Enumerations of Effectively Closed Sets

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Outline

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   - Effectively Closed Sets
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   - Acceptable Equivalence
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Outline

Motivation

Results for c.e. Sets

Many results in classical computability theory are derived from a study of indices of partial computable functions.

Examples

(1) **Enum. Theorem**– indices may be treated as arguments
(2) S^m_n **Theorem**– arguments may be treated as indices

So results should be indep. of the chosen system of indices! The following will be shown:

Fact

An enumeration of all effectively closed sets may be based on a system of indices arising from the partial computable functions.

Motivation

Expected Results for Eff. Closed Sets

Example

Suppose $W_0, W_1, \ldots$ is an enum. of the c.e. sets (domains of the p.c. funct.). Consider the follow. unif. enum. $T_0, T_1, \ldots$ of trees:

$$\sigma \in T_e \iff (\forall \tau \sqsubseteq \sigma)\ \tau \notin W_{e,|\sigma|}$$

This provides for an enum. $[T_0], [T_1], \ldots$ of all eff. closed sets.

Therefore have the following:

Expectations

(1) Many results for c.e. sets have corr. results for eff. cl. sets.
(2) Results should be indep. of the chosen system of indices.

This talk addresses points (1) and (2).

Notation

(1) $\mathfrak{C}$ is the space of closed $P \subseteq 2^\omega$ $(P \neq \emptyset)$

(2) ω^* is the set of finite strings of elements of ω

(3) $|\sigma|$ equals the length of $\sigma \in \omega^*$

(4) $\sigma \sqsubseteq \tau$ means $|\sigma| \leqslant |\tau|$ and $\sigma(i) = \tau(i)$ $(\forall i < |\sigma|)$

(5) $\sigma^\frown i$ means σ concatenated with i

(6) $x \restriction n$ equals $(x(0), x(1), ..., x(n-1))$

(7) $I(\sigma)$ equals $\{x \in \omega^\omega : \sigma \sqsubset x\}$

Trees

Definition

A **tree** T is a subset of ω^* closed under initial segments.

Example: If $63972 \in T$ then $6, 63, 639, 6397 \in T$

Definition

Given $P \in \mathfrak{C}$, define (the unique tree without dead ends) T_P by

$$T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$$

Definition

$\sigma \in T$ is a **dead end** $\iff$ $\sigma^\frown n \notin T$ for all n.

Example: If $567 \in T$ but $567^\frown n \notin T$ for all n.

Closed Sets

Definition

Given a tree $T \subseteq \omega^*$ define $[T] \subseteq \omega^\omega$ by

$$[T] = \{x \in \omega^\omega : (\forall n)\ x{\restriction}n \in T\}$$

Fact

$K \subseteq \omega^\omega$ is **closed** $\iff$ $K = [T]$ for some T

Definition

$K \subseteq \omega^\omega$ is **effectively closed** $\iff$ $K = [T]$ for some comp. T

Definition

$K \subseteq \omega^\omega$ is **decidable effectively closed** $\iff$
$K = [T]$ for some comp. T without dead ends

Applications

Effectively Closed Sets show up in:

- **Model Theory:** The set of complete consistent extensions of an axiomatizable theory.
- **Computability Theory:** The class of separating sets of two c.e. sets.
- **Algebra:** The set of prime ideals of a c.e. Boolean algebra or commutative ring with unity.
- **Graph Theory:** The set of solutions to many problems for computable graphs
(Ex: Hamiltonian circuits, vertex partitions)
- **Orderings:** The set of decompositions of a computable poset into chains and antichains or into finitely many linear orderings.

Enumeration Theory

A **numbering** of a collection C of objects is a surjective map $F : \omega \rightarrow C$.

An **enumeration without repetition** is an injective numbering.

Given two numberings ν and u, we say that u is **acceptable with respect to** ν, denoted $\nu \leq u$, iff there is a total computable function f such that $\nu = u \circ f$.

Then u is **acceptable** if it acceptable with respect to all numberings.

We say that ν and u are **acceptably equivalent**, denoted $\nu \equiv u$, iff $\nu \leq u$ and $u \leq \nu$.

