Calculus II – Solutions to Review Problems Two

1. Approximate $\int_0^\pi sin^2\ xdx$ with $n = 6$, using the Midpoint, Trapezoidal and Simpson's Rules and find upper bounds for the errors in each.

For the Trapezoidal and Simpson's Rules, we have $\frac{b-a}{n} = \pi/6$,

$x_0 = 0$, $x_1 = \pi/6$, $x_2 = \pi/3$, $x_3 = \pi/2$, $x_4 = 2\pi/3$, $x_5 = 5\pi/6$ and $x_6 = \pi$, so that

$y_0 = sin^2(0) = 0$, $y_1 = sin^2(\pi/6) = .25$, $y_2 = sin^2(\pi/3) = .75$, $y_3 = sin^2(\pi/2) = 1$, $y_4 = sin^2(2\pi/3) = .75$, $y_5 = sin^2(5\pi/6) = .25$ and $y_6 = sin^2(\pi) = 1$.

Thus $T = \frac{1}{2}\frac{\pi}{6}[0 + .5 + 1.5 + 2 + 1.5 + .5 + 0] = \pi/2 = 1.5708$ and

$S = \frac{1}{3}\frac{\pi}{6}[0 + 1 + 1.5 + 4 + 1.5 + 1 + 0] = \pi/2 = 1.5708$ also.

For the Midpoint rule,

$y_1 = sin^2(\pi/12) = .067$, $y_2 = sin^2(\pi/4) = .5$, $y_3 = sin^2(5\pi/12) = .933$, $y_4 = sin^2(7\pi/12) = .933$, $y_5 = sin^2(3\pi/4) = .5$ and $y_6 = sin^2(11\pi/12) = .067$.

Thus $M = \frac{\pi}{6}[.067 + .5 + .933 + .933 + .5 + .067] = \pi/2 = 1.5708$.

For the error estimates, $y' = 2sin\ x\ cos\ x = sin\ 2x$, y" = 2 cos 2x, $y''' = -4sin\ 2x$, and $y^{(iv)} = -8cos\ 2x$, so that $K_2 = 2$ and $K_4 = 8$.

Thus $|E_M| \le \frac{K_2(b-a)^3}{24n^2} = \pi^3/432 \le .6$.

$|E_M| \le \frac{K_2(b-a)^3}{12n^2} = \pi^3/216 \le 1.2$.

$|E_M| \le \frac{K_4(b-a)^5}{180n^4} = 8\pi^5/(180)(1296) \le .15$.

2. Use the Comparison Property to explain whether the following improper integrals converge.

(a) $\int_1^\infty sin^2 x\ dx/x^4$ (b) $\int_1^\infty e^x\ dx/x$.

(a) $0 \le sin^2 x/x^4 \le 1/x^4$, so $\int_1^\infty sin^2 x\ dx/x^4 \le \int_1^\infty dx/x^4 = -1/3x^3]_1^\infty = 1/3$– integral converges.

(b) $0 \le 1/x \le e^x/x$, so $\int_1^\infty e^x\ dx/x \ge \int_1^\infty dx/x = ln(x)]_1^\infty = \infty$–integral diverges.

3. Evaluate the following improper integrals.

(a) $\int_{-1}^8 x^{-1/3}dx$ (b) $\int_2^\infty dx/(x^2+4)$. (c) $\int_4^\infty dx/x^{\frac{3}{2}}$.

(a) $\int_{-1}^8 x^{-1/3}dx = \int_{-1}^0 x^{-1/3}dx + \int_0^8 x^{-1/3}dx = [\frac{3}{2}x^{2/3}]_{-1}^0 + [\frac{3}{2}x^{2/3}]_0^8 = \frac{-3}{2} + \frac{3}{2}4 = 4.5$.

(b) $\int_2^\infty dx/(x^2+4) = \frac{1}{2}arctan(x/2)]_2^\infty = \frac{1}{2}(\frac{\pi}{2} - \frac{\pi}{4} = \pi/8$

(c) $\int_4^\infty dx/x^{3/2} = -2x^{\frac{-1}{2}}]4^\infty = 1 - 2\lim_{x\to\infty} x^{\frac{-1}{2}} = 1$.

4. Evaluate the improper integral $\int_5^\infty dx/x^2\sqrt{x^2-9}$.

Let $x = 3sec\ \theta$, so that $dx = 3sec\ theta\ tan\ \theta\ d\theta$ and $\sqrt{x^2-9} = 3tan\ \theta$. Then $\int dx/x^2\sqrt{x^2-9} = \int \frac{cos\ \theta}{9}\ d\theta = \frac{sin\ \theta}{9}$.

Now $sec\ \theta = x/3$, so that $sin\ \theta = \frac{\sqrt{x^2-9}}{x}$.

