On Computable Compact Operators on Banach Spaces

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Outline

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Definition (Compact operator)

Let X, Y be Banach spaces. An operator $T : X \to Y$ is called *compact*, if

- it is a linear operator such that
- the closure of the image TB_X of the closed unit ball $B_X := \{x \in X : ||x|| \leq 1\}$ is compact in Y.

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$$\begin{aligned} \mathcal{B}_\infty(X,Y) &:= \{T : X \to Y \ : \ T \text{ compact}\} \\ \mathcal{B}(X,Y) &:= \{T : X \to Y \ : \ T \text{ linear and bounded}\} \end{aligned}$$

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Proposition

Any compact operator is necessarily bounded: $\mathcal{B}_\infty(X, Y) \subseteq \mathcal{B}(X, Y)$.

Proposition

Let X, Y, Z be Banach spaces.

- *If $T, S \in \mathcal{B}_\infty(X, Y)$ and $a \in \mathbb{F}$ then*
  - *$aT \in \mathcal{B}_\infty(X, Y)$ and*
  - *$S + T \in \mathcal{B}_\infty(X, Y)$.*
- *If $S \in \mathcal{B}(Y, Z)$, $T \in \mathcal{B}(X, Y)$ and S or T is compact then $ST \in \mathcal{B}_\infty(X, Z)$.*
- *Theorem of Schauder:*

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Proof.

Use of sequential convergence. □

Definition (Schauder Basis)

Let X be a Banach space. Let $e := (e_i)_{i\in\mathbb{N}}$ be a sequence in X. Then e is called a *Schauder basis* of X (or *basis* for short), if any point $x \in X$ can be uniquely represented as

$$x = \sum_{i=0}^{\infty} x_i e_i$$

with $x_i \in \mathbb{F}$.

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- Natural projections: $P_n : X \to X$, $\sum_{i=0}^{\infty} x_i e_i \mapsto \sum_{i=0}^{n} x_i e_i$.

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- Basis constant: $c := \sup_{n\in\mathbb{N}} \|P_n\|$.
- Coordinate functionals: $e'_n : X \to \mathbb{F}$, $\sum_{i=0}^{\infty} x_i e_i \mapsto x_n$.

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Definition (Computable Banach space)

A *computable Banach space* $(X, || \, ||, e)$ is

- a separable Banach space $(X, || \, ||)$
- together with a fundamental sequence $e : \mathbb{N} \to X$ (i.e. the linear span of range(e) is dense in X) such that
- the induced metric space is a computable metric space
- that makes the linear operations (addition and multiplication with scalars) computable.

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The *induced computable metric space* is the space (X, d, α_e) where

- d is given by $d(x, y) := ||x - y||$ and
- $\alpha_e : \mathbb{N} \to X$ is defined by $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$.

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A tupel (X, d, α) is called *computable metric space*, if

- (X, d) is a separable metric space,
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Definition (Cauchy representation)

The *Cauchy representation* $\delta_X : \Sigma^\omega \to X$ of a computable metric space X is defined such that a sequence $p \in \Sigma^\omega$ represents a point $x \in X$ ($\delta_X(p) = x$), if

- p encodes a sequence $(\alpha(n_i))_{i\in\mathbb{N}}$ and
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Here *rapid* means that $d(\alpha(n_i), \alpha(n_j)) < 2^{-j}$ for all $i > j$.

Outline

Definition (Image representation)

Let X, Y be separable Banach spaces. We define a representation $\delta_{\mathcal{B}_\infty(X,Y)}$ of the set $\mathcal{B}_\infty(X,Y)$ of compact operators $T : X \to Y$ by

$$\delta_{\mathcal{B}_\infty(X,Y)}\langle p, q\rangle = T \quad :\iff \quad [\delta_X \to \delta_Y](p) = T \text{ and } \delta_{\mathcal{K}(Y)}(q) = \overline{TB_X}\ .$$

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A name $\langle p, q\rangle$ of a compact operator T contains two types of information,

- a name p of T as a continuous map and
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Computably compact vs. computable and compact:

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Computably compact vs. computable and compact:

- Computably compact operators are compact and computable.
- Is any compact and computable operator necessarily computably compact?

