STRANGE ADDING MACHINES

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Abstract. We show that given a type α of an adding machine, for a dense set of parameters s in the interval $[\sqrt{2}, 2]$, if f is the tent map with slope s, then the restriction of f to the closure of the orbit of the turning point is topologically conjugate to the adding machine map of type α.

1. Introduction and Preliminaries

There is a great deal of literature, including several books (for example [3] [8] [10]), concerning the dynamics of unimodal maps of an interval to itself. A major focus of attention is the behavior of the restriction of the map to the closure of the orbit of the turning point.

In this paper we show that a type of behavior not previously known to occur does indeed occur. We show that it is possible for a tent map that the restriction of the map to the closure of the orbit of the turning point is topologically conjugate to an adding machine map. In fact, this occurs for a dense set of parameters. Our precise results are given in Theorem 3.1 and Corollary 3.2.

Previously, the only known examples of unimodal maps with the restriction of the map to the closure of the orbit of the turning point topologically conjugate to an adding machine map were the so called infinitely renormalizable maps [3] [8] [10]. Of course, there are no infinitely renormalizable maps in the tent family [3, Proposition 3.4.26]. We will call adding machines embedded in unimodal maps that are not infinitely renormalizable *strange adding machines* (SAM).

Note that every adding machine (strange or not) embedded in a unimodal map has to contain the turning point. This is easily proved in Proposition 1.3 at the end of this section. Since adding machines are minimal, such an adding machine has to be equal to the closure of the trajectory of the turning point.

Date: March 8, 2004, revised October 21, 2005.

1991 *Mathematics Subject Classification.* 37E05, 54H20.

Key words and phrases. adding machine, tent maps.

The third author was partially supported by NSF grant DMS 0139916. We are grateful to the referee for suggestions about shortening the paper.

One might make a preliminary guess that the type of behavior we describe can not occur. One might reason that for adding machine maps there must be sets of small diameter which are cyclically permuted, while for tent maps pairs of points mostly move further apart under iteration. This paper shows that such a preliminary guess is wrong.

To put our result in perspective, we recall that for almost every tent map (in the sense of Lebesgue measure) with slope at least $\sqrt{2}$, the closure of the orbit of the turning point is dense in a certain interval [4]. On the other hand, it is already known that there are tent maps f with the closure of the orbit of the turning point an infinite set, such that the restriction of f to this set is a homeomorphism [5] or is a map similar to an adding machine, although not invertible everywhere [6].

Finally, we mention that our results can be applied to other families of unimodal maps, since many unimodal maps are topologically conjugate to tent maps [3] [8] [10]. For example, there are many (in fact, uncountably many, since there are uncountably many types of adding machines) quadratic maps with strange adding machines.

We now proceed with some preliminary definitions and results. First, we recall the definition of the α-adic adding machine Δ_α. Let $\alpha = (p_1, p_2, \dots)$ be a sequence of integers where each $p_i \geq 2$. Let Δ_α denote the set of all sequences $(x_1, x_2, \dots)$ where $x_i \in \{0, 1, \dots, p_i - 1\}$ for each i. We use the product topology on Δ_α.

Addition in Δ_α is defined as follows. We set

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (z_1, z_2, \dots)$$

where $z_1 = x_1 + y_1 \mod p_1$, $z_2 = x_2 + y_2 + t_1 \mod p_2$, etc. Here $t_1 = 0$ if $x_1 + y_1 < p_1$ and $t_1 = 1$ if $x_1 + y_1 \geq p_1$. So, we *carry* a one in the second case. Continue adding and carrying in this way for the whole sequence.

We define $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ by

$$f_\alpha(x_1, x_2, \dots) = (x_1, x_2, \dots) + (1, 0, 0, \dots).$$

We will refer to the map $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ as the *adding machine map* (see, e.g., [9]).

The following two theorems are well known (see, e.g., [2] and [7]).

