

Surprising symmetries in objects counted by Catalan numbers

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Abstract

We prove that the total number $S_{n,132}(q)$ of copies of the pattern q in all 132-avoiding permutations of length n is the same for $q = 231$, $q = 312$, or $q = 213$. We provide a combinatorial proof for this unexpected threefold symmetry. We then significantly generalize this result by proving a large family of non-trivial equalities of the type $S_{n,132}(q) = S_{n,132}(q')$.

Dedicated to the memory of Herb Wilf

1 Introduction

1.1 Background and Definitions

Let $q = q_1q_2 \dots q_k$ be a permutation in the symmetric group S_k . We say that the permutation $p = p_1p_2 \dots p_n \in S_n$ *contains a q -pattern* if and only if there is a subsequence $p_{i_1}p_{i_2} \dots p_{i_k}$ of p whose elements are in the same relative order as those in q , that is,

$$p_{i_t} < p_{i_u} \text{ if and only if } q_t < q_u$$

whenever $1 \leq t, u \leq k$. If p does not contain q , then we say that p *avoids* q . For instance, 214653 contains 231 (consider the third, fourth, and sixth entries), but avoids 4321. See Chapter 14 of [1] for an introduction to pattern avoiding permutations, and Chapters 4 and 5 of [2] for a somewhat more detailed treatment.

It is straightforward to compute, using the linear property of expectation, that the average number of q -patterns in a randomly selected permutation of length n is $\frac{1}{k!} \binom{n}{k}$, where k is the length of q .

Joshua Cooper [5] has raised the following interesting family of questions. Let r be a given permutation pattern. What can be said about the average number of occurrences of q in a randomly selected r -avoiding permutation of a given length? Equivalently, can we determine the *total number* $S_{n,r}(q)$ of all q -patterns in all r -avoiding permutations of length n ?

1.2 Earlier Results

In [4], the present author found formulae for the generating functions of the sequence $S_{n,132}(q)$ for the cases of monotone q , that is, for $q = 12 \cdots k$ and $q = k(k-1) \cdots 1$, for any k . He also proved that if n is large enough, then for any fixed k , among all patterns q of length k , it is the monotone decreasing pattern that maximizes $S_{n,132}(q)$ and it is the monotone increasing pattern that minimizes $S_{n,132}(q)$.

1.3 The Outline of our Paper

In this paper, we first present a computational proof of the surprising fact that for all n , the equalities

$$S_{n,132}(231) = S_{n,132}(312) = S_{n,132}(213) \tag{1}$$

hold. The first equality is trivial, since taking the inverse of a 132-avoiding permutation keeps that permutation 132-avoiding, and turns 231-patterns into 312-patterns. However, the second equality is non-trivial. (The reverse or complement of a 132-avoiding permutation is not necessarily 132-avoiding.) In particular, if $a(p)$ denotes the number of 213-copies in p , and $b(p)$ denotes the number of 231-copies in p , then the statistics $a(p)$ and $b(p)$ are *not* equidistributed over the set of all 132-avoiding permutations of length n , but their average values are equal over that set.

In other words, we will prove that a randomly selected non-monotonic pattern of length three in a 132-avoiding permutation is equally likely to be a 231-pattern, a 312-pattern, or a 213-pattern. It is well-known (see Chapter 14 of [1]) that 132-avoiding permutations of length n are counted by the Catalan numbers $c_n = \binom{2n}{n}/(n+1)$, and as such, they are one of more than 150 distinct kinds of objects counted by those numbers. However, we do not know of any other example when a natural statistic on objects counted by Catalan numbers shows a similar threefold symmetry.

In the next part of the paper we provide a bijective proof of (1). Finally, we will significantly generalize this result by showing more than c_{h-2} pairs of patterns q and q' of length h that behave as 213 and 231, that is, for which $S_{n,132}(q) = S_{n,132}(q')$, and the equality is non-trivial.