Enumeration Theory

Note

$\equiv$ is an equivalence relation. Furthermore, if we denote

$$\mathcal{L}(C) = \{ \text{ all numberings of } C \text{ modulo } \equiv \}$$

then $\mathcal{L}(C)$ is an upper semilattice under $\leq$. In fact:

- acceptable enum. occur only in the greatest element
- enum. without repetition occur only in minimal elements

These types of enumerations exist for c.e. sets.
Which exist for effectively closed sets?

First we consider accept. and accept. equivalent numberings.
Then we will consider different distinctive numberings
(with/without repetition) of (diff. families of) eff. closed sets.

Outline

Numberings of Eff. Closed Sets

Theorem

An eff. closed set $P \subset \omega^\omega$, has the following equiv. definitions:

(a) $P = [T]$ for some primitive recursive tree $T \subset \omega^*$

(b) $P = [T]$ for some computable tree T

(c) $P = \{x : (\forall n) R(n, x)\}$ for some computable relation R

(d) $P = [T]$ for some Π^0_1 tree $T \subset \omega^*$

From the theorem, we can get numberings of the effectively closed sets based on:

- Primitive Recursive Functions [from (a)]
- Total Computable Functions [from (b)] (non-eff.)
- Universal Π^0_1 Relations [from (c)]
- C.E. Sets (2 Vers.), Halting Problem [from (d)]

Numberings of Eff. Closed Sets

Using (a): $P = [T]$ for some prim rec. T

Numbering 1: Prim. Rec. Functions π_e

For each e, $U_e = \{\emptyset\} \cup \{\sigma : (\forall \tau \sqsubseteq \sigma)\ \pi_e(\langle\tau\rangle) = 1\}$ defines a prim. rec. tree. This enum. contains all prim. rec. trees: observe that if $\{\sigma : \pi_e(\sigma) = 1\}$ is a tree then U_e is that tree.

Using (b): $P = [T]$ for some comp. T

Numbering 2: Total Comp. Functions ϕ_e

Let $T_e = \{\sigma : \phi_e(\sigma) = 1\}$ is a tree. There exist prim. rec. functions ϕ, π s.t. for each e, $T_{\pi(\phi(e))}$ is always of a tree (Cenzer; Remmel). We obtain the following enum., $e \mapsto T_e$ if ϕ_e is total (i.e. $e \in$Tot) and T_e is a tree. Otherwise $e \mapsto T_{\pi(\phi(e))}$. (This is non-effective since Tot is Π^0_2.)

Numberings of Eff. Closed Sets

Using (c): $P = \{x : (\forall n)R(n, x)\}$ for some comp. rel. R

Numbering 3: Universal Π^0_1 Relation U
There is a universal Π^0_1 relation $U \subseteq \omega \times 2^\omega$ such that if $D(x)$ is a Π^0_1 relation then there is an $e \in \omega$ such that $D(x) \leftrightarrow U(e, x)$ (Hinman). Then $e \mapsto \{x : U(e, x)\}$ is a numb. of the eff. cl. sets.

Using (d): $P = [T]$ for some Π^0_1 tree T

Num. 4: C.E. Sets (Vers. 1) $P_e = \omega^\omega \setminus \bigcup_{n \in W_e} I(\sigma_n)$.
Num. 5: C.E. Sets (Vers. 2) $T_e = \{\sigma : (\forall \tau \sqsubseteq \sigma)\ \langle\tau\rangle \notin W_{e,|\sigma|}\}$
Num. 6: Halting Problem $T_e = \{\sigma : (\forall s)\phi^\sigma_{e,s}(e)\uparrow\}$.
Now Im(ψ) codes all eff. cl. sets: let φ be any tree num. of the eff. cl. sets. $\forall n$, let $\phi^\sigma_{g(e)}(n)$ be defined only if $\sigma \notin \varphi(e)$. Then,

$$\sigma \in (\psi \circ g)(e) \iff \phi^\sigma_{g(e)}(g(e))\uparrow \iff \sigma \in \varphi(e)$$

Acceptable Equivalence

Theorem

Let φ and ψ be two of the previously mentioned numberings. If both are not based on total computable functions, then they are acceptably equivalent. If one is based on the total comp. functions, then they are Δ_3^0 acceptably equivalent.

Proof.