Theorem 1.1. *Let $\alpha = (p_1, p_2, \dots)$ be a sequence of integers with $p_i \geq 2$ for each i. Let $j_i = p_1 \cdot p_2 \cdot \ldots \cdot p_i$ for each i. Let $f : X \to X$ be a continuous map of a compact metric space X. Then f is topologically conjugate to f_α if and only if* (1), (2), *and* (3) *hold.*

(1) *For each positive integer i, there is a cover P_i of X consisting of j_i pairwise disjoint, nonempty, clopen sets which are cyclically permuted by f.*

(2) *For each positive integer* i, P_{i+1} *partitions* P_i.
(3) *If* $\mathrm{mesh}(P_i)$ *denotes the maximum diameter of an element of the cover* P_i, *then* $\mathrm{mesh}(P_i) \to 0$ *as* $i \to \infty$.

We remark that if there is a positive integer K such that the conditions in Theorem 1.1 hold for each integer $i \geq K$, then the conditions hold for each integer i.

Theorem 1.2. *Let* $\beta = (p_1, p_2, \dots)$ *and* $\gamma = (r_1, r_2, \dots)$ *be sequences of integers with* $p_i \geq 2$ *and* $r_i \geq 2$ *for each* i. *We let* M_β *denote a function whose domain is the set of all prime numbers and which maps to the extended natural numbers* $\{0, 1, 2, \dots, \infty\}$. *The function* M_β *is defined by*

$$M_\beta(q) = \sum_{i=1}^{\infty} n_i$$

where n_i *is the power of the prime* q *in the prime factorization of* p_i.

Then f_β *and* f_γ *are topologically conjugate if and only if* $M_\beta = M_\gamma$.

A *tent map* of slope s is the map $T : [0,1] \to [0,1]$ given by $T(x) = sx$ if $x \leq \frac{1}{2}$ and $T(x) = s(1-x)$ if $x \geq \frac{1}{2}$.

A *unimodal map* is a continuous map $f : [0,1] \to [0,1]$ such that $f(0) = f(1) = 0$ and there is a point $c \in (0,1)$ such that f is increasing on $[0,c]$ and decreasing on $[c,1]$. In particular, tent maps are unimodal. The *itinerary* of a point $x \in [0,1]$ is the sequence $i_0, i_1, i_2, \dots$, where i_n is L if $f^n(x) < c$, C if $f^n(x) = c$ and R if $f^n(x) > c$. The *kneading invariant* $\mathcal{K}(f)$ of f is the itinerary of $f(c)$. The reader can find the basics of the kneading theory for instance in [8].

For a map $f : X \to X$ we will call the set $\{f^n(x) : n = 0, 1, 2, \dots\}$ the *orbit* of x.

We conclude this section with a basic proposition mentioned earlier.

Proposition 1.3. *Let* $f : [0,1] \to [0,1]$ *be a unimodal map with turning point* c. *Suppose that* X *is a closed invariant subset of* $[0,1]$, *such that* $f|_X$ *is topologically conjugate to an adding machine map. Then* $c \in X$.

Proof. Let us use the notation of Theorem 1.1. Suppose that c does not belong to X. Then, by Theorem 1.1 (3), there exists n such that every element of P_n is contained in either $[0,c)$ or $(c,1]$. By Theorem 1.1 (1), f maps each element of P_n monotonically onto another element of P_n. Let S denote the set of points which are either the largest point of an element of P_n or the smallest point point of an element of P_n. Then S is a non-empty, proper, closed subset of X, and is invariant under f. This is a contradiction, as X is a minimal set under f. □

2. SAM-schemes

In this section we consider tent maps with turning point $c = \frac{1}{2}$. Let A be a subset of $[0, 1]$. We say that A *straddles* c if A contains points to the left of c and the right of c, but $c \notin A$.

For a subset B of the real line, we use the notation $[B]$ to denote the convex hull of B. Also, if A and B are subsets of the real line we use the notation $A < B$ to mean that for each pair of points $x \in A$ and $y \in B$ we have $x < y$.

Definition 2.1. Let f be a tent map. A *SAM-scheme* (strange adding machine scheme) for f of length $n = m + k + 1$ is a collection $\mathcal{C}$ of disjoint, closed, subintervals of [0,1],

$$\mathcal{C} = \{L_2, L_1, R_1, R_2, A_1, \ldots, A_m, B_1, \ldots, B_m, Y_1, \ldots, Y_k\}$$

such that each of the following holds.