2 Arguments Using Generating Functions

Let d_n be the total number of inversions (in other words, copies of the pattern 21) in all 132-avoiding n -permutations. It is proved in [4] that

$$D(x) = \sum_{n \geq 1} d_n x^n = \frac{x}{1-4x} \cdot \left(\frac{1}{\sqrt{1-4x}} - \frac{1-\sqrt{1-4x}}{2x} \right). \quad (2)$$

2.1 Counting Copies of 213

Let a_n be the total number of all 213-patterns in all 132-avoiding permutations of length n . Clearly, then $a_0 = a_1 = a_2 = 0$.

There are three ways that a 132-avoiding permutation p of length n can contain a 213-pattern q . Either q is entirely on the left of the entry n , or q is entirely on the right of n , or q ends in n .

For $n \geq 3$, this leads to the recurrence relation

$$a_n = \sum_{i=1}^n a_{i-1} c_{n-i} + \sum_{i=1}^n c_{i-1} a_{n-i} + \sum_{i=3}^n d_{i-1} c_{n-i}.$$

Let $A(x)$ (resp. $C(x)$) be the ordinary generating function for the sequence of the numbers a_n (resp. c_n). Then the last displayed formula yields the functional equation

$$A(x) = 2xA(x)C(x) + xD(x)C(x),$$

which is equivalent to

$$A(x) = \frac{x D(x) C(x)}{1 - 2x C(x)} = \frac{x}{2(1-4x)^2} + \frac{x-1}{2(1-4x)^{3/2}} + \frac{1}{2(1-4x)}. \quad (3)$$

From here, we get that if $n \geq 3$, then

$$a_n = \frac{n}{2} 4^{n-1} + \frac{1}{2} 4^n - (2n+1) \binom{2n-1}{n-1} + (2n-1) \binom{2n-3}{n-2},$$

which simplifies to

$$a_n = (n+4) \cdot 2^{2n-3} - (2n+1) \binom{2n-1}{n-1} + (2n-1) \binom{2n-3}{n-2}. \quad (4)$$

2.2 Counting Copies of 231

Let h_n be the total number of all non-inversions (in other words, copies of the pattern 12) in all 132-avoiding permutations of length n . It is proved in [4] that

$$H(x) = \sum_{n \geq 0} h_n x^n = \frac{1}{2(1-4x)} + \frac{1}{2x} - \frac{1-x}{2x\sqrt{1-4x}}. \quad (5)$$

Let b_n be the total number of all 231-copies in all 132-avoiding permutations of length n , and let $B(x) = \sum_{n \geq 0} b_n x^n$.

Let

$$Z(x) = \sum_{n \geq 0} n c_n x^n = \sum_{n \geq 0} \binom{2n}{n} \frac{n}{n+1} x^n = \frac{1}{\sqrt{1-4x}} - \frac{1 - \sqrt{1-4x}}{2x}. \quad (6)$$

Note that $Z(x)$ is the generating function for the number of entries (which are copies of the pattern 1) in all 132-avoiding n -permutations.

If p is a 132-avoiding n -permutation, and q is a 231-pattern contained in p , then either q is entirely on the left of the entry n , or q is entirely on the right of the entry n , or the entry n is the largest entry of q , or the first and second entries of q form a 12-pattern on the left of n , while the third entry of q is on the right of n .

For $n \geq 3$, this leads to the recurrence relation

$$b_n = \sum_{i=1}^n b_{i-1} c_{n-i} + \sum_{i=1}^n c_{i-1} b_{n-i} + \sum_{i=2}^{n-1} (i-1)(n-i) c_{i-1} c_{n-i} + \sum_{i=3}^{n-1} h_{i-1} c_{n-i} (n-i).$$

In terms of generating functions, this yields

$$B(x) = 2xB(x)C(x) + xZ^2(x) + xH(x)Z(x),$$

$$B(x) = \frac{xZ^2(x) + xH(x)Z(x)}{1 - 2xC(x)} = \frac{xZ^2(x) + xH(x)Z(x)}{\sqrt{1-4x}}.$$

Given the explicit formulae (5) and (6) for $H(x)$ and $Z(x)$, the last displayed equation yields the formula

$$B(x) = \frac{x D(x) C(x)}{1 - 2x C(x)} = \frac{x}{2(1-4x)^2} + \frac{x-1}{2(1-4x)^{3/2}} + \frac{1}{2(1-4x)}. \quad (7)$$

The proof of the main result of this section is now immediate.

