, (0)^\frown f : f \in P \,\} \\ &\qquad \cup \{\, (1)^\frown g : g \in Q \,\}) \end{aligned}$$

$\mathbf{0}_M = \mathsf{dg}_M(P)$ for any P with a recursive element

$$\mathcal{D}_k := \{\, \mathsf{dg}_M(P) : P \subseteq {}^{\omega}k \text{ is a } \Pi^0_1 \text{ class} \,\}$$

$$\begin{aligned} \mathsf{S}_2(A_0, A_1) = \{ f \in {}^{\omega}2 : \\ A_0 \subseteq \{\, x : f(x) = 1 \,\} \subseteq \omega \setminus A_1 \} \\ = \{\, f \in {}^{\omega}2 : (\forall x \in \omega)\; x \notin A_{f(x)} \,\} \end{aligned}$$

$$\mathsf{S}_k(A_0, \ldots, A_{k-1}) = \{\, f \in {}^{\omega}k : (\forall x \in \omega)\; x \notin A_{f(x)} \,\}$$

$$\begin{aligned} \mathsf{DNR}_k &= \mathsf{S}_k(\mathsf{K}_0, \ldots, \mathsf{K}_{k-1}) \\ &= \{\, f \in {}^{\omega}k : (\forall x \in \omega)\{x\}(x) \not\simeq f(x) \,\} \end{aligned}$$

DNR_2 is largest in $\mathcal{D}_2 = \mathcal{D}_k$

$$\mathsf{DNR}_2 >_M \mathsf{DNR}_3 >_M \cdots >_M \mathsf{DNR}_k >_M \cdots >_M \mathbf{0}_M$$

$(A_0, \ldots, A_{k-1})$ is $\leq m$-***intersecting*** iff
for all $i_0 < i_1 < \cdots < i_m < k$

$$A_{i_0} \cap A_{i_1} \cap \cdots \cap A_{i_m} = \emptyset$$

$$\mathcal{S}_k^m := \{\mathsf{dg}_M(\mathsf{S}_k(A_0, \ldots, A_{k-1})) : (A_0, \ldots, A_{k-1}) \text{ is } \leq m\text{-intersecting}\}$$

DNR_k is largest in $\mathcal{S}_k^1$

$$\mathcal{S}_k^1 \subseteq \mathcal{S}_k^2 \subseteq \cdots \subseteq \mathcal{S}_k^{k-1}$$

$\mathcal{S}_k^m$ is an upper semi-lattice but not a lattice

Main Results

(1) for $q = \lceil \frac{k}{m} \rceil$,

$$\mathcal{S}_k^m = \mathcal{S}_q^1 \ \mathbb{V} \ \mathcal{S}_{q+1}^1 \ \mathbb{V} \ \cdots \ \mathbb{V} \ \mathcal{S}_k^1$$

(2) for $l < \lceil \frac{k}{m} \rceil$,

$$\mathcal{S}_l^1 \cap \overline{\mathcal{S}_k^m} = \{\mathbf{0}_M\}, \text{ in particular}$$
$$l < k \quad \Longrightarrow \quad \mathcal{S}_l^1 \cap \overline{\mathcal{S}_k^1} = \{\mathbf{0}_M\}$$

(3) $\mathcal{S}_k^m$ and $\mathcal{L}_k^m$ have the ***splitting property***:

$$\mathbf{p} < \mathbf{q} \Longrightarrow \exists \mathbf{q}^+, \mathbf{q}^-$$
$$\left[\mathbf{p} < \mathbf{q}^+, \mathbf{q}^- < \mathbf{q} \text{ and } \mathbf{q}^+ \ \mathbb{V} \ \mathbf{q}^- = \mathbf{q}\right]$$

in particular, they are densely ordered

(1) for $q = \lceil \frac{k}{m} \rceil$,

$$\mathcal{S}_k^m \supseteq \mathcal{S}_q^1 \ \Cup \ \mathcal{S}_{q+1}^1 \ \Cup \ \cdots \ \Cup \ \mathcal{S}_k^1$$