(1) $L_2 < L_1 < \{c\} < R_1 < R_2$.
(2) For each $i = 1, \ldots, m$, $A_i \cap [L_2 \cup R_2] = \emptyset$ and $B_i \cap [L_2 \cup R_2] = \emptyset$. Also, for each $i = 1, \ldots, k$, $Y_i \cap [L_2 \cup R_2] = \emptyset$.
(3) $f(L_1) = f(R_1) = A_1$ and $f(L_2) = f(R_2) = B_1$.
(4) For each $i = 1, \ldots, m-1$, $f(A_i) = A_{i+1}$ and $f(B_i) = B_{i+1}$.
(5) $f(A_m) = f(B_m) = Y_1$.
(6) For each $i = 1, \ldots, k-1$, $f(Y_i) = Y_{i+1}$.
(7) The set $A_m \cup B_m$ straddles c, but for each $i = 1, \ldots, m-1$, the set $A_i \cup B_i$ does not straddle c.
(8) $f(Y_k) = [L_2 \cup R_2]$.

See Figure 1 to help visualize the above definition.

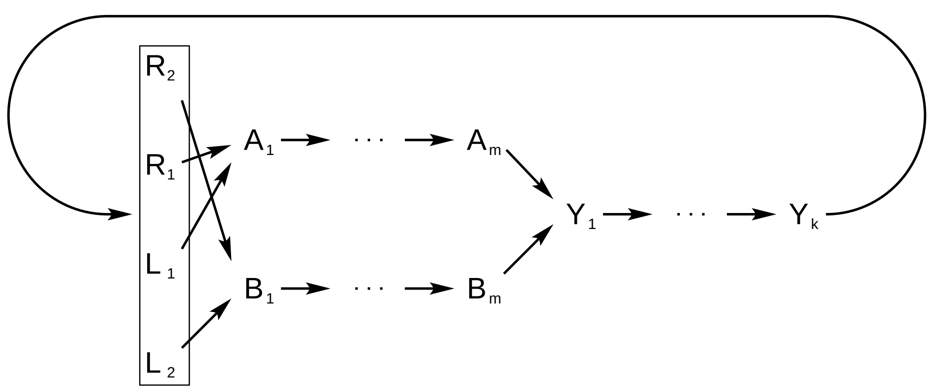

Figure 1. SAM-scheme

Definition 2.2. Let f be a tent map. Let

$$\mathcal{C} = \{L_2, L_1, R_1, R_2, A_1, \ldots, A_m, B_1, \ldots, B_m, Y_1, \ldots, Y_k\}$$

and

$$\mathcal{C}' = \{L_2', L_1', R_1', R_2', A_1', \ldots, A_s', B_1', \ldots, B_s', Y_1', \ldots, Y_t'\}$$

be SAM-schemes for f. Then $\mathcal{C}'$ is a *refinement* of $\mathcal{C}$ provided the following four conditions hold.

(1) Each interval in the collection $\mathcal{C}'$ is a subset of an interval in the collection $\mathcal{C}$.
(2) $L_2' \subset L_1, L_1' \subset L_1, R_1' \subset R_1$, and $R_2' \subset R_1$.
(3) For each $i = 1, \ldots, s-1$, $A_i' \cup B_i'$ is contained in one of the intervals in the collection $\mathcal{C}$.
(4) One of the two intervals A_s', B_s' is contained in L_2, and the other interval is contained in R_2.

Definition 2.3. Let f be a tent map. A *path* is a finite sequence of intervals, $G_0, G_1, \ldots, G_n$ such that for each $i = 0, 1, \ldots, n-1$, f is strictly monotone on G_i and $f(G_i) \supset G_{i+1}$. The *interval determined by this path* is the unique subinterval J of G_0 such that $f^i(J) \subset G_i$ for $i = 1, \ldots, n-1$ and $f^n(J) = G_n$.