Modify the proof that verifies the equivalent definitions of the effectively closed sets. Use the fact that the set of indices for total computable functions is Π_2^0. □

Computable Permutations

Theorem

Suppose that $\{P_i\}_{i\in\mathbb{N}}$ and $\{Q_i\}_{i\in\mathbb{N}}$ are two enumerations of the eff. closed sets resulting from the numberings previously given. Then there is a permutation p s.t. $(\forall x)\ P_x = Q_{p(x)}$. The perm. is Δ^0_3 if either numb. given by the total comp. functions. Otherwise the perm. is computable.

Proof (Outline):

This theorem is true in the case acc. equiv. numberings for c.e. sets, with at least one being the standard numb. for c.e. sets. We will modify this proof for eff. closed sets. We may assume that $\exists$ comp. (Δ^0_3) f, g s.t. $(\forall x)\ P_{f(x)} = Q_x$ and $Q_{g(x)} = P_x$.

1. Split the indices of the eff. closed sets into two sets:

   $$G_e = \{i : P_e = P_i\} \text{ and } H_e = \{i : P_e = Q_i\}$$

   Then $f(H_n) \subset G_n$, $g(G_n) \subset H_n$, and $\mathbb{N} = \cup G_n = \cup H_n$.

Computable Permutations

Proof (Outline) continued ...

2. From [Soa87], it is sufficient to convert f and g into 1-1 computable functions f_1 and g_1 satisfying 1.

Convert f to f_1

Use the Padding Lemma to get a comp. function h s.t:
If $\{i, j, a, b\} \subset \mathbb{N}$ $(i \neq j)$, then $P_a = P_{h(i,a)}$ and $h(i, a) \neq h(j, b)$.

Convert g to g_1

- Select two disjoint comp. insep. c.e. sets A, B.
- Define comp. trees that provide for each x, an infinite no. i s.t. $Q_i = Q_{g(x)}$. If not, then some comp. C sep. A and B! □

Q : These numb. are not only are acc. equiv., but have perm. betw. them pres. the comp. content. Are they acceptable?

Outline

Enumeration (without Repetition): C.E. Sets

Existence of Enumeration without Repetition

- The c.e. sets [Fri58]
- The n-c.e. sets [GonLemSol02]
- Any c.e. class containing all finite sets [PouPut65]
- The computable sets [Muc58, Suz59]

Existence of Enumeration but always with Repetition

- An example of a class of finite sets [Lac65, PouPut65]

Nonexistence of Enumerations

- Classes of infinite c.e. (comp.) sets [Usp55, DekMyh58]

Enumeration (without Repetition): Eff. Closed Sets

Existence of Enumeration without Repetition

- The effectively closed sets [Rai06]
- Conj: n differences of eff. closed sets
- Conj: Any family of eff. cl. sets containing all clopen sets
- Decidable effectively closed sets

Existence of Enumeration but (so far) with Repetition [Bro06]

- *Strongly* homogeneous effectively closed sets
- More generally, any family of eff. closed sets that satisfies a finite set of *string verifiable* relations

Nonexistence of Enumerations

- Conj: Nonclopen eff. closed sets

Decidable Eff. Closed Sets

Enumerating the Trees without Dead Ends

Definition

A tree $T \subseteq 2^*$ (and $[T]$) are **clopen** iff $\exists$ nonempty finite $S \subseteq \mathbb{N}$ such that $T = \emptyset$ or $T = \{\sigma : \sigma \sqsubseteq \sigma_i$ or $\sigma_i \sqsubseteq \sigma$ for some $i \in S\}$.

An enumeration without repetition exists of the decidable eff. closed sets. Furthermore, we have:

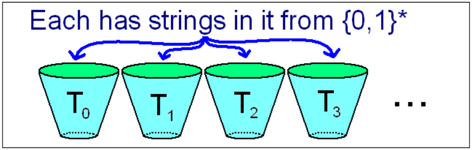
Theorem

Given any eff. enum. of unif. comp. trees, for each n there is an enum. without repet. containing all clopen trees along with all comp. trees with $\leq$ n dead ends that occur in the enumeration.

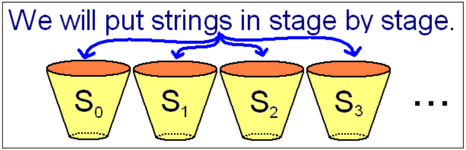
We prove this in what follows for trees without dead ends, that is for $n = 0$. A simple modification gives the general result.

