Looking in reverse at Specker's theorem and continuity

Douglas S. Bridges

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Reverse mathematics: a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

Constructive reverse mathematics: reverse mathematics carried out in Bishop-style constructive math.—that is, using intuitionistic logic (and, where necessary, constructive ZF).

There are two primary foci of the latter:

first, investigating which constructive principles are necessary to prove a given constructive theorem;

secondly, investigating what non-constructive principles are necessary additions to BISH in order to prove a given non-constructive theorem.

Uniform continuity theorem (**UCT**):

Every pointwise continuous mapping from a compact (i.e., complete, totally bounded) metric space into a metric space is uniformly continuous.

This is

intuitionistically valid as a consequence of the full fan theorem;

recursively false: there exists a pointwise continuous $f : [0,1] \longrightarrow [0,1]$ that is not uniformly continuous.

One classical proof of UCT:

Let $f : X \to Y$ be pointwise continuous, where X is compact. Suppose f is not uniformly continuous. Then

- there exist sequences $(x_n), (x'_n)$ in X, and $\alpha > 0$, such that $\rho(x_n, x'_n) \to 0$ as $n \to \infty$, but

$$\rho\left(f(x_n), f(x'_n)\right) \geqslant \alpha$$

for each n.

Apply the Bolzano–Weierstraß theorem:

- there exists a subsequence $(x_{n_k})_{k \geqslant 1}$ converging to a limit x_∞ in X; then $(y_{n_k})_{k \geqslant 1}$ converges to x_∞.

Apply the pointwise continuity of f at x_∞ to obtain k such that $\rho\Big(f(x_{n_k}), f(x'_{n_k})\Big) < \alpha$, a contradiction.

A function $f : X \to Y$ between metric spaces is **uniformly sequentially continuous** if

> for all sequences $(x_n), (x'_n)$ in X such that $\rho(x_n, x'_n) \to 0$ we have $\rho(f(x_n), f(x'_n)) \to 0$.

Uniform continuity implies uniform sequential continuity.

Classically, the second part of the proof of **UCT** shows that the converse holds if X is compact.

For a pointwise continuous function, what is necessary/ sufficient to prove (i) uniform sequential continuity and (ii) uniform continuity?

Recall that the recursive counterexample to **UCT** is based on **Specker's theorem**:

> There exists an increasing sequence of rational points of $[0,1]$ that is eventually bounded away from each point of $[0,1]$.

We introduce the **anti**(thesis of)-**Specker property (AS)**:

> Every sequence of real numbers that is eventually bounded away from each point of $[0,1]$ is eventually bounded away from the entire interval $[0,1]$.

This is classically equivalent to the Bolzano–Weierstraß theorem.

Theorem 1 **AS** *implies that every pointwise continuous map* $f : [0,1] \to \mathbb{R}$ *is uniformly sequentially continuous.*

Lemma 2 *Assume* **AS.** *Let* $f : [0,1] \to \mathbb{R}$ *be pointwise continuous, let* $(x_n)\,(x'_n)$ *be sequences in* $[0,1]$ *such that* $|x_n - x'_n| \to 0$, *and let* $\varepsilon > 0$. *Then*

either $\rho\left(f(x_n), f(x'_n)\right) < \varepsilon$ *for all sufficiently large* n,

or else $\rho\left(f(x_n), f(x'_n)\right) > \varepsilon/2$ *for infinitely many* n.

Lemma 3 *Under the hypotheses of the preceding lemma, if* $\rho\left(f(x_n), f(x'_n)\right) > \varepsilon/2$ *for infinitely many* n, *then we can derive* **LPO.**

Proof of Theorem 1:

Under the hypotheses of Lemma 2, suppose that

$$\rho\left(f(x_n), f(x'_n)\right) > \varepsilon/2$$

for infinitely many n.

Then **LPO** holds, by Lemma 3. But **LPO** implies the Bolzano–Weierstraß theorem, which (as in the classical proof of **UCT**) suffices to rule out the case that $\rho\left(f(x_n), f(x'_n)\right) > \varepsilon/2$ for infinitely many n.

It follows from Lemma 1 that $\rho\left(f(x_n), f(x'_n)\right) < \varepsilon$ for all sufficiently large n.

What about the converse of Theorem 1?

Theorem 4 *If every pointwise continuous map of* $[0,1]$ *into* $\mathbb{R}$ *is uniformly sequentially continuous, then* **AS** *holds.*

Proof: Let (x_n) be a sequence of reals eventually bounded away from each point of $[0,1]$, and construct a spike function with spikes of height 1 centred on the points x_n. This function is pointwise continuous. If it is uniformly sequentially continuous, then (x_n) is eventually bounded way from $[0,1]$.