Lemma 2.4. *Suppose that f is a tent map and*

$$\mathcal{C} = \{L_2, L_1, R_1, R_2, A_1, \ldots, A_m, B_1, \ldots, B_m, Y_1, \ldots, Y_k\}$$

is a collection of disjoint, closed, subintervals of $[0,1]$ *such that properties* (1) *through* (7) *of Definition 2.1 hold. Suppose also that* $f(Y_k) \supset [L_2 \cup R_2]$.

Then there is a SAM-scheme for f

$$\mathcal{C}' = \{L_2', L_1', R_1', R_2', A_1', \ldots, A_m', B_1', \ldots, B_m', Y_1', \ldots, Y_k'\}$$

such that each interval in $\mathcal{C}'$ is a subset of the corresponding interval in $\mathcal{C}$.

Proof. We form a collection

$$\mathcal{C}^i = \{L_2^i, L_1^i, R_1^i, R_2^i, A_1^i, \ldots, A_m^i, B_1^i, \ldots, B_m^i, Y_1^i, \ldots, Y_k^i\}$$

for each positive integer i. Let $\mathcal{C}^1 = \mathcal{C}$. There is a unique subinterval Y_k^2 of Y_k^1 with $f(Y_k^2) = [L_2^1 \cup R_2^1]$. Continuing, we obtain a collection

$$\mathcal{C}^2 = \{L_2^2, L_1^2, R_1^2, R_2^2, A_1^2, \ldots, A_m^2, B_1^2, \ldots, B_m^2, Y_1^2, \ldots, Y_k^2\}$$

satisfying (1) through (7) of Definition 2.1. Inductively, we obtain $\mathcal{C}^3, \mathcal{C}^4, \ldots$.

Finally, set

$$\mathcal{C}' = \{L_2', L_1', R_1', R_2', A_1', \ldots, A_m', B_1', \ldots, B_m', Y_1', \ldots, Y_k'\}$$

where $L_2' = \bigcap_{i=1}^{\infty} L_2^i$, etc. Then $\mathcal{C}'$ satisfies the desired properties. In particular, none of its elements can be degenerate (that is, consisting of one point only), since $[L_2' \cup R_2']$ is not degenerate. □

Lemma 2.5. *Suppose f is a tent map, and m and k are positive integers. Suppose $\mathcal{C}$ is a SAM-scheme for f of length $n = m + k + 1$,*

$$\mathcal{C} = \{L_2, L_1, R_1, R_2, A_1, \dots, A_m, B_1, \dots, B_m, Y_1, \dots, Y_k\}.$$

Suppose that $n' = j \cdot n$ for some integer $j \geq 2$. Then there are positive integers s and t such that $s + t + 1 = n'$ and a SAM-scheme $\mathcal{C}'$ for f of length n',

$$\mathcal{C}' = \{L_2', L_1', R_1', R_2', A_1', \dots, A_s', B_1', \dots, B_s', Y_1', \dots, Y_t'\}$$

such that $\mathcal{C}'$ is a refinement of $\mathcal{C}$.

Proof. It may be helpful in following this proof to look at Figure 1. Each arrow in Figure 1 indicates that f maps the first interval linearly onto the second.

Set $t = k + m$, and set $s = (j-1)(m+k+1)$. Let Y_1' be the interval determined by the path

$$B_1, B_2, \dots, B_m, Y_1, \dots, Y_k, [L_1 \cup R_1].$$

Let $Y_i' = f^{i-1}(Y_1')$ for $i = 2, \dots, t$. Let S represent the finite sequence of intervals $L_2, B_1, \dots, B_m, Y_1, \dots, Y_k$. Let C be the interval determined by the path

$$A_1, \dots, A_m, Y_1, \dots, Y_k, S, \dots, S, L_2, Y_1' \tag{2.1}$$

where the finite sequence S appears $j - 2$ consecutive times. Similarly, let D be the interval determined by the path

$$A_1, \dots, A_m, Y_1, \dots, Y_k, S, \dots, S, R_2, Y_1'. \tag{2.2}$$

Then C and D are disjoint closed intervals. Let A_1' denote the interval C or D which is to the right of the other, and let B_1' denote the interval which is to the left of the other. Set $A_i' = f^{i-1}(A_1')$ and $B_i' = f^{i-1}(B_1')$ for $i = 2, \dots, s$.

