

**Primitive Recursiveness  
of Real Numbers  
under Different Representations**

Qingliang Chen  
Department of Computer Science  
Sun Yat-sen University, China

(joint work with Kaile Su and Xizhong Zheng)

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## Abstract

- In mathematics, various **representations** of real numbers have been investigated, such as nested interval representations, Cauchy sequences, ...
- The **effective** versions of these representations are equivalent in the sense that they define the same notion of computability of real numbers.
- However, the **primitive recursive (p.r., for short)** versions of these representations can lead to different notions of p.r. real numbers.
- We summarize the known results about the primitive recursiveness of real numbers for different representations as well as show some new relationships.

Our goal is to clarify systematically how the primitive recursiveness depends on the representations of the real numbers.

# The Encoding of Real Numbers

In order to compute a real number, we need to represent it in some way as input or output.

The concrete means to represent a real number:

- by decimal expansion,
- by Cauchy sequences,
- by Dedekind cut,
- by continued fraction,
- by nested intervals,
- ...

# The Definition of Computable Real Numbers

The original definition of **computable real numbers** [Turing, 1936]:

*“the ‘computable’ numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means”*

**Turing-Church Thesis:** “Finite means = Procedure of a Turing Machine”.

Namely,  $x$  is *computable* if there is a computable function  $f : \mathbb{N} \rightarrow \{0, 1, \dots, 9\}$  such that

$$x = \sum_{n=0}^{\infty} f(n) \cdot 10^{-(n+1)}$$

Here we consider only the real numbers in the interval  $[0, 1]$ .

## Equivalent Definitions of Computable Real Numbers

The computability of real numbers can be equivalently defined by means of Cauchy sequences, Dedekind cuts and other representations of real numbers [Robinson51, Myhill53, Rice54].

That is,

**the computability of reals is independent of their representations.**

The class of computable reals will be denoted by **EC** (for Effectively Computable).

## The Subrecursive Real Numbers

Besides the computability, the subrecursive real numbers like

- primitive recursive real numbers,
- polynomial time computable real numbers,
- ...

have also been discussed.

The result is that:

**The different notions of subrecursive real numbers could be defined if different representations are used.**

## The Subrecursive Real Numbers

Specker [Speker49] is the first who investigates this problem and he shows that

p.r. Dedekind cuts  $\subsetneq$  p.r. decimal expansions  
 $\subsetneq$  p.r. Cauchy sequences.

Later on, Peter [Peter51], Mostowski [Mostowski57], and Lehman [Lehman60] investigated other versions of p.r. reals and showed some more relations between the notions of p.r. real numbers based on different representations.

However, not every important representation of real numbers have been discussed and there is no a systematical overview about the subrecursiveness of real numbers so far.

# The Representations of Real Numbers

Now we give the precise definition of the relativization of all above representations to a class  $\mathcal{F}$  of functions.

**Definition 1** Let  $\mathcal{F}$  be a class of functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  or  $f : \mathbb{N} \rightarrow \mathbb{Q}$ , and let  $x \in [0, 1]$  be a real number.

1.  $x$  has an  $\mathcal{F}$ -Cauchy representation ( $x \in \mathbf{CS}(\mathcal{F})$ ) if there is a function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  in  $\mathcal{F}$  such that the sequence  $f(s)$  converges to  $x$  effectively.
2.  $x$  has an  $\mathcal{F}$ -Dedekind cut representation ( $x \in \mathbf{DC}(\mathcal{F})$ ) if there is a function  $f : \mathbb{N}^2 \rightarrow \{0, 1\}$  in  $\mathcal{F}$  such that  $f(n, m) = 1$  if and only if  $n/(m + 1) < x$
3.  $x$  has a  $b$ -adic representation ( $x \in \mathbf{bAE}(\mathcal{F})$ ) if there is a function  $f : \mathbb{N} \rightarrow \{0, 1, \dots, b - 1\}$  in  $\mathcal{F}$  such that  $x = \sum_{s=0}^{\infty} f(s) \cdot b^{-(s+1)}$ . Especially, for  $b = 10$  and  $b = 2$ , they are a decimal and binary representation, respectively.
4.  $x$  has a continued fraction representation ( $x \in \mathbf{CF}(\mathcal{F})$ ) if there is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  in  $\mathcal{F}$  such that  $x = f(1) + \frac{1}{f(2) + \frac{1}{f(3) + \dots}}$ .
5.  $x$  has a nested interval representation ( $x \in \mathbf{NI}(\mathcal{F})$ ) if there are two functions  $f, g : \mathbb{N} \rightarrow \mathbb{Q}$  in  $\mathcal{F}$  such that  $f(s) \leq f(s + 1) \leq x \leq g(s + 1) \leq g(s)$  for all  $s$  and  $\lim_{s \rightarrow \infty} (g(s) - f(s)) = 0$ .