Proposition (Functionals as compact operators)

Let X be a computable Banach space. Then $X' = \mathcal{B}_\infty(X, \mathbb{F})$ and the representations $\delta_{X'}$ and $\delta_{\mathcal{B}_\infty(X,\mathbb{F})}$ are computably equivalent.

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Definition (Dual space representation)

Let X be a separable Banach space. We define a representation $\delta_{X'}$ of the dual space X' by

$$\delta_{X'}\langle p, q\rangle = f :\iff [\delta_X \to \delta_{\mathbb{F}}](p) = f \text{ and } \delta_{\mathbb{R}}(q) = ||f||.$$

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Proof.

- For any functional $f : X \to \mathbb{F}$ we obtain $\overline{fB_X} = \overline{B}(0, ||f||)$.
- The map $R : \mathbb{R} \to \mathcal{K}(\mathbb{F}), r \mapsto \overline{B}(0, r)$ is computable and has a computable right inverse.

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- There exists a computable functional $f : \ell_2 \to \mathbb{R}$ without computable norm (Brattka and Yoshikawa, 2006).

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- There exists a computable functional $f : \ell_2 \to \mathbb{R}$ without computable norm (Brattka and Yoshikawa, 2006).
- Therefore there exists a computable compact operator that is not computably compact.

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Computably compact vs. computable and compact:

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- There are computable and compact operators that are not computably compact.

Proposition (Operator norm of compact operators)

Let X, Y be computable Banach spaces. Then the map

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For any linear bounded operator $T : X \to Y$ we obtain

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Proposition (Operator norm of compact operators)

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- Then we can compute $\sup ||\overline{TB_X}|| \in \mathbb{R}$, as the suprema over compact subsets of $\mathbb{R}$ is computable (Weihrauch, 2000).

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- Computably compact operators are compact and computable and have a computable norm.
- Is any compact and computable operator with computable norm necessarily computably compact?

Example

Let $a = (a_k)_{k \in \mathbb{N}}$ be a computable sequence of reals such that $\|a\|_2 < 1$, but $\|a\|_2$ is non-computable.

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$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Then T is compact since its range is two-dimensional, T is obviously computable as the sequence a is computable, the norm $||T|| = 1$ is computable, but T is not computably compact and the image TB_{ℓ_2} is not even located since $d_{TB_{\ell_2}}(e_1) = 1 - ||a||_2$ is not computable.

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Computably compact vs. computable and compact with computable norm:

- Computably compact operators are compact and computable and have a computable norm.
- There are computable and compact operators with computable norm that are not computably compact.

Theorem (Computable Theorem of Schauder)

Let X, Y be computable Banach spaces with computable dual spaces X', Y'. Then the map

$$A : \mathcal{B}_\infty(X, Y) \to \mathcal{B}_\infty(Y', X'), T \mapsto T'$$

that maps any compact operator $T : X \to Y$ to its adjoint T', is computable.

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Definition (Computable dual space)

Let X be a computable Banach space. We say that X has a *computable dual space* X', if there exists a sequence $e^* : \mathbb{N} \to X'$ such that

1. $(X', || \, ||, e^*)$ is a computable Banach space,
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Proof.

- Computable composition from left and functionals as compact operators.
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Proposition (Composition)

Let X, Y, Z be computable Banach spaces. Then composition

$$\circ :\subseteq \mathcal{C}(Y,Z) \times \mathcal{B}_\infty(X,Y) \to \mathcal{B}_\infty(X,Z), (S,T) \mapsto ST,$$

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Proof.