So we get $[\frac{\sqrt{x^2-9}}{9x}]_0^5 = \frac{1}{9}[lim_{x\to\infty}\frac{\sqrt{x^2-9}}{x} - \frac{4}{5}] = \frac{1}{9}(1 - \frac{4}{5}) = \frac{1}{45}$.

5. Evaluate the improper integral $\int_0^1 x\ ln\ x\ dx$. (Hint: Use integration by parts.)

Let $u = ln\ x$ and $dv = x\ dx$, so $du = \frac{dx}{x}$ and and $v = \frac{1}{2}x^2$.

Then $I = [\frac{1}{2}x^2\ ln\ x]_0^1 - \int_0^1 \frac{1}{2}x\ dx = [\frac{1}{2}x^2\ ln\ x - \frac{1}{4}x^2]_0^1 = (0-\frac{1}{4}) - (lim_{x\to 0}\frac{1}{2}x^2\ ln\ x - 0) = -\frac{1}{4}$.

For the last part, use L'Hopital's Rule to get

$lim_{x\to 0}x^2\ ln\ x = lim_{x\to 0}\frac{ln\ x}{x^{-2}} = lim_{x\to 0}\frac{x^{-1}}{-2x^{-3}} = lim_{x\to 0} - 2x^2 = 0$.

6. Show that any convergent sequence is bounded.

Suppose that $lim_{n\to\infty}a_n = L$. Then, applying the definition of limit with $\epsilon = 1$, there exists some N such that $|a_n - L| < 1$ for all $n > N$. Thus for $n > N$, $a_n < 1 + |L|$. Now let $B = max\{1 + |L|, a_0, a_1, \dots, a_N\}$. Then for every n, $a_n \leq B$.

7. Show that if $\sum_{n=1}^{\infty}$ converges, then $lim_{n\to\infty}a_n = 0$.

By definition, $\sum_{n=1}^{\infty} a_n = lim_{n\to\infty}S_n$, where $S_n = a_1 + a_2 + \cdots + a_n$.

Then $S_n - S_{n-1} = a_n$ for all $n > 1$ and $lim_{n\to\infty}S_{n-1} = \lim_{n\to\infty} S_n$.

So $lim_{n\to\infty}a_n = lim_{n\to\infty}S_n - S_{n-1} = \lim_{n\to\infty} S_n - lim_{n\to\infty}S_{n-1} = 0$.

8. State the (Direct) Comparison Test and show how it follows from the Monotone Bounded Convergence Theorem.

Comparison Test: Suppose that $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Proof: Since each $a_n \geq 0$, the sequence $S_n = a_1 + \dots a_n$ of partial sums is monotone increasing. Let $B_n = b_1 + \dots b_n$ and let $B = \sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} B_n$. Since each $b_n \geq 0$, $B_n \leq B$ for each n and since each $a_n \leq b_n$, $S_n \leq B_n \leq B$ for all n. Thus the sequence $\{S_n\}$ is increasing and bounded above and therefore converges by the Monotone Bounded Convergence Theorem. Since $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$, this means that the sum converges.

9. Use the Squeeze Theorem to find $\lim_{n\to\infty} \frac{nsin(n)}{n^2+1}$.

We have $-1 \leq sin(n) \leq 1$, so that

$\frac{-n}{n^2+1} \leq \frac{nsin(n)}{n^2+1} \leq \frac{n}{n^2+1}$.

But $lim_{n\to\infty\frac{\pm n}{n^2+1} = \lim_n \to\infty} \frac{\frac{\pm 1}{n^2}}{1+\frac{1}{n^2}} = \frac{0}{1} = 0$.

Thus $\lim_{n\to\infty} \frac{nsin(n)}{n^2+1} = 0$ by the Squeeze Theorem.

10. Find a formula for the n'th partial sum S_n of the series $(1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{7}) + (\frac{1}{7} - \frac{1}{10}) + \dots$, and use to find the sum of the series.

$a_n = \frac{1}{3n+1} - \frac{1}{3n+4}$ for $n \geq 0$. This is a telescoping series, so

$S_n = (1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{7}) + \cdots + (\frac{1}{3n+1} - \frac{1}{3n+4}) = 1 - \frac{1}{3n+4}$.

Thus the sum S of the series is $S = \lim_{n\to\infty} S_n = 1$.

11. Find a formula for the n'th term of the series $100 + 20 + 4 + \frac{4}{5} + \cdots$ and use to find the sum of the series.

$S_n = 100(1 + .2 + (.2)^2 + \cdots + (.2)^{n-1}) = 100(1 - (.2)^n)/(1 - .2)$.

$S = \lim_{n\to\infty} 100(1 - (.2)^n)/(1 - .2) = 100/.8 = 125$.

