TOPOLOGICAL ENTROPY AND ADDING MACHINE MAPS

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Abstract. We prove two theorems which extend known results concerning periodic orbits and topological entropy in one-dimensional dynamics. One of these results concerns the adding machine map (also called the odometer map) f_α defined on the α-adic adding machine Δ_α. We let $H(f_\alpha)$ denote the greatest lower bound of the topological entropies of F, taken over all continuous maps F of the interval which contain a copy of f_α. We prove that if α is a sequence of primes such that 2 appears in the sequence exactly k times, then $H(f_\alpha) = \frac{\log 2}{2^{k+1}}$.

1. Introduction

Many important results in one-dimensional dynamics deal with coexistence of periodic points and connections between periodic points and topological entropy. It is a natural problem to try to extend these results to infinite closed, invariant sets. One result along these lines has been obtained by Ye [14]; this result deals with coexistence of minimal sets of certain types.

In this paper we focus attention on extending two basic results involving periodic points and topological entropy. These two results are the following. Let I denote the interval $[0, 1]$. For a continuous map $F : I \to I$, let $h(F)$ denote the topological entropy of F. The first of these basic results gives for each positive integer t, the minimal value of the topological entropy of a map of the interval having a periodic point of period t. A proof may be found in [2], Corollary 4.4.18, page 230, or [6], Proposition VIII.21, page 206.

Theorem 1.1. *Let $t = 2^s \cdot n$ where n is an odd integer with $n \geq 3$ and s is a non-negative integer. Let λ_n denote the largest root of the polynomial $x^n - 2x^{n-2} - 1$. If $F : I \to I$ is continuous and has a periodic point of period t, then $h(F) \geq \frac{\log \lambda_n}{2^s}$. Moreover, there are examples where equality holds.*

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The second of these basic results gives for a particular cyclic permutation π of a finite subset X of I, a class of maps which have minimum entropy among the set of all continuous maps of I to itself which agree with π on X. A proof may be found in [2], Theorem 4.4.3, page 222 and Theorem 4.4.5, page 223, or [6], Proposition VIII.20, page 206.

Theorem 1.2. *Let X be a finite subset of I which contains the endpoints of I. Let $F : I \to I$ and $G : I \to I$ be continuous. Suppose that X is invariant under both F and G and $F|_X = G|_X$. Suppose also that F is monotonic on each interval joining adjacent points of X. Then $h(F) \leq h(G)$.*

In section 3, we extend Theorem 1.1 to a special class of minimal sets. The map on one of these minimal sets we call an adding machine map. These maps have also been called odometer maps and solenoids. Adding machine maps play a major role in the description of the dynamics of maps of a one-dimensional space. The spectral decomposition theorem of Blokh [5] shows this. Of course, adding machine maps arise in other settings in dynamics. Two examples are the papers by Bell and Meyer [3] and Marcus and Meyer [11]. In [7] the authors characterize adding machine maps as infinite minimal sets in which every point is regularly recurrent.

Given any sequence α of primes, one may define a corresponding adding machine map f_α (see section 2). We say that a continuous map $F : I \to I$ contains a copy of f_α if the restriction of F to some closed invariant subset of I is topologically conjugate to f_α. We let $H(f_\alpha)$ denote the greatest lower bound of $h(F)$ taken over all continuous maps $F : I \to I$ which contain a copy of f_α. We show that if α is a sequence of primes such that 2 appears in the sequence exactly k times (where $k \geq 0$), then $H(f_\alpha) = \frac{\log 2}{2^{k+1}}$. Moreover, $h(F) > \frac{\log 2}{2^{k+1}}$ for any continuous map $F : I \to I$ which contains a copy of f_α.

Finally, in section 4, we extend Theorem 1.2 to the case where X is an arbitrary closed invariant set.

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2. Preliminary results

In this section we give definitions and elementary properties of adding machines. There are also results that will be needed in the subsequent sections. First we define the *adding machines*, Δ_α.

Definition 2.1. Let $\alpha = (j_1, j_2, \dots)$ be a sequence of integers where each $j_i \geq 2$. Let Δ_α denote all sequences $(x_1, x_2, \dots)$ where $x_i \in$

$\{0, 1, \ldots, j_i - 1\}$ for each i. We put a metric d_α on Δ_α given by

$$d_\alpha((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{\delta(x_i, y_i)}{2^i},$$

where $\delta(x_i, y_i) = 1$ if $x_i \neq y_i$ and $\delta(x_i, y_i) = 0$ if $x_i = y_i$.