## The Hierarchy inside p.r. Reals

When we limit the function class  $\mathcal{F}$  to be p.r. functions, it will lead to the definitions of various versions of “p.r. real numbers”.

Denote by

- $\mathbf{R}_4$ =the classes of real numbers which have p.r. continued fraction,
- $\mathbf{R}_3$ =the classes of real numbers which have p.r. Dedekind cut,
- $\mathbf{R}_2^b$ =the classes of real numbers which have p.r.  $b$ -adic expansion,
- $\mathbf{R}_1$ =the classes of real numbers which have p.r. Cauchy representation,
- $\mathbf{R}_0$ = the classes of real numbers which have p.r. nested interval representations.

That is,

$$\mathbf{R}_4 = \mathbf{CF}(\mathcal{F}), \mathbf{R}_3 = \mathbf{DC}(\mathcal{F}), \mathbf{R}_2^b = \mathbf{bAE}(\mathcal{F}), \mathbf{R}_1 = \mathbf{CS}(\mathcal{F}), \mathbf{R}_0 = \mathbf{NI}(\mathcal{F}).$$

We will see that the relationship among these classes is as follows.

$$\mathbf{R}_4 \subsetneq \mathbf{R}_3 \subsetneq \mathbf{R}_2^b \subsetneq \mathbf{R}_1 \subsetneq \mathbf{R}_0 = \mathbf{EC}$$

## The Nested Interval Representation

First of all, we give an equivalent definitions of nested interval representation of a real.

**Definition 2** *A p.r. approximation of a real  $x$  is a pair  $(a, e)$  of p.r. functions  $a, e : \mathbb{N} \rightarrow \mathbb{Q}$  such that*

- 1.  $e$  is monotonically decreasing and converges to 0; and*
- 2.  $|a(n) - x| \leq e(n)$  holds for all  $n \in \mathbb{N}$ .*

**Lemma 3** *A real number has a p.r. nested interval representation if and only if it has a p.r. approximation.*

## The Nested Interval Representation

**Theorem 4 (Skordev01)** *A real number is computable if and only if it has a p.r. approximation.*

Proof sketch: “ $\Leftarrow$ ”: trivial;

“ $\Rightarrow$ ”: if  $x$  is computable, then there is a computable function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  such that  $|f(n) - x| \leq 2^{-n}$ .

Let  $M$  be a Turing machine which computes the function  $f$ , then there is a p.r. predicate  $T$  such that  $T(n, y, s)$  holds if and only if the machine  $M$  with the input  $n$  outputs  $y$  in  $s$  steps. Therefore,  $f(n) = y$  if and only if  $T(n, y, s)$  for some  $s$ . Then we can easily construct a p.r. approximation for  $x$ .

**Corollary 5** *A real number has a p.r. nested interval representation if and only if it is computable. That is,*

$$\mathbf{R}_0 = \mathbf{EC}.$$

## Cauchy Representation

There are several different but equivalent versions of p.r. “Cauchy representation” of a real.