Construction: Preparatory Comments

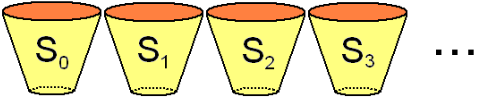
Proof: Let $T_0, T_1, \ldots$ be an eff. enumeration of unif. computable trees. Assume without loss of generality that all clopen trees occur in this enumeration. Now:



We will construct a sequence $S_0, S_1, \ldots$ of trees as follows:

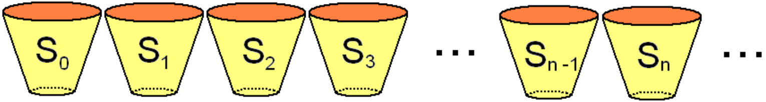


Construction: At Stage 0

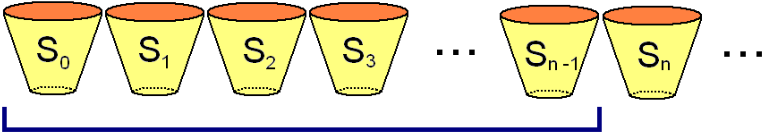


All S_i are empty.

Construction: At Stage n

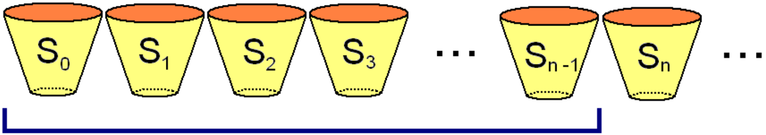


Construction: At Stage n



These already have strings of length $\leqslant 2^{n-1}$

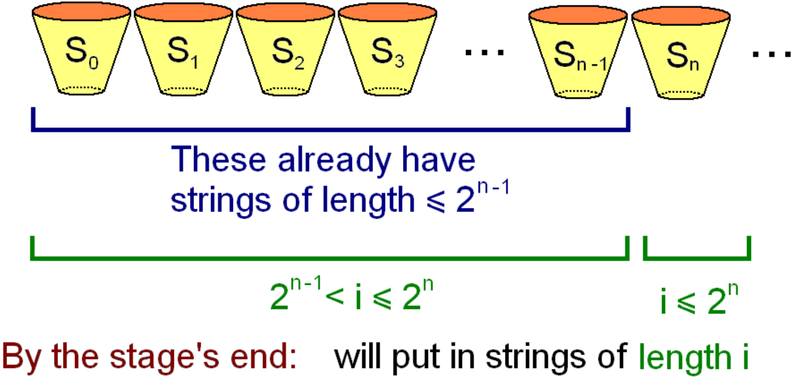
Construction: At Stage n



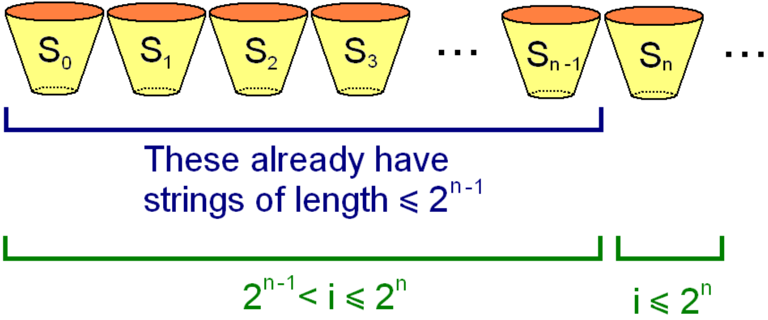
These already have strings of length $\leqslant 2^{n-1}$

By the stage's end: will put in strings of length i

Construction: At Stage n



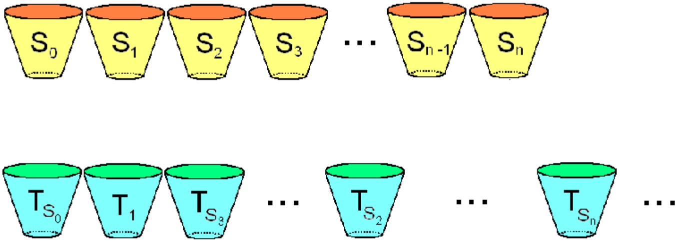
Construction: At Stage n



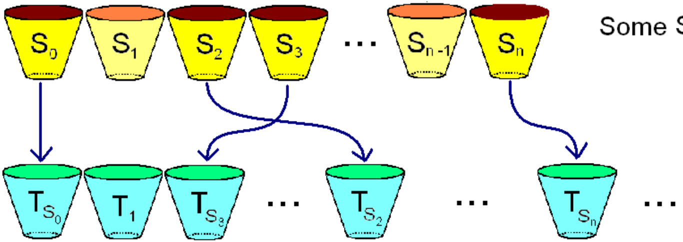
By the stage's end: will put in strings of length i

The length ensures each will be different!