Finally, there are unique subintervals L_2', L_1' of L_1 and R_1', R_2' of R_1 such that $f(L_1') = f(R_1') = A_1'$ and $f(L_2') = f(R_2') = B_1'$. Let

$$\mathcal{C}' = \{L_2', L_1', R_1', R_2', A_1', \dots, A_s', B_1', \dots, B_s', Y_1', \dots, Y_t'\}.$$

Then $\mathcal{C}'$ is a collection of disjoint closed subintervals of $[0, 1]$ satisfying the hypothesis of Lemma 2.4. In particular, condition (7) of Definition 2.1 is satisfied since the paths (2.1) and (2.2) are identical except at the penultimate place. By Lemma 2.4 we may assume that $\mathcal{C}'$ is a SAM-scheme, and by construction, $\mathcal{C}'$ is a refinement of $\mathcal{C}$. □

3. Main Theorem

In this section we prove the main results of the paper.

Theorem 3.1. *Let $\alpha = (p_1, p_2, \dots)$ be a sequence of integers greater than 1. The set of parameters s, such that for the tent map f_s, the restriction of f_s to the closure of the orbit of $c = \frac{1}{2}$ is topologically conjugate to $f_\alpha : \Delta_\alpha \to \Delta_\alpha$, is dense in $[\sqrt{2}, 2]$.*

Proof. Let $i_0, i_1, \dots, i_n$ be a finite sequence of L's and R's. It suffices to show that if $f = f_s$ is a tent map with $s > \sqrt{2}$ such that $\mathcal{K}(f)$ begins with $i_0, i_1, \dots, i_n$, then there is a tent map G such that $\mathcal{K}(G)$ also begins with $i_0, i_1, \dots, i_n$, and the restriction of G to the closure of the orbit of c under G is topologically conjugate to f_α.

So, suppose we are given $f = f_s$ as specified. Since $s > 1$, there is a unique fixed point p of f with $p > c$. We may assume that p is not in the orbit of c. Also, $f^2(c) < c < p < f(c)$.

We construct G as desired. We divide the proof into steps.

Step 1. In this step we construct a SAM-scheme for f.

Choose a closed interval $K \subset (f^2(c), p)$ symmetric around c, such that each point in $f(K)$ has an itinerary which begins with $i_0, i_1, \dots, i_n$. We choose K so that p is not in the orbit of the endpoints of K. There is a unique closed interval $Y \subset (p, f(c))$ such that $f(Y) = K$. Since $f^3(c) < p$ (as $s > \sqrt{2}$), there is a unique closed interval $V \subset (f^2(c), c)$ such that $f(V) = Y$. Also, there is a unique closed interval $W \subset [c, p]$ such that $f(W) = Y$. We may choose K small enough to insure that $f^3(K) < \{p\}$ and also that the collection of intervals $\{K, f(K), f^2(K), V, W, Y\}$ is pairwise disjoint. Then

$$f^2(K) < V < K < W < \{p\} < Y < f(K).$$

There are unique closed intervals V_1 and W_1 such that $\{p\} < W_1 < Y < V_1 < f(K)$, $f(V_1) = V$, and $f(W_1) = W$. Then there are unique closed intervals V_2 and W_2 such that $K < V_2 < W < W_2 < \{p\}$, $f(V_2) = V_1$, and $f(W_2) = W_1$. By induction we obtain for each positive integer i closed intervals $V_i, W_i > \{c\}$ such that $f(V_{i+1}) = V_i$ and $f(W_{i+1}) = W_i$. Also, the length of V_i (as well as W_i) goes to zero, and the intervals V_i (as well as the intervals W_i) become arbitrarily close to p as i increases. Moreover, the intervals $K, f(K), f^2(K), Y, V, W, V_i, W_i$ are all pairwise disjoint.