$\lceil \frac{11}{3} \rceil = 4$ so $\mathcal{S}_4^1, \mathcal{S}_5^1, \ldots, \mathcal{S}_{11}^1 \subseteq \mathcal{S}_{11}^3$:

$$\mathsf{S}_5 \underbrace{(A_0, \ldots, A_4)}_{\leq \text{ 1-intersecting}} \equiv_M \mathsf{S}_{11} \underbrace{(A_0, \ldots, A_4, A_0, \ldots, A_4, A_0)}_{\leq \text{ 3-intersecting}}$$

$$\mathsf{S}_5(A_0, \ldots, A_4) \subseteq \quad \mathsf{S}_{11}(A_0, \ldots, A_4, A_0, \ldots, A_4, A_0)$$

hence $\geq_M$

$\leq_M$ via $f \mapsto (x \mapsto f(x) \pmod 5)$

$\mathcal{S}^2_3 \subseteq \mathcal{S}^1_2 \ \Cup \ \mathcal{S}^1_3$

$\mathsf{S}_3 \underbrace{(A_0, A_1, A_2)}_{\leq \text{ 2-intersecting}} \equiv_M$

$$\bigvee\!\!\!\!\bigvee_{i<3} \underbrace{\mathsf{S}_2(A_i, \bigcap_{j\neq i} A_j)}_{\in \mathcal{S}^1_2} \ \Cup \ \underbrace{\mathsf{S}_3(A^*_0, A^*_1, A^*_2)}_{\in \mathcal{S}^1_3}$$

$A^*_i = \{\, x \in A_i : x \text{ not enum earlier in any other } A_j \,\}$

$\geq_M\colon \quad f \mapsto f_0 \oplus f_1 \oplus f_2 \oplus f \quad \text{where}$

$$f_i(x) = \begin{cases} 0, & \text{if } f(x) = i \\ 1, & \text{otherwise} \end{cases}$$

$$x \notin A_{f(x)} \implies \begin{cases} f_i(x) = 0 \implies x \notin A_i \\ f_i(x) = 1 \implies x \notin \bigcap_{j\neq i} A_j \end{cases}$$

$$\mathsf{S}_3(A_0, A_1, A_2) \leq_M$$

$$\bigvee_{i<3} \mathsf{S}_2(A_i, \bigcap_{j \neq i} A_j) \;\vee\; \mathsf{S}_3(A_0^*, A_1^*, A_2^*)$$

$$f_0 \oplus f_1 \oplus f_2 \oplus g \mapsto f \quad \text{where}$$

$$f(x) = \begin{cases} i, & \text{if some (least) } f_i(x) = 0 \\ g(x), & \text{otherwise} \end{cases}$$

$$f_i(x) = 0 \implies x \notin A_i \implies x \notin A_{f(x)}$$

$$\bigwedge_{i<3} f_i(x) = 1 \implies x \in \text{at most one } A_i \text{ so}$$

$$x \notin A^*_{g(x)} \implies x \notin A_{g(x)}$$

$$\mathcal{S}_4^2 \subseteq \mathcal{S}_2^1 \;\Cup\; \mathcal{S}_3^1 \;\Cup\; \mathcal{S}_4^1$$

$$\mathsf{S}_4 \underbrace{(A_0, A_1, A_2, A_3)}_{\leq \text{ 2-intersecting}} \equiv_M$$

$$\bigcurlyvee_{i<j<4} \underbrace{\mathsf{S}_3(A_i, A_j, \bigcap_{k\neq i,j} A_k)}_{\in \mathcal{S}_3^2 \subseteq \mathcal{S}_2^1 \Cup \mathcal{S}_3^1} \;\Cup\; \underbrace{\mathsf{S}_4(A_0^*, A_1^*, A_2^*, A_3^*)}_{\in \mathcal{S}_4^1}$$

$$\mathcal{S}_4^3 \subseteq \mathcal{S}_2^1 \;\Cup\; \mathcal{S}_3^1 \;\Cup\; \mathcal{S}_4^1$$

$$\mathsf{S}_4 \underbrace{(A_0, A_1, A_2, A_3)}_{\leq \text{ 3-intersecting}} \equiv_M$$