A subset A of $\mathbb{N}$ is said to be **pseudobounded** if

$$n^{-1}a_n \to 0$$

for each sequence $(a_n)_{n \geqslant 1}$ in A.

Ishihara introduced the following principle, which has become very significant in constructive reverse mathematics:

BD-$\mathbb{N}$ Every countable, pseudobounded subset of $\mathbb{N}$ is bounded.

BD-$\mathbb{N}$ is provable classically, intuitionistically, and in recursive mathematics, but does not appear to be provable within BISH.

Ishihara and Schuster have shown that the proposition

> Every uniformly sequentially continuous function from a compact metric space into $\mathbb{R}$ is pointwise continuous.

is equivalent to **BD**-$\mathbb{N}$.

Theorem 5 *Assume that* **AS** *and* **BD**-$\mathbb{N}$ *hold. Then every pointwise continuous mapping of* $[0,1]$ *into a metric space is uniformly continuous.*

Let $(q_n)_{n \geqslant 1}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Consider any pointwise continuous mapping $f : X \to Y$. Define a triple sequence $\left(\alpha_{m,n,k}\right)_{m,n,k \geqslant 1}$ such that

$$\begin{aligned} \alpha_{m,n,k} &= 0 \Rightarrow \rho(f(q_m), f(q_n)) > \varepsilon/2 \wedge |q_m - q_n| < 1/(k+1), \\ \alpha_{m,n,k} &= 1 \Rightarrow \rho(f(q_m), f(q_n)) < \varepsilon \vee |q_m - q_n| > 1/k. \end{aligned}$$

If $\alpha_{m,n,k} = 0$, set $\nu_{m,n,k} = k$.

If $\alpha_{m,n,k} = 1$, set $\nu_{m,n,k} = 0$.

Then

$$A = \left\{\nu_{m,n,k} : m, n, k \geqslant 1\right\}$$

is a countable subset of $\mathbb{N}$.

The next two lemmas enable us to prove that A is pseudobounded.

Lemma 6 *Assuming* **AS***, let* $(a_n)_{n \geqslant 1}$ *be a sequence in* A*, and let* $0 < \alpha < \beta$*. Then either there exists* n *such that* $n^{-1}a_n > \alpha$*, or else* $n^{-1}a_n < \beta$ *for all* n*.*

Lemma 7 *Under the hypotheses of* Lemma 6*,* $k^{-1}a_k \longrightarrow 0$ *as* $k \longrightarrow \infty$*.*

Proof of Theorem 5:

It follows from Lemma 7 that A is pseudobounded.

Hence, by **BD**-$\mathbb{N}$, there exists a positive integer K such that $\nu_{m,n,k} < K$ for all m, n, k.

Consider any (m, n) such that $\rho(f(q_m), f(q_n)) > \varepsilon$.

If $|q_m - q_n| < 1/K$, then $\alpha_{m,n,K} \neq 1$, so $\nu_{m,n,K} = K$, which contradicts our choice of K.

Hence $|q_m - q_n| \geqslant 1/K$.

It follows from the pointwise continuity of f that if $x, y \in X$ and $\rho(f(x), f(y)) > \varepsilon$, then $|q_m - q_n| \geqslant 1/K$.

Generalised anti-Specker property for a subset X of a metric space E : Every sequence in E that is eventually bounded away from each point of X is eventually bounded away from X.

Theorems 1 and 5 generalise as follows:

- If X is compact and has the anti-Specker property, then every pointwise continuous mapping of X into a metric space is uniformly sequentially continuous.

- If X is separable and has the anti-Specker property, and if **BD**-$\mathbb{N}$ holds, then every pointwise continuous mapping of X into a metric space is uniformly continuous.

A proof similar to that of Theorem 5 shows that if X is separable and has the anti-Specker property, and if **BD**-$\mathbb{N}$ holds, then X is **pseudo-compact:**

> Every pointwise continuous mapping of X into $\mathbb{R}$ has totally bounded range (dsb 1976).

In that case, every pointwise continuous mapping of X into a metric space has totally bounded range.

In particular, X itself is totally bounded.

Theorem (dsb and Hannes Diener) The following are equivalent.

(i) $[0,1]$ is pseudocompact.

(ii) Every pointwise continuous mapping of $[0,1]$ into $\mathbb{R}$ is uniformly continuous.

(iii) Every pointwise continuous mapping of $[0,1]$ into $\mathbb{R}$ is Lebesgue integrable.

(iv) Every pointwise continuous mapping of a compact metric space into a metric space is uniformly continuous.

The equivalence of (ii) and (iii) strengthens a result of Iris Loeb (2005).

Outline of proof that (i) $\Rightarrow$ (ii): Assume that $[0,1]$ is pseudocompact.

Step1: $[0,1] \times [0,1]$ is pseudocompact.