Since $s > \sqrt{2}$, there is a point $q \in f(K)$ such that p is in the orbit of q. We may choose q so that for some positive integer j, $f^j(q) = p$, while for each $i = 0, 1, \dots, j-1$, $f^i(q) \neq p$ and $f^i(q) \notin K$. It follows that the orbit of q is disjoint from the union of the intervals K, V, W, Y,

V_i, W_i. So, there is a closed interval U containing q in its interior with $U \subset f(K)$ such that for each $i = 0, 1, \ldots, j-1$, $f^i(U)$ is disjoint from the union of these intervals. We may choose U so that the intervals $U, f(U), \ldots, f^{j-1}(U)$ are pairwise disjoint.

Now, f^j maps U linearly onto an interval containing p in its interior. So, $f^j(U)$ contains all of the intervals V_i and W_i for i sufficiently large. In particular, there is a positive integer r such that for some positive integer d, $j+r+3$ is the product $p_1 \cdot p_2 \cdot \ldots \cdot p_d$ of the first d integers in the sequence α, and such that $f^j(U) \supset V_r \cup W_r$. By construction the intervals $K, Y, V, V_1, \ldots, V_r, W_1, \ldots, W_r, U, f(U), \ldots, f^{j-1}(U)$ are pairwise disjoint.

There are unique closed intervals U_V and U_W in U with $f^j(U_V) = V_r$ and $f^j(U_W) = W_r$. Then $U_V \cap U_W = \emptyset$. We may assume without loss of generality that $U_V > U_W$.

There are closed intervals $L_2 < L_1 < \{c\} < R_1 < R_2$ in K such that $f(L_1) = f(R_1) = U_V$ and $f(L_2) = f(R_2) = U_W$. Set $m = j + r + 1$. Let $A_1 = U_V$ and $B_1 = U_W$. Let $A_i = f^{i-1}(A_1)$ and $B_i = f^{i-1}(B_1)$ for each $i = 1, \ldots, m$. Then $A_m = V$ and $B_m = W$.

By construction we have a collection of pairwise disjoint closed subintervals of $[0, 1]$

$$\mathcal{C} = \{L_2, L_1, R_1, R_2, A_1, \ldots, A_m, B_1, \ldots, B_m, Y\}$$

such that properties (1) through (7) of Definition 2.1 hold (with $k = 1$), and in addition, $f(Y) \supset [L_2 \cup R_2]$. By Lemma 2.4, we may assume that $\mathcal{C}$ is a SAM-scheme for f.

Finally, recall that $(A_1 \cup B_1) \subset U \subset f(K)$. Hence, we have the following additional property.

(*) Each point of $A_1 \cup B_1$ has itinerary which begins $i_0, i_1, \ldots, i_n$.

Step 2. In this step we obtain for each positive integer n a SAM-scheme $\mathcal{C}_n$ for f such that $\mathcal{C}_{n+1}$ is a refinement of $\mathcal{C}_n$. We use this to obtain a closed set C which is invariant under f. Finally, we make a truncation to obtain a map F, and then make an identification to obtain a new unimodal map G of an interval J to itself.

We let $\mathcal{C}_1$ denote the collection of closed intervals obtained in Step 1. From now on we denote the intervals $L_2, L_1, \ldots$ in $\mathcal{C}_1$ by $L_2^1, L_1^1, \ldots$ and let m be denoted by m_1.

We apply Lemma 2.5 to get a SAM-scheme

$$\mathcal{C}_2 = \{L_2^2, L_1^2, R_1^2, R_2^2, A_1^2, \ldots, A_{m_2}^2, B_1^2, \ldots, B_{m_2}^2, Y_1^2, \ldots, Y_{k_2}^2\}$$

which is a refinement of $\mathcal{C}_1$. For the positive integer j which appears in Lemma 2.5 we take $j = p_{d+1}$, where p_{d+1} is the next term term in the sequence α. In the first step we used the product of the terms $p_1, \ldots, p_d$ in the sequence α. In the subsequent steps we only use one of the p_i's at each successive step.