**Proposition 6** *Let  $x$  be a real number. Then the following conditions are equivalent.*

1.  $x$  has a p.r. Cauchy representation  $f$ , i.e.,  $|x - f(n)| \leq 2^{-n}$  for all  $n$ ;
2. There is a p.r. function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  and a p.r. function  $e : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$(\forall n, m \in \mathbb{N})(m \geq e(n) \implies |f(m) - x| \leq 2^{-n}). \quad (1)$$

3. There is a p.r. function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  such that  $\lim_{n \rightarrow \infty} f(n) = x$  and

$$(\forall n, m \in \mathbb{N})(n \leq m \implies |f(n) - f(m)| \leq 2^{-n}). \quad (2)$$

4. There is a p.r. function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  and a p.r. function  $e : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\lim_{n \rightarrow \infty} f(n) = x$  and

$$(\forall n, m \in \mathbb{N})(n, m \geq e(k) \implies |f(n) - f(m)| \leq 2^{-k}). \quad (3)$$

## The Cauchy Representation

**Theorem 7** *The class of Cauchy p.r. real numbers is closed under arithmetical operations. That is, it is a field.*

By a simply diagonalization against all p.r. Cauchy representations, it is easy to construct a computable real number which does not have p.r. Cauchy representation. That is we have

**Theorem 8** *The class of Cauchy p.r. reals is a proper subset of computable reals. That is,*

$$\mathbf{R}_1 \subsetneq \mathbf{R}_0 = \mathbf{EC}.$$

## Representations by $b$ -adic Expansion

Notice that, if  $x$  has a p.r.  $b$ -adic expansions, i.e.,  $x = \sum_{n=0}^{\infty} f(n) \cdot b^{-(n+1)}$  for a p.r. function  $f$ , then the function  $g$  defined by  $g(n) = \sum_{i=0}^n f(i) \cdot b^{-(i+1)}$  is a p.r. Cauchy representation of  $x$ . This observation implies immediately that  $\mathbf{R}_2 \subseteq \mathbf{R}_1$ .

On the other hand, Specker [Specker49] shows that  $\mathbf{R}_2^{10} \neq \mathbf{R}_1$

**Lemma 9 (Specker49)** *There are p.r. functions  $u$  and  $v$  such that the function  $q$  defined by*

$$q(n) := u((\mu t \geq n)(v(t) = 0))$$

*is not a p.r. function.*

## Representations by $b$ -adic Expansion

**Theorem 10 (Specker49)** *There exists a decimal p.r. real number  $x$  such that  $3x$  is not decimal p.r.*

Proof sketch: We need to find a real number  $x := 0.a_0a_1a_2a_3 \cdots$  such that the function  $a$  defined by  $a(i) := a_i$  is p.r., but for  $3x = b.b_0b_1b_2b_3 \cdots$ , the function  $b$  defined by  $b(i) := b_i$  is not p.r. Let's look first at an example. For simplicity, we restrict that  $a_i \in \{1, 3, 5\}$  for all  $i$ .

$$x = 0.a_0a_1a_2 \cdots := 0.335131155331511 \cdots$$

$$3x = b.b_0b_1b_2 \cdots := 1.00539346599453? \cdots$$

where “?” can be 3 or 4 depending on the first non-3 digit of  $x$  after this place is equal to 1 or 5.

$$b(n) \equiv 0 \pmod{2} \iff a((\mu t > n)(a(t) \neq 3)) = 5. \quad (4)$$

Then it is not hard to prove it by Lemma 9.

## Representations by $b$ -adic Expansion

**Remark 11** *By the same kind of construction of Specker's, it is not hard to see that any p.r.  $b$ -adic expansion reals is not closed under arithmetical operations.*

**Corollary 12** *The class of  $b$ -adic expansions p.r. reals is a proper subset of the class of Cauchy p.r. real numbers, i.e.,*

$$\mathbf{R}_2^b \subsetneq \mathbf{R}_1.$$

## The Relationship among the Class $\mathbf{R}_2^b$

Now we explore the relationship among the class  $\mathbf{R}_2^b$  for different  $b$ 's.

**Theorem 13 (Mostowski57)** *Let  $b, d > 1$ . If a power of  $b$  is divisible by  $d$ , then any  $b$ -adic p.r. real is also  $d$ -adic p.r., i.e.,  $\mathbf{R}_2^b \subseteq \mathbf{R}_2^d$ .*

The inverse of the Theorem 13 was an open question in Mostowski. A positive answer to this question was given by Lachlan [Lachlan63].