Step 2: If $F : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is pointwise continuous, then for all but countably many $r \in \mathbb{R}$ the set

$$\{x \in [0,1] : F(x) \leqslant r\}$$

is either compact or empty.

Step 3: If $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is pointwise continuous and positive-valued, then f is bounded away from 0.

Step 4: Let $f : [0,1] \to \mathbb{R}$ be pointwise continuous. Given $\varepsilon > 0$ and using Step 2, we may assume that

$$S = \left\{ \left(x, x'\right) : 0 \leqslant x, x' \leqslant 1 \wedge \left| f(x) - f(x') \right| \geqslant \varepsilon \right\}$$

is compact and therefore located. Then

$$\begin{aligned} h(x) &= \rho\left((x,x), S\right) \\ &= \inf \left\{ \max \left\{ |x - s|, \left|x' - s'\right| \right\} : \left(s, s'\right) \in S \right\} \end{aligned}$$

exists and defines a pointwise continuous map $h : [0,1] \to \mathbb{R}^+$.

By Step 3, there exists $\delta > 0$ such that $h(x) > \delta$ for all $x \in [0,1]$.

If $x, x' \in [0,1]$ and $|x - x'| < \delta$, then $(x, x') \notin S$, so

$$\left| f(x) - f(x') \right| \leqslant \varepsilon.$$

Now connect **AS** with the fan theorem.

Complete binary fan—the set 2^* of all finite sequences in $\{0,1\}$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a finite or infinite binary sequence. Then for each applicable $n \in \mathbb{N}$,

$$\overline{\alpha}n = (\alpha_1, \ldots, \alpha_n)$$

is called a **restriction** of α.

By a **path** in 2^* we mean a finite or infinite binary sequence.

A path α is **blocked** by a subset B of 2^* if some restriction of α is in B.

$B \subset 2^*$ is a **bar** for 2^* if each infinite path of 2^* is blocked by B.

A bar B for 2^* is **uniform** if there exists a positive integer n such that each finite path of length n is blocked by B.

A subset S of a set X is **detachable** from X if

$$\forall_{x \in X} \left(x \in S \vee x \notin S \right).$$

A c-**subset** of 2^* is a subset B of 2^* such that

$$B = \{ u \in 2^* : \forall_{v \in 2^*} \left(u * v \in D \right) \}$$

for some detachable subset D of 2^*.

Thus a finite sequence u belongs to B if and only if each of its extensions belongs to D.

Three versions of the fan theorem: the **fan theorem for detachable bars**,

$\mathbf{FT}_D$ *Every detachable bar of the complete binary fan is uniform;*

the **fan theorem for** c-**bars** (that is, bars that are c-sets),

$\mathbf{FT}_{\mathrm{c}}$ *Every* c-*bar of the complete binary fan is uniform;*

and **Brouwer's fan theorem:**

FT *Every bar of the complete binary fan is uniform.*

dsb and Josef Berger have proved the following.

Theorem 8 *Let B be a c-subset of 2^*. Then there exists a sequence $(x_n)_{n\geqslant 1}$ in $\mathbb{Q}$ with the following properties.*

$\triangleright$ *If B is a bar for 2^*, then $(x_n)_{n\geqslant 1}$ is eventually bounded away from each element of $[0,1]$.*

$\triangleright$ *If $(x_n)_{n\geqslant 1}$ is eventually bounded away from $[0,1]$, then B is a uniform bar for 2^*.*

Theorem 9 *Let $(x_n)_{n \geqslant 1}$ be a rational sequence. Then there exists a* c*-subset B of* 2^* *with the following properties.*

$\lhd$ *If $(x_n)_{n \geqslant 1}$ is eventually bounded away from each element of* $[0, 1]$, *then B is a bar for* 2^*.

$\lhd$ *If B is a uniform bar for* 2^* *and $(x_n)_{n \geqslant 1}$ is eventually bounded away from each point of* $[0, 1]$, *then $(x_n)_{n \geqslant 1}$ is eventually bounded away from* $[0, 1]$.

It follows that

$$\mathbf{AS} \Leftrightarrow \mathbf{FT}_c.$$

Berger has proved that $\mathbf{FT}_c$ is equivalent to the uniform continuity theorem for continuous functions from Cantor space $2^{\mathbb{N}}$ to $\mathbb{N}$.

Thus

> **AS** is equivalent to **UCT** for integer-valued functions from Cantor space;
>
> **AS** $+$ **BD**-$\mathbb{N}$ implies **UCT** for functions from $[0, 1]$ to a metric space.

We also know that

(W)UCT for functions from $[0, 1]$ to $\mathbb{R}$ implies **AS.**

Does **UCT** imply **BD**-$\mathbb{N}$?

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