Theorem 1. *For all positive integers n , the equalities*

$$S_{n,132}(231) = S_{n,132}(312) = S_{n,132}(213)$$

hold.

Proof. As we mentioned in the Introduction, the first equality is trivially true since there is a natural bijection between the 231-copies of the 132-avoiding permutation p and the 312-copies of the 132-avoiding permutation p^{-1} . Indeed, let $p = p_1 p_2 \cdots p_n$ be a 132-avoiding permutation, and let $1 \leq i < j < k \leq n$. Then $p_i p_j p_k$ is a 231-copy in p if and only if ijk is a 312-copy in p^{-1} .

The equality $S_{n,132}(231) = S_{n,132}(213)$ holds since we have seen in formulae (3) and (7) that the two sides of this equality have identical generating functions. \square

3 A Bijective Proof

In this section we provide a bijective proof for the surprising identity $S_{n,132}(213) = S_{n,132}(231)$.

3.1 Binary Plane Trees

In our proof, we will identify a 132-avoiding permutation p with its *binary plane tree* $T(p)$ using a very well-known bijection. We will briefly describe this bijection now. For more details, the reader may consult Chapter 14 of [1]. The tree $T(p)$ will be a binary plane tree, that is, a rooted unlabeled tree in which each vertex has at most two children, and each child is a left child or a right child of its parent, even if it is the only child of its parent.

The root of $T(p)$ corresponds to the entry n of p , the left subtree of the root corresponds to the string of entries of p on the left of n , and the right subtree of the root corresponds to the string of entries of p on the right of n . Both subtrees are constructed recursively, by the same rule. Note that since p is 132-avoiding, the position of the entry n of p determines the set of entries that are on the left (resp. on the right) of n . In fact, if n is in the i th position, the set of entries on the left of n must be $\{n - i + 1, n - i + 2, \dots, n - 1\}$, and the set of entries on the right of n must be $\{1, 2, \dots, n - i\}$.

We point out that in the process of constructing $T(p)$, each vertex of $T(p)$ is associated to an entry of p . Indeed, each vertex is added to $T(p)$ as the root of a subtree S , and so each vertex is associated to the entry that is the largest among the entries that belong to S . However, it is important to point out that $T(p)$ is an *unlabeled tree* since the way in which the entries of p correspond to the vertices of $T(p)$ is completely determined by the unlabeled tree $T(p)$ as long as p is 132-avoiding.

See Figure 1 for an illustration.

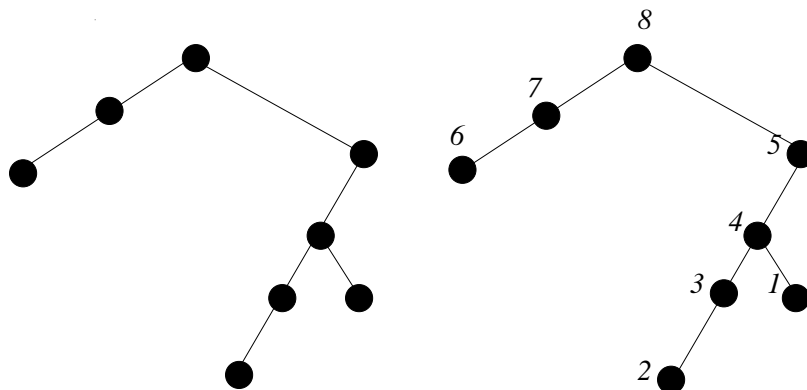


Figure 1: The tree $T(p)$ for $p = 67823415$, and the entries of p associated to the vertices of $T(p)$.

Note that in order to get p from $T(p)$, it suffices to read the vertices of $T(p)$ *in-order*,

that is, by first reading the left subtree of the root, then the right subtree of the root, and then the right subtree of the root. The respective subtrees are read recursively, by this same rule. Therefore, it is meaningful to talk about the first, second, etc, last vertex of $T(p)$, since that means the first, second, etc, last vertex of $T(p)$ in the *in-order* reading.