**Theorem 14 (Lachlan63)** *Let  $b, d > 1$ ,  $\mathbf{R}_2^b \subseteq \mathbf{R}_2^d$  if and only if  $d$  divides a power of  $b$ , i.e.,  $(\exists k, s)(b^k = s \cdot d)$ .*

## The Dedekind Cut Representation

By definition, the (left) Dedekind cut of a real number  $x$  is the set  $C_x := \{r \in \mathbb{Q} : r < x\}$  of rational numbers. However, we can avoid this by considering the relation  $L_x$  defined by

$$L_x(m, n) \iff m/(n+1) < x$$

and say that  $x$  has a *p.r. Dedekind cut* if the relation  $L_x$  is p.r.

**Definition 15** For any positive real number  $x$  and natural numbers  $n, m$ , we have

$$m/(n+1) < x \iff m \leq \lfloor (n+1) \cdot x \rfloor$$

where  $\lfloor y \rfloor$  denotes the integer part of the real number  $y$ , i.e., the maximal natural number  $t$  such that  $t \leq y$ . The function  $f(n) := \lfloor n \cdot x \rfloor$  is also called *Beatty function* or *Beatty sequence* of  $x$ .

By the above observation, we have immediately the following description of Dedekind p.r. real numbers.

**Theorem 16 (Peter50)** A real number is Dedekind p.r. if and only if its Beatty function is p.r.

## Another Description of Dedekind p.r. Real Numbers

Another description of Dedekind p.r. real numbers uses the Hurwitz's characteristic of real numbers based on the Farey sequences [Hurwitz1894].

**Definition 17** *The Farey sequence of order  $n$  is the increasing sequence of irreducible fractions between 0 and 1 which have denominators less than or equal to  $n$ . In general, the Farey sequence  $F_n$  of order  $n$  is an increasing sequence of irreducible fractions  $s/t$  such that  $0 \leq s \leq t \leq n$  and  $t \neq 0$ .*

**Example 18** *For example, the Farey sequence of order four is  $F_4 = \{0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1\}$ .*

**Remark 19** *For any three successive terms  $s_1/t_1, s_2/t_2$  and  $s_3/t_3$  of  $F_n$ , the middle one is always the “mediant” of its neighborhoods, i.e.,  $s_2/t_2 = (s_1 + s_3)/(t_1 + t_3)$*

## Another Description of Dedekind p.r. Real Numbers

*Hurwitz characteristic* of  $x \in (0, 1)$  is defined as follows.

- Initially, let  $\gamma_x(0) = 0$ .  $\gamma_x(1) := 0$  if  $x < 1/2$  and  $\gamma_x(1) := 1$  if  $x > 1/2$ .
- Suppose that  $\gamma_x(n)$  is defined and  $x$  is located between two adjacent fractions in some Farey sequence of the lowest order which have been used so far, say  $s_1/t_1$  and  $s_2/t_2$ . Then defined  $\gamma_x(n+1) := 0$  if  $x < (s_1 + s_2)/(t_1 + t_2)$ , and  $\gamma_x(n+1) := 1$  if  $x > (s_1 + s_2)/(t_1 + t_2)$ .

**Theorem 20 (Lehman60)** *A real number  $x \in (0, 1)$  is Dedekind p.r. if and only if its Hurwitz characteristic  $\gamma_x$  is p.r.*

## The p.r. Dedekind Cut and p.r. $b$ -adic Reals

Now we discuss the relation between Dedekind p.r. and  $b$ -adic real numbers.

**Theorem 21 (Specker49)** *Let  $b > 1$ . Any Dedekind p.r. real is  $b$ -adic p.r. But there is a  $b$ -adic p.r. real which is not Dedekind p.r. That is,  $\mathbf{R}_3 \subsetneq \mathbf{R}_2^b$ .*

Proof sketch: If  $x \in [0, 1]$  is Dedekind p.r., then, by Theorem 16, the Beatty function  $\lfloor n \cdot x \rfloor$  is a p.r. function. The  $b$ -adic expansion  $f$  of  $x$  can be defined recursively by  $f(0) := \lfloor x \rfloor$  and, for any  $n$ ,

$$\begin{aligned} f(n+1) &:= \max \left\{ t \leq b : \sum_{i=0}^n f(i) \cdot b^{-(i+1)} + t \cdot b^{-(n+2)} \leq x \right\} \\ &= \max \left\{ t \leq b : \sum_{i=0}^n f(i) \cdot b^{(n+1-i)} + t \leq \lfloor b^{(n+2)} x \rfloor \right\} \end{aligned}$$

Thus,  $x$  is  $b$ -adic expansion p.r. and hence  $\mathbf{R}_3 \subseteq \mathbf{R}_2^b$ .