- Composition of continuous operators is $([\delta_Y \to \delta_Z], [\delta_X \to \delta_Y], [\delta_X \to \delta_Z])$–computable (Weihrauch, 2000).
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Computably Compact Operators

Composition from Left

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Composition from left is computable and x' and $\mathcal{B}_\infty(X,\mathbb{F})$ are computably equivalent. □

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Theorem (Computable Theorem of Schauder)

Let X, Y be computable Banach spaces with computable dual spaces X', Y'. Then the map

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Outline

Definition (Cauchy representation)

Let X be a Banach space with a shrinking basis e and let $(Y, || \, ||, f)$ be a separable Banach space. We define a numbering α_{ef} of some finite rank operators $T : X \to Y$ by

$$\alpha_{ef}\langle k, \langle n_0, \dots, n_k\rangle, \langle l_0, \dots, l_k\rangle\rangle(x) := \sum_{i=0}^{k} \alpha_{e'}(n_i)(x)\alpha_f(l_i).$$

Then range(α_{ef}) is dense in the set $\mathcal{B}_\infty(X, Y)$ of compact operators. We denote by $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}$ the Cauchy representation of $\mathcal{B}_\infty(X, Y)$ induced by α_{ef}.

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Definition (Shrinking basis)

Let X be a Banach space with basis e. For $f \in X'$ let

$$\|f\|_{(n)} := \sup\left\{ \frac{|fx|}{||x||} : x \in \overline{\operatorname{span}\{e_i : i > n\}}, x \neq 0 \right\}.$$

The e is called *shrinking*, if $\lim_{n\to\infty} \|f\|_{(n)} = 0$ for all $f \in X'$.

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In case that a basis e is shrinking, the coordinate functionals e' form a basis of X' and one obtains a Cauchy representation of X'.

For a Banach space X with a shrinking basis e the dual space X' has the approximation property, and the set $\mathcal{F}(X, Y)$ of finite rank operators is dense in $\mathcal{B}_\infty(X, Y)$.

Definition (Cauchy representation)

Let X be a Banach space with a shrinking basis e and let $(Y, || \, ||, f)$ be a separable Banach space. We define a numbering α_{ef} of some finite rank operators $T : X \to Y$ by

$$\alpha_{ef}\langle k, \langle n_0, ..., n_k\rangle, \langle l_0, ..., l_k\rangle\rangle(x) := \sum_{i=0}^{k} \alpha_{e'}(n_i)(x)\alpha_f(l_i).$$

Then range(α_{ef}) is dense in the set $\mathcal{B}_\infty(X, Y)$ of compact operators. We denote by $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}$ the Cauchy representation of $\mathcal{B}_\infty(X, Y)$ induced by α_{ef}.

Questions

- Under which conditions is the space $(\mathcal{B}_\infty(X,Y), \|\ \|, \alpha_{ef})$ a computable Banach space and

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- Under which conditions is the space $(\mathcal{B}_\infty(X, Y), || \, ||, \alpha_{ef})$ a computable Banach space and
- when is the corresponding Cauchy representation $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}$ equivalent to $\delta_{\mathcal{B}_\infty(X,Y)}$?

Proposition

Let X be a computable Banach space with a shrinking and computable basis e and let Y be a computable Banach space. Then the injection

$$\mathrm{inj} : \mathcal{B}_\infty(X, Y) \hookrightarrow \mathcal{C}(X, Y)$$

is $(\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}, [\delta_X \to \delta_Y])$–computable.

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Proposition

Let X be a computable Banach space with computable basis e. Then the corresponding sequence $(e'_n)_{n\in\mathbb{N}}$ of coordinate functionals is a computable sequence in $\mathcal{C}(X)$.

Proposition

Let X be a computable Banach space with a monotone, shrinking and computable basis e and let $(Y, || \, ||, f)$ be a computable Banach space. Then the image map

$$\mathrm{im} : \mathcal{B}_\infty(X, Y) \to \mathcal{K}(Y), T \mapsto \overline{TB_X}$$

is $(\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}, \delta_{\mathcal{K}(Y)})$–computable.

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is $(\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}, \delta_{\mathcal{K}(Y)})$–computable.

A basis $e = (e_i)_{i \in \mathbb{N}}$ is called *monotone*, if its basis constant $c := \sup_{n \in \mathbb{N}} ||P_n|| = 1$.