Construction: After Stage n

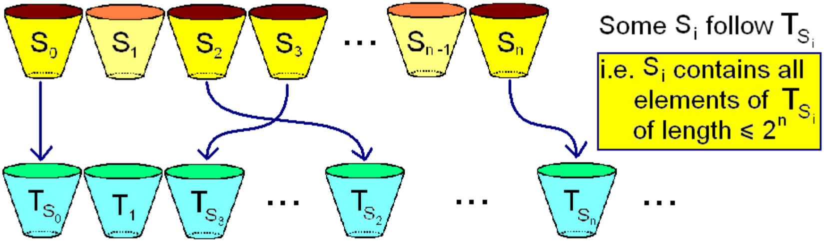


Construction: After Stage n

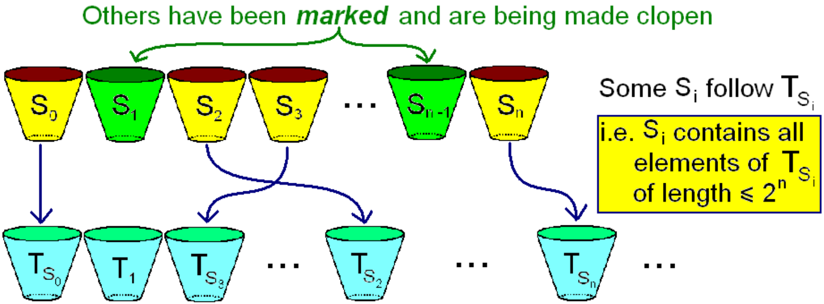


Some S_i follow T_{S_i}

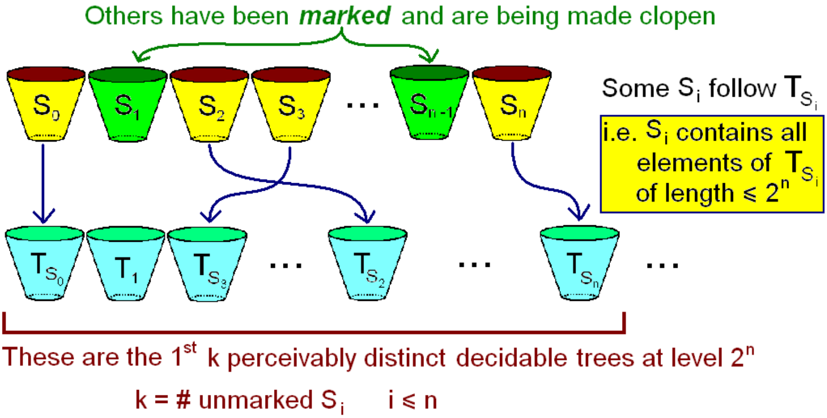
Construction: After Stage n



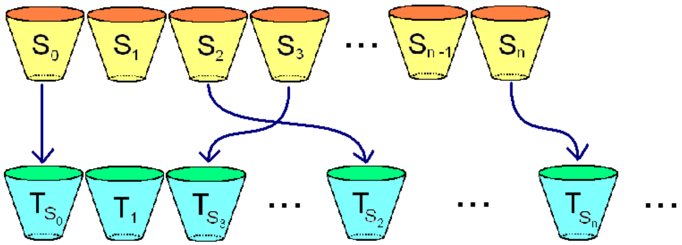
Construction: After Stage n



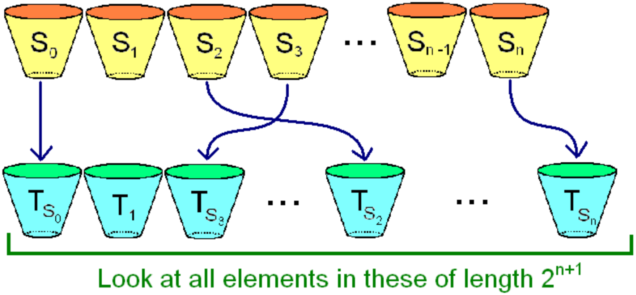
Construction: After Stage n



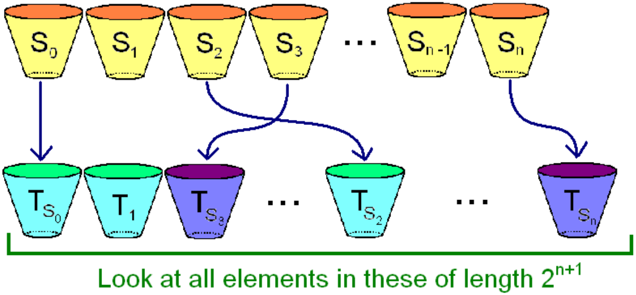
Construction: At Stage n+1



Construction: At Stage n+1