A *left descendant* (resp. *right descendant*) of a vertex x in a binary plane tree is a vertex in the left (resp. right) subtree of x . The left (resp. right) subtree of x does *not* contain x itself.

It is straightforward to see that $p_i p_j$ is a 12-pattern in p if and only if p_i is a left-descendant of p_j in $T(p)$. On the other hand, $p_j p_i$ is a 21-pattern in p if and only if either p_i is a right descendant of p_j in $T(p)$ or there is a vertex x in $T(p)$ so that p_j is a left descendant of x and p_i is a right descendant of x . In the previous section we gave an exhaustive list of the ways in which 213-patterns and 231-patterns can occur in a 132-avoiding permutation. The reader is invited to translate that list into the language of binary plane trees.

3.2 Our Bijection

Let p be a 132-avoiding n -permutation, and let Q be an occurrence of the pattern 213 in p . Let Q_2, Q_1, Q_3 be the three vertices of $T(p)$ that correspond to Q , going left to right. Let us color these three entries black. There are then two possibilities.

1. Either Q_1 is a right descendant of Q_2 and Q_2 is a left descendant of Q_3 , or
2. there exists a lowest left descendant Q_x of Q_3 so that Q_2 is a left descendant of Q_x and Q_1 is a right descendant of Q_x .

Let A_n be the set of all binary plane trees on n vertices in which three vertices forming a 213-pattern are colored black. Let B_n be the set of all binary plane trees on n vertices in which three vertices forming a 231-pattern are colored black.

Now we are going to define a map $f : A_n \rightarrow B_n$. We will then prove that f is a bijection. The map f will be defined differently in the two cases described above.

- *Case 1.* If $T \in A_n$ is in the first case, then let $f(T)$ be the tree obtained by interchanging the right subtree of Q_2 and the right subtree of Q_3 . Keep all three black vertices Q_i black, even as Q_1 gets moved.

See Figure 2 for an illustration.

Note that in $f(T)$, in the set of black vertices, there is one that is an ancestor of the other two, namely Q_3 .

- *Case 2.* If $T \in A_n$ is in the second case, then let $f(T)$ be the tree obtained by interchanging the right subtrees of the vertices Q_x and Q_3 , and coloring Q_2, Q_x and Q_1 black. See Figure 3 for an illustration.

Note that in $f(T)$, there is no black vertex that is an ancestor of the other two black vertices. Also note that in $f(T)$, the lowest common ancestor of Q_x and Q_1 is Q_3 .

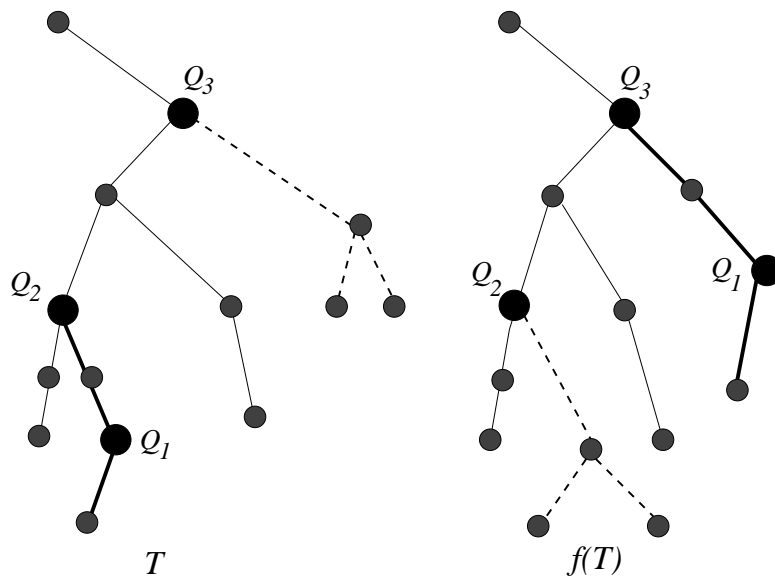


Figure 2: Interchanging the right subtrees of Q_2 and Q_3 .