By Lemma 2.5 and induction we obtain for each positive integer n a SAM-scheme $\mathcal{C}_n$ such that $\mathcal{C}_{n+1}$ is a refinement of $\mathcal{C}_n$. Let C_n denote the set of x such that each point in the orbit of x lies in some interval in $\mathcal{C}_n$. Let $C = \bigcap_{n=1}^{\infty} C_n$. Then C is closed and invariant under f.

Let a denote the rightmost element of C. For each positive integer n, there are points in L_1^n and R_1^n which map to a. It follows that the two preimages of a are both in C. We denote these preimages by b and b' where $b < b'$. Observe that there are no elements of C in the open interval (b, b').

Next, we define a truncated tent map F by $F(x) = \min\{f(x), a\}$. Then $F(x) = f(x)$ for each $x \in C$.

Finally, consider all of the components of $F^{-n}([b, b'])$ where $n = 0, 1, 2, \ldots$. Since b and b' are not periodic, and their orbits are disjoint from the open interval (b, b'), this yields a collection of pairwise disjoint closed intervals. We create a new interval J with a natural order by collapsing each one of these intervals to a point. Then the natural projection $\varphi : [0, 1] \to J$ is increasing.

There is a unique continuous map $G : J \to J$ such that $G(\varphi(x)) = \varphi(F(x))$ for each $x \in [0, 1]$. Let $\tilde{c} = \varphi(b)$. Then G is a unimodal map with turning point $\tilde{c}$. Moreover, $\mathcal{K}(G)$ begins with $i_0, i_1, \ldots, i_n$. This completes Step 2.

Step 3. In this step we make some observations concerning the action of f on C and the action of G on $\varphi(C)$.

Let $\mathcal{D}_n$ be the collection of intervals obtained from $\mathcal{C}_n$ by replacing the four intervals $L_2^n, L_1^n, R_1^n, R_2^n$ by one interval $M_n = [L_2^n \cup R_2^n]$. Then $\mathcal{D}_n$ is a collection of disjoint intervals. Moreover, the set E_n consisting of each endpoint of each interval in $\mathcal{D}_n$ is invariant under f. Since $[b, b']$ is contained in the interior of M_n, $F|_{E_n} = f|_{E_n}$. So, $F(E_n) \subset E_n$. It follows that no element of E_n is contained in an interval which is identified to a point in forming the new interval J. Hence, if A and B are intervals in $\mathcal{D}_n$ with $A < B$, then $\varphi(A)$ and $\varphi(B)$ are intervals with $\varphi(A) < \varphi(B)$.

Let j_i be defined as in Theorem 1.1, and recall the remark following Theorem 1.1. By construction, for each positive integer $i \geq d$, the points of C may be grouped into j_i non-empty disjoint sets which are cyclically permuted by f and F as in Theorem 1.1 (see the columns of Figure 1). Of course, the mesh of these covers does not go to zero as required in Theorem 1.1. It follows that the points of $\varphi(C)$ may be grouped into disjoint sets cyclically permuted by G as in Theorem 1.1. We will ultimately show that in this case the mesh does go to zero. This completes Step 3.

Step 4. Let $M_n = [v_n, w_n]$. We will show that $\varphi(v_n) \to \tilde{c}$ and $\varphi(w_n) \to \tilde{c}$.

To do this, it suffices to show that $v_n \to b$ and $w_n \to b'$. By construction, each of the intervals L_1^n and R_1^n is mapped linearly onto M_n by $f^{j_1 \cdots j_n}$. Since the length of M_n is bounded from above, the lengths of L_1^n and R_1^n must go to zero. But $M_{n+1} \subset L_1^n \cup R_1^n \cup [b, b']$, $b \in L_1^n$, and $b' \in R_1^n$. It follows that $v_n \to b$ and $w_n \to b'$. This completes Step 4.

Step 5. In this step we show that the map $G : J \to J$ is topologically conjugate to a tent map.

It follows from our construction and Step 4 that for any open subinterval D of $[G^2(\tilde{c}), G(\tilde{c})]$ which contains the critical point $\tilde{c}$, there is a positive integer k such that $G^k(D) = [G^2(\tilde{c}), G(\tilde{c})]$. We leave the straightforward proof of this to the reader.