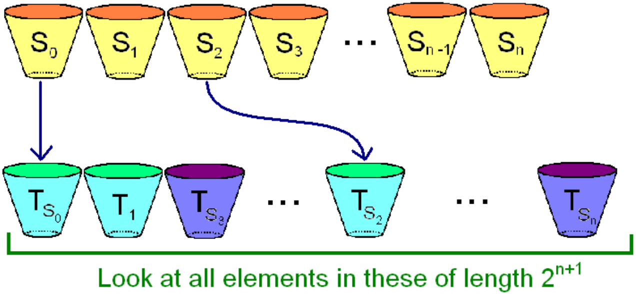


Construction: At Stage n+1



Some now have dead ends.

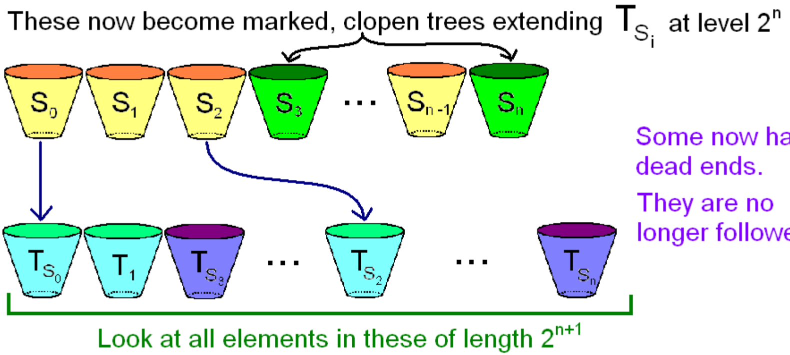
Construction: At Stage n+1



Some now have dead ends.

They are no longer followed!

Construction: At Stage n+1



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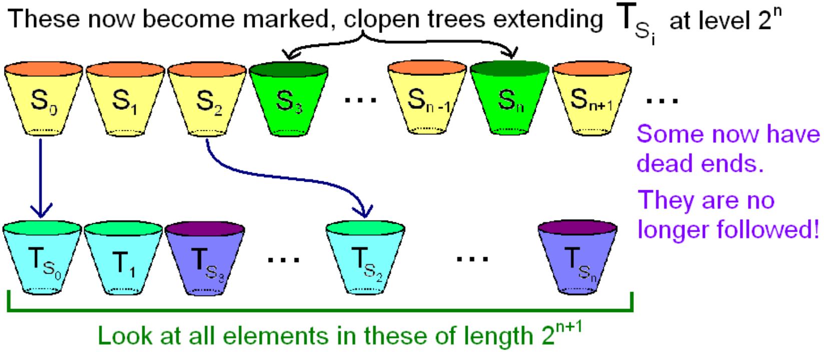
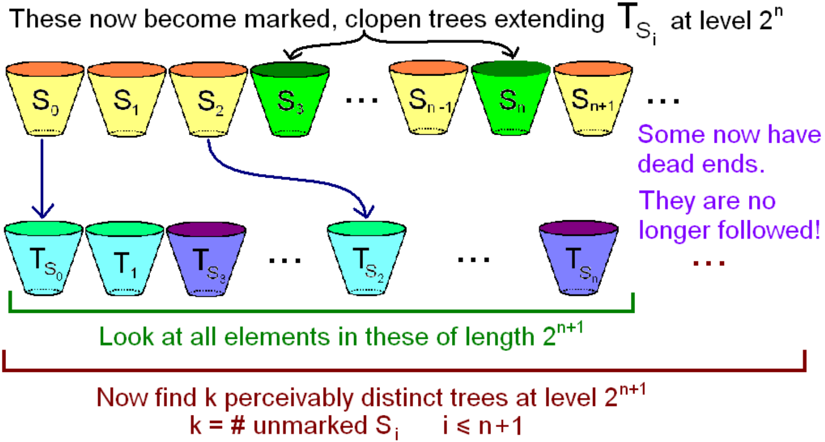
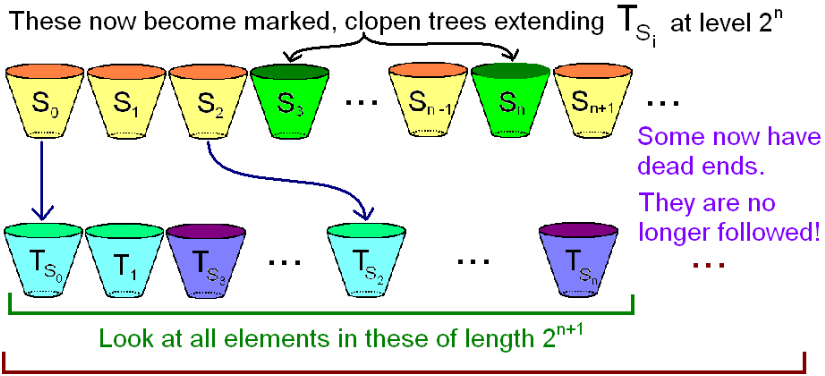
Construction: At Stage n+1