We define $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ by

$$f_\alpha(x_1, x_2, \ldots) = (x_1, x_2, \ldots) + (1, 0, 0, \ldots).$$

Here the addition is defined as in [10], page 108.

The term *adding machine* is used to refer to the compact topological group Δ_α whose underlying space is homeomorphic to a Cantor set. See Hewitt and Ross [10] for background on these topological groups. We will refer to the map $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ as the *adding machine map.*

We now give a necessary and sufficient condition for a continuous map of a compact topological space to itself to be topologically conjugate to $f_\alpha : \Delta_\alpha \to \Delta_\alpha$. The following theorem is proved in [7].

Theorem 2.2. *Let $\alpha = (j_1, j_2, \ldots)$ be a sequence of integers with $j_i \geq 2$ for each i. Let $m_i = j_1 \cdot j_2 \cdot \cdots \cdot j_i$ for each i. Let $f : X \to X$ be a continuous map of a compact topological space X. Then f is topologically conjugate to f_α if and only if* (1)*,* (2)*, and* (3) *hold.*
(1) *For each positive integer i, there is a cover P_i of X consisting of m_i pairwise disjoint, nonempty, clopen sets which are cyclically permuted by f.*
(2) *For each positive integer i, P_{i+1} partitions P_i.*
(3) *If $W_1 \supset W_2 \supset W_3 \supset \cdots$ is a nested sequence with $W_i \in P_i$ for each i, then $\cap_{i=1}^{\infty} W_i$ consists of a single point.*

Moreover, in this case statement (4) *also holds.*
(4) *X is metrizable and if* $\mathrm{mesh}(P_i)$ *denotes the maximum diameter of an element of the cover P_i, then* $\mathrm{mesh}(P_i) \to 0$ *as $i \to \infty$.*

We conclude this section with the following classification of adding machine maps up to topological conjugacy. This result was obtained by Buescu and Stewart [8]. A different proof may be found in [7].

Theorem 2.3. *Let $\beta = (j_1, j_2, \ldots)$ and $\gamma = (k_1, k_2, \ldots)$ be sequences of integers with $j_i \geq 2$ and $k_i \geq 2$ for each i. We let M_β denote a function whose domain is the set of all prime numbers and which maps to the extended natural numbers $\{0, 1, 2, \ldots, \infty\}$. The function M_β is defined by*

$$M_\beta(p) = \sum_{i=1}^{\infty} n_i$$

where n_i is the power of the prime p in the prime factorization of j_i.

Then f_β and f_γ are topologically conjugate if and only if $M_\beta \equiv M_\gamma$.

In light of Theorem 2.3, there is no loss of generality in assuming that the sequence α used in defining Δ_α is a sequence of primes. However, in some contexts it will be more convenient to let α be a sequence of integers which are not required to be prime.

3. Computation of $H(f_\alpha)$

We begin this section with several basic definitions concerning cycles.

Definition 3.1. Let $P = \{p_1 < p_2 < \cdots < p_k\}$ and $Q = \{q_1 < q_2 < \cdots < q_t\}$ be finite ordered sets, and let $\pi : P \to P$ and $\psi : Q \to Q$ be cyclic permutations. We say that π is equivalent to ψ if $k = t$ and for each $i = 1, \ldots, k$ and $j = 1, \ldots, k$, we have that $\pi(p_i) = p_j$ if and only if $\psi(q_i) = q_j$.

We use the notation $\#\pi$ to denote the cardinality of P. We will call a cyclic permutation of a finite ordered set a *cycle*.

For a continuous map $F : I \to I$ we let $h(F)$ denote the topological entropy of F as defined in [1]. See also [2] or [6] for the definition and basic properties of topological entropy.

Definition 3.2. For a cycle π we let $h(\pi)$ denote $\inf h(F)$ where the infimum is taken over all continuous maps $F : I \to I$ such that F has a period orbit equivalent to π (i.e., there is a finite subset P of I which is cyclically permuted by F such that $F|P : P \to P$ is equivalent to π).

Notation 3.3. Let $q \geq 3$ be an odd positive integer. Let λ_q denote the largest root of the polynomial $x^q - 2x^{q-2} - 1$. It is easy to verify that

$$\sqrt{2} < \lambda_q < 2$$

and $\lim_{n\to\infty} \lambda_n = \sqrt{2}$. See page 233 of [2].

