Tutorial on Mass Problems

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Let P be a set of reals viewed as a *mass problem*, i.e., a "decision problem with more than one solution." Here the "solutions" of P are the elements of P. Many unsolvable mathematical problems are best viewed as mass problems. One says that a mass problem P is *weakly reducible* to a mass problem Q if for every solution Y of Q there exists a solution X of P such that X is Turing reducible to Y. A *weak degree* is an equivalence class of mass problems under mutual weak reducibility. Weak degrees are also known as *Muchnik degrees*.

Let $\mathcal{P}_w$ be the set of weak degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω, partially ordered by weak reducibility. Algebraically, it is easy to see that $\mathcal{P}_w$ is a countable distributive lattice with top and bottom. Simpson and others have studied the lattice $\mathcal{P}_w$ in a series of publications beginning in 1999. Our principal findings are as follows.

(1) There is a natural embedding of $\mathcal{E}_T$, the countable semilattice of recursively enumerable Turing degrees, into the lattice $\mathcal{P}_w$. This embedding is one-to-one and preserves the semilattice structure and the top and bottom. We identify $\mathcal{E}_T$ with its image in $\mathcal{P}_w$ under this embedding.

(2) Like the semilattice $\mathcal{E}_T$, the lattice $\mathcal{P}_w$ is structurally rich. In particular, any countable distributive lattice is lattice-embeddable in any nontrivial initial segment of $\mathcal{P}_w$. Moreover, the $\mathcal{P}_w$ analog of the Sacks Splitting Theorem holds. These structural results are proved by means of priority arguments. The $\mathcal{P}_w$ analog of the Sacks Density Theorem remains as an open problem.

(3) Unlike $\mathcal{E}_T$, the lattice $\mathcal{P}_w$ contains a large number of specific, natural degrees other than the top and bottom degrees. These specific, natural degrees in $\mathcal{P}_w$ arise from foundationally interesting topics such as reverse mathematics, algorithmic randomness, subrecursive hierarchies, computational complexity, and hyperarithmeticity.

(4) The known specific, natural degrees in $\mathcal{P}_w$ are disjoint from the recursively enumerable Turing degrees in $\mathcal{E}_T$. The only exceptions are the top and bottom degrees in $\mathcal{E}_T$, which are the same as the top and bottom degrees in $\mathcal{P}_w$.