Construction: At Stage n+1



Construction: At Stage n+1



Now find k perceivably distinct trees at level 2^{n+1}
k = # unmarked S_i $\quad i \leq n+1$

Repeat the Procedure

Construction: Verification

$(\forall i)(\exists k)\quad \lim_j S_i^j = T_k$ for some T_k without dead ends

Idea: S_i follows a tree without dead ends or becomes marked and made clopen.

$(\forall i)\quad T_i$ has no dead ends $\longrightarrow$ $(\exists c)\ T_i = S_c$

Idea: If $\widehat{i}$ is the least index such that $T_{\widehat{i}} = T_i$ then $T_{\widehat{i}}$ will eventually be followed by some S_j or else some S_j will 'duplicate' $T_{\widehat{i}}$ as a clopen set.

$i \neq j \quad \longrightarrow \quad S_i \neq S_j$

Idea: At stage n, all the S_i's were ensured to be different at level 2^n.

This completes the proof. Some index set issues result.

Decidable Eff. Closed Sets

Index Set Issues

Now although,

T_e has no dead ends $\Rightarrow P = [T_e]$ is a decidable eff. cl. set

We get that,

P is a dec. eff. cl. set $\not\Rightarrow P = [T_e]$ for some T_e without d. ends

Proof.

Let $S_0, S_1, \ldots$ be an enum. without rep. of all T_i without d. ends.
Constuct T so that $(\forall n)\;\; T \cap 2^{n+1} \neq S_n \cap 2^{n+1}$.

Stage 0. Let $T \cap 2^0 := \{\emptyset\}$.

Stage n+1. Now $T \cap 2^n \neq \emptyset$.
So $\exists$ 2 diff. subtrees of 2^{n+1} ext. $T \cap 2^n$.
Let $T \cap 2^{n+1} :=$ ext. diff. from $S_n \cap 2^{n+1}$. □

Future Directions

We get that $\{e : T_e \text{ has no d. ends}\} \neq \{e : P_e = [T_e] \text{ is decid.}\}$.

A Future Revisitaton of Results

All results in [CenRem98] rel. to dec. Π^0_1 cl. will be revisited.

More generally: What is the complexity of index sets related to different enum. of (different families of) eff. closed sets?

General Enumeration Questions

Which enumerations are acceptable? Which families can or can't be enumerated? Can they be enum. without repetition?

Some specific examples to consider for all of the above include differences of eff. cl. sets, those that contain *n* computable elements, or those that are *thin*. These are all questions I am working on now.

Recognitions and References

Much appreciation and recognition goes to my advisor Douglas Cenzer, Rick Smith, and Seyyed Dashti for many excellent discussions. Thank you all for your time and attention.

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