Low for Random and Domination

Work on progress

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Cantor Space, Topology and Measure

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  $$[\sigma] = \{\sigma X \ : \ X \in 2^\omega\}$$

  for all $\sigma \in 2^{<\omega}$
- We use a standard Lebesgue measure on the Cantor space: the measure of a basic open set $[\sigma]$ is

  $$\mu([\sigma]) = 2^{-|\sigma|}$$

  This uniquely determines the Lebesgue measure of sets in 2^ω.

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Definition

A Martin-Löf test $\mathcal{M}$ is a uniform sequence (E_i) of c.e. sets of binary strings such that $\mu(E_i) \leq 2^{-i}$. A real α avoids $\mathcal{M}$ if some for i, $\alpha \notin E_i\Sigma^\omega$. A real number is called random if it avoids all Martin-Löf tests. W.l.o.g. assume $E_{i+1} \subset E_i$.

Π^0_1 Classes and Randomness

Definition

A Σ^0_1 class is a collection C of reals such that

$$C = \{\alpha \ : \ \exists n(R(\alpha \upharpoonright n)\},$$

where R is a computable relation. Equivalently,

$$C = \{[\sigma] \ : \ \sigma \in W\},$$

where $W \subseteq 2^{<\omega}$ is c.e. (identify sets of sets of reals with their union).

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- Π^0_1 classes are the complements of Σ^0_1 classes.

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- If (E_i) is a Martin-Löf test, E_i is a Σ^0_1 class
- The random numbers is a Σ^0_2 class of measure 1, a countable union of Π^0_1 classes of measure tending to 1.

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- Relativizing we get: A is *low for random w.r.t. B* ($A \leq_{LR} B$) if every B-random is A-random.
- $\leq_{LR}$ is transitive, Σ^0_3 and it contains $\leq_T$.

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- For some member U^A of a universal Martin-Löf test relative to A there is $V^B \in \Sigma_1^{0,B}$ with $\mu V^B < 1$ and

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- There is a least degree containing the low for random reals. Some of them are not computable (Kučera).
- It is not known if there is a least upper bound for any two degrees.
- The usual $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$ is not a supremum (Nies).

A Splitting theorem for the c.e. LR degrees

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- $B \cap C = \emptyset$
- $B \cup C = A$
- $B \not\leq_{LR} C$ *and* $C \not\leq_{LR} B$.

$2^{\aleph_0}$ predecessors

Theorem

The LR degree of the halting problem bounds $2^{\aleph_0}$ LR degrees. Also, every LR degree containing a superhigh set bounds $2^{\aleph_0}$ LR degrees.

Randomness and the Arithmetical Hierarchy

Theorem

For every $n > 1$ there exists a $(n-1)$-random set which is properly Σ^0_n and a $(n-1)$-random set which is properly Π^0_n.

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- A Turing degree is a.e. dominating if it contains an a.e. dominating function.

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- There are c.e. sets which compute an a.e. dominating function but don't compute the halting problem (Cholak and others).
- The *LR* degree of the halting problem contains halfs of minimal pairs in the c.e. Turing degrees (B. and Montalbán).

A non-cuppable a.e. dominating c.e. degree

Theorem

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Corollary

If A is c.e. and is computed by every a.e. dominating function of c.e. Turing degree then A is non-cuppable in the c.e. degrees.

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- strategies need better and better approximations of the measure of the domain of functional(s)
- only minimal restraints are allowed on the dominating function (at most one at each level of the tree)

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- How about the local structure of c.e. *LR* degrees? (density, minimal pairs etc.)
- Global constructions: minimal *LR*-degrees?

End

Thank you!