Proposition

Let X be a computable Banach space with a monotone, shrinking and computable basis e and let $(Y, || \, ||, f)$ be a computable Banach space. Then the image map

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Lemma

Let X be a Banach space with a monotone basis e. Then

$$P_n B_X = B_n := \left\{ x = \sum_{i=0}^{n} x_i e_i \in X : ||x|| \leq 1 \right\} \subseteq B_X.$$

Proposition

Let X be a computable Banach space with a monotone, shrinking and computable basis e and let $(Y, || \, ||, f)$ be a computable Banach space. Then the image map

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is $(\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}, \delta_{\mathcal{K}(Y)})$–computable.

Corollary

Let X be a computable Banach space with a computable and monotone basis e. Then $(P_n B_X)_{n\in\mathbb{N}}$ is a computable sequence in $\mathcal{K}(X)$.

Proposition

Let X be a computable Banach space with a monotone, shrinking and computable basis e and let Y be a computable Banach space. Then $(\mathcal{B}_\infty(X,Y), ||\,||, \alpha_{ef})$ is a computable Banach space and $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \leq \delta_{\mathcal{B}_\infty(X,Y)}$.

Proposition

Let X be a computable Banach space with a monotone, shrinking and computable basis e and let Y be a computable Banach space. Then $(\mathcal{B}_\infty(X,Y), ||\ ||, \alpha_{ef})$ is a computable Banach space and $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \leq \delta_{\mathcal{B}_\infty(X,Y)}$.

Proof.

- The injection into $\mathcal{C}(X,Y)$ and the closure of the image of the unit ball are computable. Therefore, the Cauchy representation is computably reducible to the image representation.
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Let X be a computable Banach space with a monotone, shrinking and computable basis e and let Y be a computable Banach space. Then $(\mathcal{B}_\infty(X,Y), || \, ||, \alpha_{ef})$ is a computable Banach space and $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \leq \delta_{\mathcal{B}_\infty(X,Y)}$.

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Definition (Computably shrinking)

Let X be a computable Banach space with a computable basis e. Then e is called *computably shrinking*, if there exists a $(\delta_{X'}, \delta_{\mathbb{N}^\mathbb{N}})$–computable map $S : X' \rightrightarrows \mathbb{N}^\mathbb{N}$ such that for each fixed $f \in X'$ any $m \in S(f)$ is a shrinking modulus for $f \in X'$.

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Definition (Shrinking modulus)

Let X be a Banach space with basis e and let $f \in X'$. Then $m : \mathbb{N} \to \mathbb{N}$ is called a *shrinking modulus* of f, if $||f||_{(m(k))} < 2^{-k}$ for all $k \in \mathbb{N}$.

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Proposition (Effective approximation property)

Let X be a computable Banach space with a computable basis e. Then there is a computable multi-valued map $\mathsf{AP} : \mathcal{K}(X) \rightrightarrows \mathbb{N}^{\mathbb{N}}$ such that for each fixed $K \in \mathcal{K}(X)$ any $m \in \mathrm{AP}(K)$ is a basis modulus of K.

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Definition (Basis modulus)

Let X be a Banach space with a basis and corresponding natural projections P_n. Let $K \subseteq X$ be a subset. Then $m : \mathbb{N} \to \mathbb{N}$ is called a *basis modulus* for K, if $||P_{m(k)}x - x|| < 2^{-k}$ for any $x \in K$.

Theorem

Let X be a computable Banach space with a monotone, computably shrinking and computable basis e and let Y be a computable Banach space with a computable basis f. Then $(\mathcal{B}_\infty(X, Y), || \, ||, \alpha_{ef})$ is a computable Banach space and $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \equiv \delta_{\mathcal{B}_\infty(X,Y)}$.

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Corollary

Let X and Y be infinite-dimensional computable Hilbert spaces with computable orthonormal bases e and f, respectively. Then $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \equiv \delta_{\mathcal{B}_\infty(X,Y)}$ and $(\mathcal{B}_\infty(X,Y), || \, ||, \alpha_{ef})$ is a computable Banach space.

Outline

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If, additionally, X and Z are computably reflexive, then the following holds:

6. *if $T : X \to Y$ and $T' : Y' \to X'$ are computable, then the operation $\widehat{T} : \mathcal{B}_\infty(Y, Z) \to \mathcal{B}_\infty(X, Z), S \mapsto ST$ is computable.*