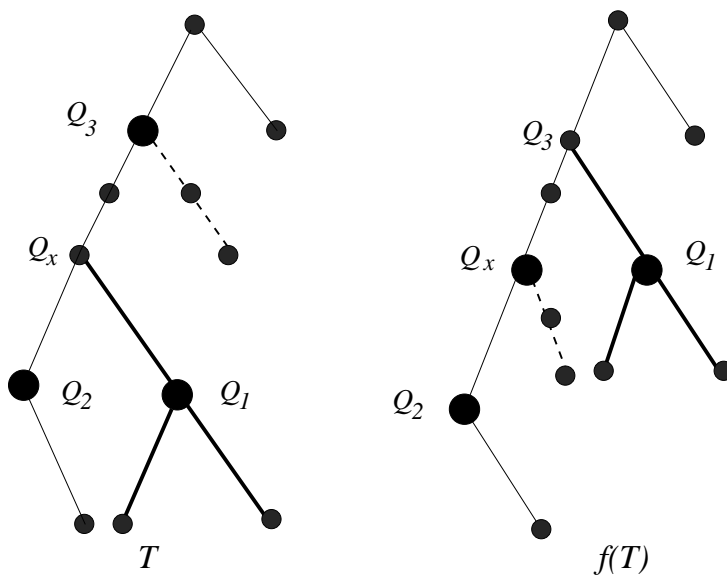


Figure 3: Interchanging the right subtrees of Q_x and Q_3 .

It is a direct consequence of our definitions that if $T \in A_n$, then $f(T) = B_n$. Now we are in a position to prove the main result of this section.

Theorem 2. *The map $f : A_n \rightarrow B_n$ defined above is a bijection.*

Proof. Let $U \in B_n$. We will show that there is exactly one $T \in A_n$ so that $f(T) = U$ holds. This will show that f has an inverse, proving that f is a bijection.

By definition, three nodes of U are colored black, and the entries of the permutation corresponding to U form a 231-pattern. Let K_2 , K_3 , and K_1 denote these three vertices, from left to right. There are two possibilities for the location of the K_i relative to each other. We will show that in both cases, U has a unique preimage under f , essentially because swapping two subtrees is an involution.

1. If K_3 is an ancestor of both other black vertices, then $f(T) = U$ implies that T belongs to Case 1. In this case, the unique $T \in A_n$ satisfying $f(T) = U$ is obtained by swapping the right subtrees of K_3 and K_2 , and keeping all three black vertices black, even if K_1 got moved.
2. If K_3 is not an ancestor of both other black vertices and then $f(T) = U$ implies that T belongs to Case 2. In this case, let K_x be the smallest common ancestor of K_3 and K_1 . Then the unique $T \in A_n$ satisfying $f(T) = U$ is obtained by swapping the right subtrees of K_3 and K_x , and coloring K_x black instead of K_3 , while keeping K_1 and K_2 black.

This completes the proof. □

4 A Generalization

In this section, we will significantly generalize the result of the previous section. The key observation is that in the proof of Theorem 2, the *left* subtrees of Q_1 and Q_2 never changed.

In order to state our result, we announce the following definitions.

Definition 3. Let q be a pattern of length k and let t be a pattern of length m . Then $q \oplus t$ is the pattern of length $k + m$ defined by

$$(q \oplus t)_i = \begin{cases} q_i & \text{if } i \leq k, \\ t_{i-k} + k & \text{if } i > k. \end{cases}$$

In other words, $q \oplus t$ is the concatenation of q and t so that all entries of t are increased by the size of q .

Example 4. If $q = 3142$ and $t = 132$, then $q \oplus t = 3142576$.

Definition 5. Let q be a pattern of length k and let t be a pattern of length m . Then $q \ominus t$ is the pattern of length $k + m$ defined by

$$(q \ominus t)_i = \begin{cases} q_i + m & \text{if } i \leq k, \\ t_{i-k} & \text{if } i > k. \end{cases}$$

In other words, $q \ominus t$ is the concatenation of q and t so that all entries of q are increased by the size of t .

Example 6. If $q = 3142$ and $t = 132$, then $q \ominus t = 6475132$.

Now we are ready to state and prove the most general result of this paper.

