Tutorial on Π^0_1 Classes

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A Π^0_1 *class* may be defined as the collection of infinite paths through a computable subtree of $2^{<\omega}$, the complete binary-branching tree. Π^0_1 classes have become a fundamental notion in computability theory because of their ability to code a wide range of constructions. For example, the collection of ideals of a computably enumerable (c.e.) commutative ring forms a Π^0_1 class. We will consider some examples of Π^0_1 class constructions, especially those related to analysis, such as are found in [Cenzer,1999], [Cenzer and Jockusch, 2000], and [Cenzer and Remmel, 1998].

A *basis theorem* for Π^0_1 classes is a theorem of the form "Every nonempty Π^0_1 class (with property Y) contains a path with property X." For example, every nonempty Π^0_1 class contains a path of low Turing degree. A number of these basis theorems relate to measure, genericity, and the recently-expanding area of algorithmic randomness; we will pay special attention to those.

The collection of all Π^0_1 classes ordered by inclusion, called $\mathcal{E}_{Pi}$ by analogy with $\mathcal{E}$, the lattice of computably enumerable (c.e.) sets, is a lattice with least upper bound equal to the union of two classes and greatest lower bound equal to their intersection. This will be the final topic of the tutorial; in particular intervals of and embedding into $\mathcal{E}_{Pi}$, questions of definability, and connections to $\mathcal{E}^*$, which is $\mathcal{E}$ modulo finite difference. This section will draw mostly on the work of Cenzer and Nies, 2001, 2004], [Cholak, Coles, Downey, and Herrmann, 2001], and [Weber, 2006].