Effective randomness for computable probability measures

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Outline

1. Introduction
    - Randomness and measures
    - Various notions of randomness
    - Equivalence of measures
2. Bernoulli measures
    - Definitions
    - Equivalence characterization
3. Arbitrary computable measures

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Randomness and measures

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Observe the two following sequences. Which one is random?

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Observe the two following sequences. Which one is random?

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It depends on the probability measure!

We work with computable probability measures.

Definition

A probability measure μ *on* 2^ω *is computable if the function*

$$\begin{array}{ccc} 2^* & \longrightarrow & [0,1] \\ u & \longmapsto & \mu\{\alpha \in 2^\omega : u \sqsubset \alpha\} \end{array}$$

is computable.

Martin-Löf randomness

Definition (Martin-Löf, 1966)

$\alpha \in 2^\omega$ is μ-Martin-Löf random if for all computable collection $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of computably enumerable open sets s.t for all n, $\mu(\mathcal{U}_n) \leq 2^{-n}$, we have:

$$\alpha \notin \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$$

Schnorr randomness

Definition (Schnorr, 1971)

$\alpha \in 2^\omega$ is μ-Schnorr random if for all computable collection $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of computably enumerable open sets s.t for all n, $\mu(\mathcal{U}_n) = 2^{-n}$, we have:

$$\alpha \notin \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$$

Computable randomness

Notion of randomness based on a game-theoretic model. Fix a computable probabilty measure μ. The μ-game works as follows:

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- We play against a sequence $\alpha \in 2^\omega$, starting with an initial capital of $V_0 = 1$.
- Initially, all the bits of α are hidden.
- At each turn, we bet some amount of money b (which does not exceed his capital) on the value of the first bit which hasn't been revealed yet.

Computable randomness

- If our guess is wrong, we loose our stake b:

  $$V_{n+1} = V_n - b$$

  If it is correct, we win our stake multiplied by a "fairness factor":

  $$V_{n+1} = V_n + b\,\frac{\mu(\text{guess was wrong})}{\mu(\text{guess was correct})}$$

- We win if

  $$\limsup_{n\to+\infty} V_n = +\infty$$

Computable randomness

Definition (Schnorr, 1971)

$\alpha \in 2^\omega$ *is* *μ-computably random* *if there is no computable strategy which wins against* α *in the* μ*-game.*

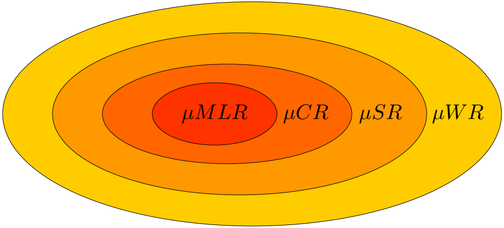
Weak randomness

One last notion of randomness....

Definition (Kurtz, 1981)

$\alpha \in 2^\omega$ is μ-weakly random if it belongs to every c.e. open set of μ-measure 1

For every computable measure μ, we have the following inclusions:



Equivalence of measures

Definition

Two probability measures μ *and* ν *are equivalent if they have the same nullsets (or equivalently the same sets of measure* 1*)*

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Definitions

Definition

A Bernoulli measure of parameters $\{p_i\}_{i\in\mathbb{N}}$ *corresponds to the probability measure obtained by tossing independent* $(0/1)$ *coins, where the probability of the* i*-th coin to outpout* 1 *is* p_i.

A Bernoulli measure is strongly positive if there is some $\varepsilon > 0$ *s.t.* $\forall i\ p_i \in [\varepsilon, 1-\varepsilon]$

Equivalence characterization

Let μ and ν be two strongly positive Bernoulli measures, respectively of parameters $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$

Theorem (Kakutani, 1948)

$$\sum_{i\in\mathbb{N}}(p_i - q_i)^2 < +\infty \iff \mu \sim \nu$$

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Theorem (Kakutani, 1948)

$\sum_{i\in\mathbb{N}}(p_i - q_i)^2 < +\infty \iff \mu \sim \nu$

Theorem (Vovk, 1987)

If μ and ν are computable:

$$\sum_{i\in\mathbb{N}}(p_i - q_i)^2 < +\infty \iff (\mu MLR = \nu MLR)$$

Equivalence characterization

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Theorem

$$\begin{aligned}\sum(p_i - q_i)^2 < +\infty \quad &\Longleftrightarrow \quad \mu \sim \nu \\ &\Longleftrightarrow \quad (\mu MLR = \nu MLR) \\ &\Longleftrightarrow \quad (\mu CR = \nu CR) \\ &\Longleftrightarrow \quad (\mu SR = \nu SR) \\ &\Longleftrightarrow \quad (\mu WR = \nu WR)\end{aligned}$$