To prove the inequality  $\mathbf{R}_3 \neq \mathbf{R}_2^b$ , assume by contradiction that  $\mathbf{R}_3 = \mathbf{R}_2^b$  for some  $b > 1$ . Choose a natural number  $d$  such that  $d$  does not divide  $b^k$  for any  $k \in \mathbb{N}$ . According to Theorem 14, we have  $\mathbf{R}_3 = \mathbf{R}_2^b \not\subseteq \mathbf{R}_2^d$ . This contradicts the fact  $\mathbf{R}_3 \subseteq \mathbf{R}_2^d$ .

## The p.r. Uniform Base Expansion

Here the uniform dependence of a  $b$ -adic expansion to its base  $b$  refers to the dependence of each digits to the base. This can be described by a “uniform digits function”.

**Definition 22** A function  $f : \mathbb{N}^2 \rightarrow \mathbb{N}$  is called a uniform base expansion of a real number  $x$  if, for all  $n \in \mathbb{N}$  and any natural number  $b \geq 2$ ,

$$0 \leq f(b, n) < b \quad \text{and} \quad x = \sum_{n=0}^{\infty} f(b, n)b^{-(n+1)}. \quad (5)$$

If this function  $f$  is p.r., then we say that  $x$  has a p.r. uniform base expansion.

## The p.r. Uniform Base Expansion

The following theorem shows the relationship between p.r. uniform base expansions and p.r. Dedekind cuts.

**Theorem 23** *A real number has a p.r. uniform base expansion if and only if it is Dedekind p.r.*

Proof sketch: “ $\Rightarrow$ ”: If  $x \in [0, 1]$  has a p.r. uniform base expansion  $f_x$ , for any natural number  $b > 1$ , we have

$$b \cdot x = f_x(b, 0) + \frac{f_x(b, 1)}{b^1} + \frac{f_x(b, 2)}{b^2} + \dots = \sum_{i=0}^{\infty} f(b, i)b^{-i}.$$

“ $\Leftarrow$ ”: If  $x$  is a Dedekind p.r. real number, then its Beatty function  $\lfloor n \cdot x \rfloor$  is p.r. The uniform base expansion  $f_x$  of  $x$  can be obviously defined inductively by

$$f_x(0) := \lfloor b \cdot x \rfloor$$

$$f_x(n + 1) := \lfloor b^{n+1} \cdot x \rfloor - b \cdot \lfloor b^n \cdot x \rfloor.$$

and hence  $f_x$  is p.r. That is,  $x$  has a p.r. uniform base expansion.

## The Cauchy p.r. Reals and Dedekind p.r. Reals

Between Cauchy p.r. reals and Dedekind p.r. reals Specker has shown the following decomposition theorem.

**Theorem 24 (Specker49)** *Every Cauchy p.r. real number is the sum of two Dedekind p.r. real numbers.*

# The Continued Fraction Expansion Representation

The continued fraction is another very interesting representation of real numbers, any irrational number  $x$  can be represented as an infinite continued fraction  $x = [b_0, b_1, b_2, \dots]$  where  $b_n \geq 1$  and  $b_n \in \mathbb{N}$  for  $n \in \mathbb{N}$ , where

$$x = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \dots}}$$

For rational numbers, there are finite  $b_n$ 's.

## The Continued Fraction Expansion Representation

**Definition 25** An irrational number  $x$  is called primitive-recursively irrational (p.r. irrational for short) if it is possible to find a primitive-recursively lower bound of the distance between  $x$  and any given rational number. More precisely, there is a p.r. function  $f$  such that for all positive integers  $m$  and  $n$

$$\left| x - \frac{m}{n} \right| > \frac{1}{f(n)}. \quad (6)$$

**Theorem 26 (Lehman60)** A real number  $x$  has a p.r. continued fraction expansion if and only if it is Cauchy p.r. and p.r. irrational.