Theorem 7. *Let q and t be any non-empty patterns that end in their largest entry. Let i_u denote the increasing pattern $12 \cdots u$. Then for all positive integers n , we have*

$$S_{n,132}((q \ominus t) \oplus i_u) = S_{n,132}((q \oplus i_u) \ominus t),$$

where 1 denotes the pattern consisting of one entry.

In particular, the result of the previous section is the special case of Theorem 7 in which $q = t = i_u = 1$ (the one-entry pattern 1).

Example 8. If $q = 3124$, $t = 213$, and $u = 2$, then Theorem 7 says that

$$S_{n,132}(645721389) = S_{n,132}(645789213).$$

Proof. (of Theorem 7) Note that we can assume that q and t are both 132-avoiding, since otherwise the statement of Theorem 7 is trivially true as both sides are equal to 0.

Let k denote the length of q , let m denote the length of t . Similarly to the proof of Theorem 2, let A_n be the set of all binary plane trees on n vertices in which h vertices forming a $((q \ominus t) \oplus i_u)$ -pattern are colored black, and let B_n be the set of all binary plane trees on n vertices in which h vertices forming a $((q \oplus i_u) \ominus t)$ -pattern are colored black.

Let $T \in A_n$. Let Q_b be the k th black vertex of T in the in-order reading, let Q_a be the $(k + m)$ th black vertex of T , and let Q_c be the rightmost black vertex of T . We are now going to construct a bijection $F : A_n \rightarrow B_n$. The construction is analogous to the one that we saw before Theorem 2.

1. If Q_a is a right descendant of Q_b , then let $F(T)$ be the tree obtained from T by swapping the right subtree of Q_b and the right subtree of Q_c . Note that in $F(T)$, the black vertices form a $(q \ominus t) \oplus i_u$ -pattern, and that Q_c is an ancestor of all other black vertices in $F(T)$. See Figure 4 for an illustration.
2. Otherwise, there exists a lowest vertex $Q_x \in T$ so that Q_b is a left descendant of Q_x and Q_a is a right descendant of Q_x . Note that in this case, it follows that Q_x is not black. Now let $F(T)$ be the tree obtained from T by swapping the right subtree of Q_x and the right subtree of Q_c , and by coloring Q_x black, instead of Q_c . Note that again, in $F(T)$, the black vertices form a $(q \ominus t) \oplus i_u$ -pattern. Also note that there is no black vertex in $F(T)$ that would be an ancestor of all other black vertices. See Figure 5 for an illustration.

It is straightforward to show that $F : A_n \rightarrow B_n$ is a bijection. Indeed, let $U \in B_n$. If there is a black vertex in U that is an ancestor of all other black vertices, then U could only be obtained by the first rule, otherwise U could only be obtained by the second rule. The unique preimage $F^{-1}(U)$ is then obtained by swapping the appropriate right

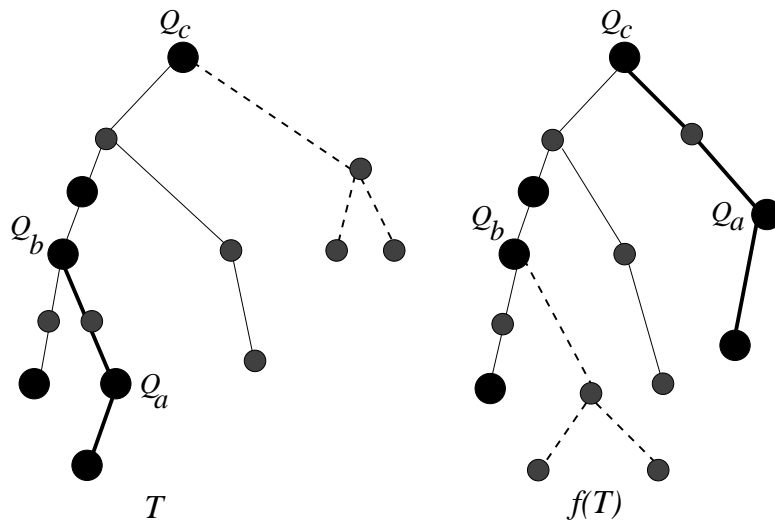


Figure 4: Interchanging the right subtrees of Q_b and Q_c , and turning a copy of 341256 into a copy of 345612.

subtrees. In the first case, swap the right subtrees of U_b and U_c , where U_b is the k th and U_c is the $(k + u)$ th black vertex of U in the in-order reading. In the second case, let U_x be the $(k + 1)$ st black vertex of U , let U_a be the last black vertex of U , and let U_c be the lowest common ancestor of U_x and U_a . Then the unique preimage $F^{-1}(U)$ is obtained by swapping the right subtrees of U_x and U_c , and coloring U_c black instead of U_x . \square

Note that by transitivity, Theorem 7 implies the following.

