Dimensions of Points in Self-Similar Fractal

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Self-similar fractals arise as the unique attractors of iterated function systems (IFSs) consisting of finitely many contracting similarities satisfying an open set condition. Each point x in such a fractal F arising from an IFS S is naturally regarded as the "outcome" of an infinite *coding sequence* T (which need not be unique) over the alphabet $\Sigma_k = \{0, \ldots, k-1\}$, where k is the number of contracting similarities in S. A classical theorem of Moran (1946) and Falconer (1989) states that the Hausdorff and packing dimensions of a self-similar fractal coincide with its similarity dimension, which depends only on the contraction ratios of the similarities.

The theory of constructive dimension introduced by Lutz (2003) uses the theory of computing to provide a meaningful notion of the *dimensions of individual points* in Euclidean space. In this paper, we use (and extend) this theory to analyze the dimensions of individual points in fractals that are *computably self-similar*, meaning that they are unique attractors of IFSs that are computable and satisfy the open set condition. Our main theorem states that, if $F \subseteq \mathbb{R}^n$ is any computably self-similar fractal and S is any IFS testifying to this fact, then the dimension identities

$$\dim(x) = \dim(F)\dim^{\pi_S}(T)$$

and

$$\mathrm{Dim}(x) = \dim(F)\,\mathrm{Dim}^{\pi_S}(T)$$

hold for all $x \in F$ and all coding sequences T for x. In these equations, $\dim(F)$ denotes the similarity dimension of the fractal F; $\dim(x)$ and $\mathrm{Dim}(x)$ denote the dimension and strong dimension, respectively, of the point x in Euclidean space; and $\dim^{\pi_S}(T)$ and $\mathrm{Dim}^{\pi_S}(T)$ denote the dimension and strong dimension, respectively, of the coding sequence T relative to a probability measure π_S that the IFS S induces on the alphabet Σ_k. The above-mentioned theorem of Moran and Falconer follows easily from our main theorem by relativization.

Along the way to our main theorem, we develop the elements of the theory of constructive dimensions relative to general probability measures. The proof of our main theorem uses Kolmogorov complexity characterizations of these dimensions.