Outline

Arbitrary computable measures

Theorem

Let μ and ν be two computable probability measures.

$$\begin{tikzcd}
\mu CR = \nu CR \arrow[d] & & \\
\mu MLR = \nu MLR \arrow[dr] & & \mu SR = \nu SR \arrow[dl] \\
& \mu \sim \nu \arrow[d] & \\
& \mu WR = \nu WR &
\end{tikzcd}$$

Sketch of the proof ($\mu MLR = \nu MLR \implies \mu \sim \nu$)

- Suppose that μ and ν are not equivalent, that is, for example, there exists X such that $\mu(X) = 0$ and $\nu(X) \geq r > 0$.
- Hence, for all n there exists an open set $\mathcal{V}_n$ s.t. $\mu(\mathcal{V}_n) < 2^{-n}$ and $\nu(\mathcal{V}_n) \geq r$
- Since the open sets of the form $u2^\omega$ $(u \in 2^*)$ form a base of the Cantor space, one can effectively find a finite union of such sets $\mathcal{V}_n^* = \bigcup_i u_i 2^\omega$ such that $\mu(\mathcal{V}_n^*) < 2^{-n}$ and $\nu(\mathcal{V}_n^*) > r(1 - 2^{-n})$
- $\mathcal{U}_n = \bigcup_{k>n} \mathcal{V}_k^*$ is a computable collection of computably enumerable open sets, and $\mu(\bigcap_n \mathcal{U}_n) = 0$ and $\nu(\bigcap_n \mathcal{U}_n) \geq r/2$.

Sketch of the proof ($\mu \sim \nu \not\Rightarrow \mu MLR = \nu MLR$)

Let μ be Lebesgue measure. We construct a computable measure ν such that:

- $\{\alpha \in 2^\omega : \limsup \frac{\nu(\alpha\restriction n)}{\mu(\alpha\restriction n)} = +\infty\} = \emptyset$
- $\{\alpha \in 2^\omega : \limsup \frac{\mu(\alpha\restriction n)}{\nu(\alpha\restriction n)} = +\infty\} = \{\Omega\}$

where Ω is Chaitin's constant.

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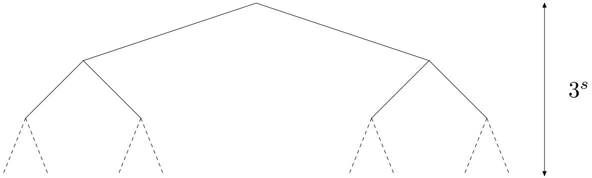
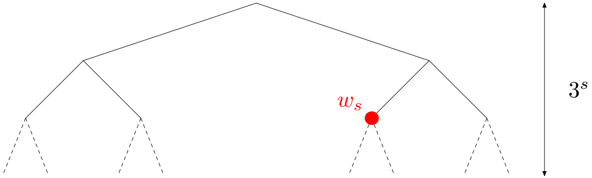
where Ω is Chaitin's constant.

Since Ω is Δ^0_2, we use the limit lemma to write

$$\Omega = \lim_{s \to +\infty} w_s$$

We can assume that w_s is infinitely often a prefix of Ω.