Proposition 3.4. *Let $t = 2^s \cdot n$ where n is an odd integer with $n \geq 3$ and s is a non-negative integer.*

(i) *If π is a cycle with $\#\pi = t$, then $h(\pi) \geq \frac{\log \lambda_n}{2^s}$.*

(ii) *There is a cycle π with $\#\pi = t$ and $h(\pi) = \frac{\log \lambda_n}{2^s}$.*

Proof. See Corollary 4.4.18 of [2] or Proposition VIII.21 of [6]. □

Lemma 3.5. *Suppose that π is a cycle and $\#\pi = 2^s$ where $s \geq 2$. Suppose that $h(\pi) > 0$. Then $h(\pi) \geq \frac{\log \lambda_3}{2^{s-2}}$.*

Proof. Since $h(\pi) > 0$, the cycle π is not simple (where simple is as defined in [4]). By Theorem B of [4], if $F : I \to I$ is any continuous map which has a periodic orbit equivalent to π, then F has a periodic point of period $3 \cdot 2^{s-2}$. Hence by Proposition 3.4

$$h(\pi) \geq \frac{\log \lambda_3}{2^{s-2}}.$$

□

Lemma 3.6. *Let $\alpha = (j_1, j_2, \dots)$ where each j_i is an integer at least two. There exists a parameter $s = s_\alpha$ such that the quadratic map $F_s : I \to I$ defined by $F_s(x) = sx(1-x)$ contains a copy of $f_\alpha : \Delta_\alpha \to \Delta_\alpha$.*

Proof. We summarize how this follows from results in [12]. By theorem 6.2 on page 156 each F_s has no wandering intervals. By Proposition 4.5 on page 242, if F_s is infinitely renormalizable and without wandering intervals, then F_s contains a copy of an adding machine map. Using Theorem 3.2 on page 146 and Theorem 5.3 on page 151, one may construct an infinitely renormalizable map in the quadratic family corresponding to any given adding machine map f_α.

□

Theorem 3.7. *Let $\alpha = (j_1, j_2, \dots)$ where each j_i is an integer at least two. Let t be a positive integer such that if a prime number q appears in the prime factorization of t with power k, then q appears with powers totalling at least k in the prime factorizations of $j_1, j_2, \dots$. Let π be a cycle with $\#\pi = t$ and $h(\pi) > 0$. There is an embedding $e : \Delta_\alpha \to I$ such that the minimal topological entropy of any extension $F : I \to I$ of $e \circ f_\alpha \circ e^{-1} : e(\Delta_\alpha) \to e(\Delta_\alpha)$ is $h(\pi)$.*

Proof. By Theorem 2.3 we may assume without loss of generality that $t = j_1$ and each j_i is prime for $i \geq 2$. Let $\beta = (j_2, j_3, \dots)$.

By Lemma 3.6 there is a quadratic map F_s which contains a copy of $f_\beta : \Delta_\beta \to \Delta_\beta$. We can construct a continuous map $g : I \to I$ such that each of the following holds.

(1) There are pairwise disjoint closed intervals $J_1, J_2, \dots, J_t$ of I which are cyclically permuted by G in a way which is equivalent to π.

(2) G is linear on each interval $J_1, J_2, \dots, J_{t-1}$, and G is quadratic on J_t.

(3) $G^t|_{J_1}$ is topologically conjugate to F_s.

(4) G is linear on each component of the complement of $J_1 \cup J_2 \cup \cdots \cup J_t$ in I.

It follows from Propoistion 2.2 that G contains a copy of $f_\alpha : \Delta_\alpha \to \Delta_\alpha$. Let $e : \Delta_\alpha \to I$ be the corresponding embedding.

It is easily seen that $h(G) = \max\{h(\pi), \frac{h(F_s)}{t}\}$. It follows from Proposition 3.4 and Lemma 3.5 that $h(\pi) > \frac{\log 2}{t}$. Since $h(F_s) \leq \log 2$, we have $h(G) = h(\pi)$.

Finally, let $F : I \to I$ be any continuous extension of $e \circ f_\alpha \circ e^{-1} : e(\Delta_\alpha) \to e(\Delta_\alpha)$. Then F has a periodic orbit equivalent to π. So, $h(F) \geq h(\pi)$.