Let D be an open subinterval of $[G^2(\tilde{c}), G(\tilde{c})]$. For some $j \geq 0$, $F^j(\varphi^{-1}(D))$ contains a neighborhood of $[b, b']$. It follows from Step 3 that no such neighborhood is collapsed to a point by φ. Therefore, $G^j(D)$ contains a neighborhood of $\tilde{c}$. Hence, some iterate of G maps D onto the interval $[G^2(\tilde{c}), G(\tilde{c})]$.

Thus, G restricted to $[G^2(\tilde{c}), G(\tilde{c})]$ is topologically transitive, and by the theorem of Parry [11] (see also [1]), it is conjugate to a tent map. This conjugacy can be extended to the whole interval J. This completes Step 5.

Step 6. We show that G satisfies the desired properties stated in the first paragraph of the proof. By Step 4, we may (and we do) assume that G is a tent map. Also, by construction $\mathcal{K}(G)$ begins with $i_0, i_1, \ldots, i_n$.

We will show that $G|_{\varphi(C)}$ satisfies the hypothesis of Theorem 1.1. It will follow that $G|_{\varphi(C)}$ is topologically conjugate to f_α. Moreover, since $c = \tilde{c} \in \varphi(C)$ and f_α is minimal, it will follow that $\varphi(C)$ is the closure of the orbit of c under G.

We have already observed in Step 3 that we have a sequence of clopen covers $(\mathcal{P}_i)$ of $\varphi(C)$ cyclically permuted as in Theorem 1.1. It remains to show that $\mathrm{mesh}(\mathcal{P}_i) \to 0$.

By construction (see Definition 2.1 (7)), for each positive integer i, there is a unique set S_i in $\mathcal{P}_i$ which straddles c. Also, if $i > 1$ we have

$S_i \subset \varphi(M_{i-1})$ (see Definition 2.2 (4)). Let T_i denote the unique set in $\mathcal{P}_i$ with $c \in T_i$. Then $T_i \subset \varphi(M_i) \subset \varphi(M_{i-1})$.

Consider any set E_i in $\mathcal{P}_i$ with $E_i \neq S_i$ and $E_i \neq T_i$. By construction, there is a smallest positive integer j such that $G^j(E_i) \subset \varphi(M_{i-1})$. Then none of the sets $E_i, G(E_i), \ldots, G^{j-1}(E_i)$ contains c or straddles c. It follows that G^j maps the convex hull of E_i linearly into $\varphi(M_{i-1})$. Since G has slope larger than 1, $\text{diam}(E_i) < \text{diam}(\varphi(M_{i-1}))$. It follows that for each $i > 1$, $\text{mesh}(\mathcal{P}_i) < \text{diam}(\varphi(M_{i-1}))$. By Step 4, $\text{diam}(\varphi(M_i)) \to 0$. Hence, $\text{mesh}(\mathcal{P}_i) \to 0$. This completes the proof. □

It is well-known that the tent map with slope $a \in (2^{2^{-(k+1)}}, 2^{2^{-k}}]$ are k times renormalizable. The k-th renormalization is a tent map with slope a^{2^k}. This observation gives us the following corollary.

Corollary 3.2. *Let $\alpha = (p_1, p_2, \ldots)$ be a sequence of integers with $p_i \geq 2$ for each i. Suppose that $M_\alpha(2) \geq k$ for some positive integer k (where M_α is defined in Theorem 1.2). Then the set of parameters s, such that for the tent map f_s, the restriction of f_s to the closure of the orbit of c is topologically conjugate to $f_\alpha : \Delta_\alpha \to \Delta_\alpha$, is dense in* $[2^{2^{-(k+1)}}, 2]$. *In particular, if $M_\alpha(2) = \infty$, then this set of parameters is dense in* $[1, 2]$.

Finally, we remark that since every tent map with slope from $(1, 2]$ and infinite orbit of c is topologically conjugate to a quadratic map, our results show that there are uncountably many quadratic maps with strange adding machines.

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