3^s


w_s
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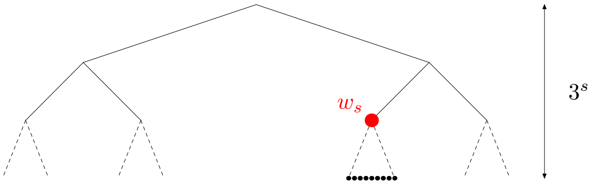
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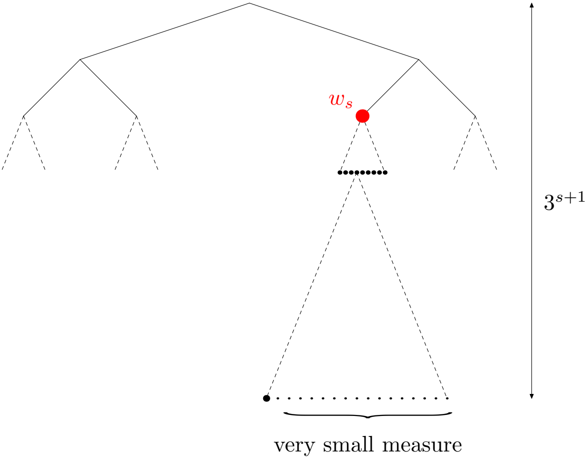
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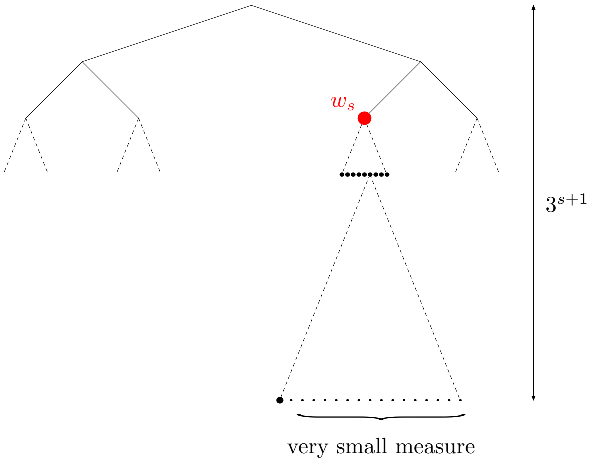
very small measure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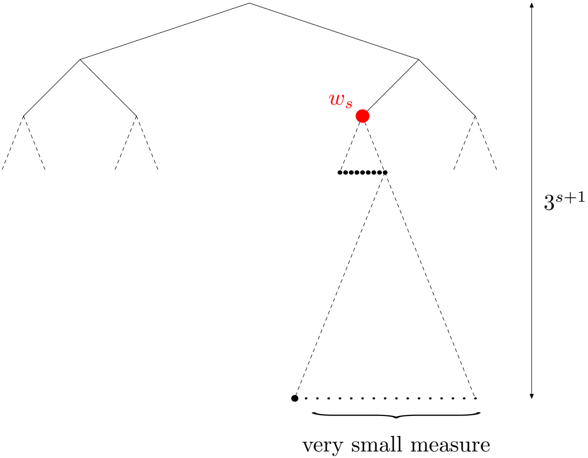
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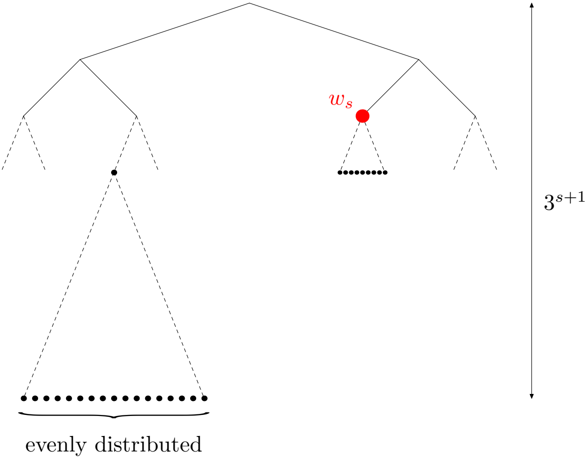
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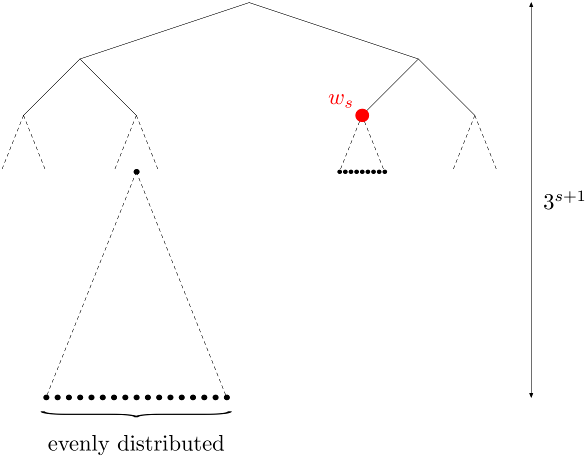
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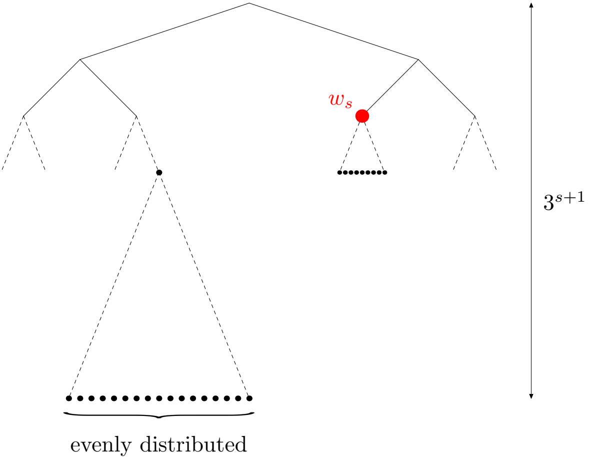
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Open question

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& \mu \sim \nu \arrow[d, shift right] \arrow[d, shift left, leftarrow, red, "?"] & \\
& \mu WR = \nu WR &
\end{tikzcd}$$

Open question

In other words....

Question

Is it possible that two computable measures have the same Π^0_1 nullsets and yet do not have the same nullsets?

THANK YOU