12. Use Integral Comparison to find lower and upper bounds for $\sum_{n=1}^{\infty} n^{\frac{-5}{2}}$ within .1 of each other.

$\int_n^{\infty} x^{-\frac{5}{2}} dx = [-\frac{2}{3}x^{-\frac{3}{2}}]_n^{\infty} = \frac{2}{3}n^{-\frac{3}{2}}$.

Then $\frac{2}{3}(n+1)^{-\frac{3}{2}} = \int_{n+1}^{\infty} x^{-\frac{5}{2}} dx \leq R_n \leq \int_n^{\infty} x^{-\frac{5}{2}} dx = \frac{2}{3}n^{-\frac{3}{2}}$.

For $n = 3$, we get $.083 \leq \frac{2}{3}4^{-\frac{3}{2}} \leq R_3 \leq \frac{2}{3}3^{-\frac{3}{2}} \leq .129$.

Also, $S_3 = 1 + \frac{1}{2^{\frac{5}{2}}} + \frac{1}{3^{\frac{5}{2}}} = 1.241$

Since $S = S_3 + R_3$, $1.32 \leq 1.241 + .083 \leq S \leq 1.241 + .129 \leq 1.38$

13. Use the Alternating Series Remainder to find lower and upper bounds L and B for $S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n+1}$ such that $B - L < .2$.

$\{\frac{1}{2^n+1}\}$ is decreasing and has limit 0, so the sum converges by the AST.
$S = \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{9} + \cdots$
In general, $S_{2n} < S < S_{2n-1}$
$L = \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{9} < S < \frac{1}{2} - \frac{1}{3} + \frac{1}{5} = B.$
with $B - L = \frac{1}{9} = a_{2n} = a_4$.

14. Determine whether each series converges or diverges, by what test and why. For (f) and (g), determine whether the series converges absolutely and/or conditionally.

(a) $\sum_{n=2}^{\infty} \frac{1}{n\ ln(n)^2}$
Converges by the Integral Comparison Test since
$\int_2^{\infty} \frac{1}{x\ (ln\ x)^2} dx = [\frac{-1}{ln\ x}]_2^{\infty} = \frac{1}{ln\ 2}$.

(b) $\sum_{n=1}^{\infty} \frac{n+2}{n^3+1}$
Converges by Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-Test with $p = 2 > 1$.
$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+2)}{n^3+1} = 1,$
Since $0 < L < \infty$, $\sum_{n=1}^{\infty} \frac{n+2}{n^3+1}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(c) $\sum_{n=1}^{\infty} \frac{1}{n\ 2^n}$
Converges by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which converges as a Geometric Series with $r = \frac{1}{2} < 1$.

(d) $\sum_{n=0}^{\infty} (\frac{n}{2n+1})^n$
Converges by the Root Test with $r = \lim_{n\to\infty} a_n^{\frac{1}{n}} = lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$.

(e) $1 + \frac{2}{3} + \frac{2\cdot 3}{3\cdot 5} + \frac{2\cdot 3\cdot 4}{3\cdot 5\cdot 7} + \cdots$
Converges by the Ratio Test, since $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} < 1$.

(f) $\sum_{n=0}^{\infty} (-1)^n (\frac{n+1}{2n+1})$
Diverges by the Limit Test, since $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} \neq 0$.

(g) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{\frac{1}{2}}}$
Converges by the Alternating Series Test with $a_n = \frac{1}{n^{\frac{1}{2}}}$, since $lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is decreasing.
Does not converge absolutely by the p-test, since $p = \frac{1}{2} < 1$.
Thus the series converges conditionally.

15. Find the interval of convergence for the power series $\sum_{n=1}^{\infty} x^n/n^{1/2}$.
By the Ratio Test, the series converges for $r < 1$, where
$r = \lim_{n\to\infty} |\frac{x^{n+1}/(n+1)^{1/2}}{x^n/n^{1/2}}| = |x|$.
Thus the series converges for $|x| < 1$, which makes the Radius of Convergence 1.
For $x = 1$, the series $\sum_{n=1}^{\infty} 1/n^{1/2}$ diverges by the p-test with $p = 1/2 < 1$.
For $x = -1$, the series $\sum_{n=1}^{\infty} (-1)^n/n^{1/2}$ converges by the Alternating Series Test. Thus the Interval of Convergence is $[-1, 1)$.

16. Write the power series for $ln(1 + x)$ (give the first 4 terms as well as the general summation formula). Then use the series to estimate ln(1.2) within .001 (give lower and upper bounds).
$ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} x^n/n$
$ln(1.2) = .2 - .04/2 + .008/3 - .0016/4.$
By Alternating Series,
$.182266 = .2 - .02 + .002666 - .0004 < ln(1.2) < .2 - .02 + .002666 = .182666.$