Corollary 9. *Let q , t , and i_u be as in Theorem 7, and let $1 \leq v < u$. Then we have*

$$S_{n,132}((q \oplus i_v) \ominus t) \oplus i_{u-v} = S_{n,132}((q \ominus t) \oplus i_u).$$

Proof. Theorem 7 shows that both sides are equal to $S_{n,132}((q \oplus i_u) \ominus t)$. \square

5 Further Directions

Formula (4) implies that $S_{n,132}(213) \sim C_1 4^n n$, while the generating functions computed in [4] imply that $S_{n,132}(321) \sim C_2 4^n n^{3/2}$ and $S_{n,132}(123) \sim C_3 4^n n^{1/2}$, where the C_i are positive constants. So occurrences of non-monotone patterns of length three are infinitely rare compared to occurrences of 321, and infinitely frequent compared to occurrences of 123; the frequency of non-monotone patterns is halfway between the two extremes.

While precise formulae like the ones given in earlier sections of this paper may not be obtainable for longer patterns, comparative results as the ones described in the previous paragraph may be possible to prove even for such patterns.

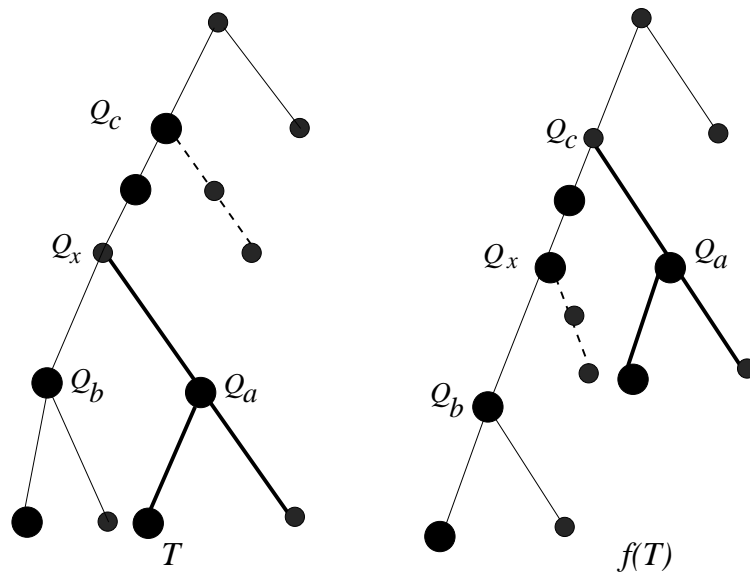


Figure 5: Interchanging the right subtrees of Q_x and Q_c , coloring Q_x black instead of Q_c , and turning a copy of 341256 into a copy of 345612.

If we set $u = 1$ and $h = k + m + 1$, then Theorem 7 provides

$$\sum_{i=2}^{h-1} c_{i-2}c_{h-i-1} = c_{h-2}$$

non-trivial examples of two patterns s and s' for which $S_{n,132}(s) = S_{n,132}(s')$ for all n . Other choices of u provide additional such pairs. However, it seems that there are other pairs of patterns whose total number of copies in all 132-avoiding permutations agree. We hope to discuss such pairs in an upcoming paper.

Are there any other such pairs? Are there any such pairs when 132 is replaced by another pattern r ? Are there any patterns r and r' for which $S_{n,r}(u) = S_{n,r'}(u')$ for all n and the equality is non-trivial?

Finally, how do the permutation statistics studied in this paper translate to the other 150 families of objects counted by the Catalan numbers listed in [7]?

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