$$\bigcurlyvee_{i<4} \underbrace{\mathsf{S}_2(A_i, \bigcap_{j\neq i} A_j)}_{\in \mathcal{S}_2^1} \;\Cup\; \underbrace{\mathsf{S}_4(A_0^{**}, A_1^{**}, A_2^{**}, A_3^{**})}_{\in \mathcal{S}_4^2 \subseteq \mathcal{S}_2^1 \Cup \mathcal{S}_3^1 \Cup \mathcal{S}_4^1}$$

$$A_i^{**} = \{\, x \in A_i : x \text{ not earlier in two other } A_j \,\}$$

$$\mathcal{S}_5^2 \subseteq \mathcal{S}_3^1 \ \Cup\ \mathcal{S}_4^1 \ \Cup\ \mathcal{S}_5^1$$

$$\mathsf{S}_5 \underbrace{(A_0, \ldots, A_4)}_{\leq \text{ 2-intersecting}} \equiv_M$$

$$\bigvee_{i<j<k<5} \underbrace{\mathsf{S}_4(A_i, A_j, A_k, \bigcap_{l \neq i,j,k} A_l)}_{\in \mathcal{S}_4^2 \subseteq \mathcal{S}_2^1 \Cup \mathcal{S}_3^1 \Cup \mathcal{S}_4^1} \ \Cup\ \underbrace{\mathsf{S}_5(A_0^*, \ldots, A_4^*)}_{\in \mathcal{S}_5^1}$$

This uses $\mathcal{S}_2^1$ so does not give the result

$(A_i, A_j, A_k, \bigcap_{l \neq i,j,k} A_l)$ is "strongly" $\leq$ 2-intersecting

$\mathcal{S}_4^{2,1} \subseteq \mathcal{S}_3^1 \ \Cup\ \mathcal{S}_4^1$ as desired

$$\mathcal{S}_7^2 \subseteq \mathcal{S}_4^1 \ \Vvee \ \mathcal{S}_5^1 \ \Vvee \ \mathcal{S}_6^1 \ \Vvee \ \mathcal{S}_7^1$$

$$\mathsf{S}_7 \underbrace{(A_0, \ldots, A_6)}_{\leq \text{2-intersecting}} \equiv_M$$

$$\bigvee_{i_0<i_1<i_2<i_3<i_4<7} \underbrace{\mathsf{S}_3(A_{i_0}, \ldots, A_{i_4}, \bigcap_{k \neq i_j} A_k)}_{\in \mathcal{S}_6^{2,\,1} \subseteq \mathcal{S}_5^{2,\,2} \Vvee \mathcal{S}_6^1 \subseteq \mathcal{S}_4^1 \Vvee \mathcal{S}_5^1 \Vvee \mathcal{S}_6^1}$$

$$\Vvee \ \underbrace{\mathsf{S}_4(A_0^*, \ldots, A_6^*)}_{\in \mathcal{S}_7^1}$$

(2) if $l < \lceil \frac{k}{m} \rceil$ so $k > lm$ and

$$\Phi : \mathsf{S}_k \underbrace{(A_0, \ldots, A_{k-1})}_{\leq\, m\text{-intersecting}} \to \mathsf{S}_l \underbrace{(B_0, \ldots, B_{l-1})}_{\leq\, 1\text{-intersecting}}$$

then exists a recursive $g \in \mathsf{S}_l(B_0, \ldots, B_{l-1})$.

$$T_j^x := \{ \sigma \in {}^{<\omega}k : (\exists f \supseteq \sigma)\Phi(f)(x) = j \}$$

T is p-***fat*** iff T includes a p-branching subtree

$$(\exists j < l)\ T_j^x \text{ is } (m+1)\text{-fat}$$

$$x \in B_j \implies \bigcup_{i \neq j} T_i^x \text{ is } (k-m)\text{-fat}$$

$$\implies T_j^x \text{ is not } (m+1)\text{-fat}$$

$$g(x) := \text{least } j < l\ \left[T_j^x \text{ is } (m+1)\text{-fat}\right]$$