## The Continued Fraction Expansion and Dedekind Cut Representation

By means of the characterization of the Theorem 26, Lehman can further show not every Dedekind p.r. real has a p.r. continued fraction expansion. To this end, we need another technical lemma which can be easily proved from the Lemma 9.

**Lemma 27 (Lehman60)** *There is a p.r. function  $\lambda : \mathbb{N} \rightarrow \{1, 2\}$  which takes value 1 infinitely many times such that the function  $\sigma$  defined by  $\sigma(n) := \mu m \geq n (\lambda(m) \neq 2)$  is not p.r.*

# The Continued Fraction Expansion and Dedekind Cut Representation

**Theorem 28 (Lehman60)** *There is a Dedekind p.r. real number  $x$  which is not primitive recursively irrational.*

Proof sketch: The real number  $x$  is given by its Hurwitz characteristic  $\gamma_x$  which is defined by  $\gamma_x(0) = 0$  and

$$\gamma_x(n + 1) := \gamma_x(n) \text{ if } \lambda(n) = 2$$

$$\gamma_x(n + 1) := 1 \div \gamma_x(n) \text{ if } \lambda(n) \neq 2$$

where  $\lambda$  is the p.r. function of Lemma 27. By Theorem 20  $x$  is Dedekind p.r.

As it is shown by Hurwitz [Hurwitz1894], the continued fraction  $x = [0, b_1, b_2, b_3, \dots]$  of  $x$  have the following form.

$$\underbrace{0 \ 0 \ \dots \ 0}_{b_1} \underbrace{1 \ 1 \ \dots \ 1}_{b_2} \underbrace{0 \ 0 \ \dots \ 0}_{b_3} \underbrace{1 \ 1 \ \dots \ 1}_{b_4} 0 \dots \dots$$

By the definition of  $\gamma_x$ , we have  $\gamma_x(n) \neq \gamma_x(n + 1)$  if and only if  $\lambda(n) \neq 2$ . Thus the function  $\sigma$  of the Lemma 27 should be p.r. because

$$\sigma(n) = (\mu m \geq n)(\lambda(m) \neq 2) \tag{7}$$

$$= (\mu m \leq \sum_{i \leq n+1} b_i)(m \geq n \ \& \ \lambda(m) \neq 2). \tag{8}$$

This contradicts Lemma 27. Hence we conclude that  $(b_n)$  is not p.r. and  $x$  does not have a p.r. continued fraction expansion. By Theorem 26,  $x$  is nor p.r. irrational.

# The Continued Fraction Expansion and Dedekind Cut Representation

**Corollary 29 (Lehman60)** *The class of real numbers which have p.r. continued fraction expansions is a proper subset of the class of Dedekind p.r. real numbers, i.e.,*

$$\mathbf{R}_4 \subsetneq \mathbf{R}_3.$$

## Conclusion

We have seen that, the p.r. reals under different representations form a comprehensive hierarchy:

$$\mathbf{CF}(\mathcal{F}) \subsetneq \mathbf{DC}(\mathcal{F}) \subsetneq \mathbf{bAE}(\mathcal{F}) \subsetneq \mathbf{CS}(\mathcal{F}) \subsetneq \mathbf{NI}(\mathcal{F}) = \mathbf{EC}.$$

for the class  $\mathcal{F}$  of p.r. functions. Among these classes, it seems that the class  $\mathbf{CS}(\mathcal{F})$  might be properly regarded as the class of *primitive recursive reals*.

## Future Work: More p.r. Representations for Reals

### Pierce Expansion Representation

$$x = \frac{1}{f(1)} - \frac{1}{f(1)f(2)} + \frac{1}{f(1)f(2)f(3)} - \dots$$

where  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a non-decreasing p.r. function.

### Engel Expansion Representation

$$x = \frac{1}{f(1)} + \frac{1}{f(1)f(2)} + \frac{1}{f(1)f(2)f(3)} + \dots$$

where  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a non-decreasing p.r. function.

### Harmonic Expansion Representation

$$x = f(1) + \frac{1}{2} \left( f(2) + \frac{1}{3} \left( f(2) + \frac{1}{4} (f(3) + \dots) \right) \right)$$

where  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a p.r. function.

**THANK YOU !**