□

Corollary 3.8. *Let $\alpha = (j_1, j_2, \dots)$ where each j_i is an integer at least two. Let r be any real number. There is an embedding $e : \Delta_\alpha \to I$ such that the minimal topological entropy of an extension $F : I \to I$ of $e \circ f_\alpha \circ e^{-1} : e(\Delta_\alpha) \to e(\Delta_\alpha)$ is at least r.*

Proof. Let E_n denote the maximal topological entropy of a cycle of length n. It follows from Theorem 11.6 of [13] that $\lim_{n\to\infty} E_n = \infty$. So, there is a positive integer N such that $E_n \geq r$ for $n \geq N$.

Choose $t \geq N$ satisfying the condition in the hypothesis of Theorem 3.7. There is a cycle π of length t with $h(\pi) = E_t \geq r$. So our conclusion follows from Theorem 3.7.

□

Theorem 3.9. *Let $f : I \to I$ be continuous. Suppose $X \subset I$, $f(X) = X$, and $f|_X$ is topologically conjugate to $f_\alpha : \Delta_\alpha \to \Delta_\alpha$, where $\alpha = (j_1, j_2, \dots)$. Let $m_i = j_1 \cdot j_2 \cdot \dots \cdot j_i$ for each positive integer i. Then for each positive integer n, there is a positive integer $t \geq n$ and a positive integer k such that*
(1) *k is a multiple of m_n.*
(2) *k divides m_t.*
(3) *f has a periodic point of period k.*

Proof. By Theorem 2.2, we have for each positive integer i a cover $P_i = \{X_{i,1}, \dots, X_{i,m_i}\}$ of X such that statements (1), (2), and (3) of Theorem 2.2 hold.

Let n be a positive integer. Let d denote the usual metric on I. Let

$$\epsilon = \min d(X_{n,j_1}, X_{n,j_2})$$

where X_{n,j_1} and X_{n,j_2} are distinct elements of P_n viewed as subsets of I. Then $\epsilon > 0$.

There is a positive integer $t \geq n$ such that the diameter of each of the sets $X_{t,j}$ is less than $\frac{\epsilon}{2}$.

Let $C_{t,j}$ denote the convex hull of $X_{t,j}$ for $j = 1, \dots, m_t$. Then $f(C_{t,1}) \supset C_{t,2}$, $\dots$, $f(C_{t,m_t-1}) \supset C_{t,m_t}$, and $f(C_{t,m_t}) \supset C_{t,1}$. Hence, there is a point $x \in C_{t,1}$ such that $f^j(x) \in C_{t,j+1}$ for $j = 1, 2, \dots, m_t - 1$ and $f^{m_t}(x) = x$.

Now, there is a unique $s \in \{1, \dots, m_n\}$ such that $X_{t,1} \subset X_{n,s}$. Since $x \in C_{t,1}$, and the length of the interval $C_{t,1}$ is less than $\frac{\epsilon}{2}$, and $C_{t,1}$ contains points of $X_{n,s}$, it follows that $d(x, X_{n,s}) < \frac{\epsilon}{2}$.

Suppose that the period of x is k. Since $f^{m_t}(x) = x$, we have that $k \big| m_t$. Let $X_{n,r}$ denote $f^k(X_{n,s})$. By the same argument as in the previous paragraph we see that $d(f^k(x), X_{n,r}) < \frac{\epsilon}{2}$. Since $f^k(x) = x$, it follows that $X_{n,r} = X_{n,s}$. Thus, k is a multiple of m_n.

□

Theorem 3.10. *Let $\alpha = (j_1, j_2, \dots)$ where each j_i is prime. Suppose that for some integer $k \geq 0$ there are exactly k values of i such that $j_i = 2$. Then $H(f_\alpha) = \frac{\log 2}{2^{k+1}}$. Moreover, if $e : \Delta_\alpha \to I$ is any embedding and $F : I \to I$ is any extension of $e \circ f_\alpha \circ e^{-1} : e(\Delta_\alpha) \to e(\Delta_\alpha)$, then $h(F) > \frac{\log 2}{2^{k+1}}$.*

Proof. First, suppose $e : \Delta_\alpha \to I$ is an embedding and $F : I \to I$ is an extension of $e \circ f_\alpha \circ e^{-1}$. By Theorem 3.9, F has a periodic orbit with period an integer of the form $n \cdot 2^k$ where $n \geq 3$ is odd. By Proposition 3.4 we have $h(F) \geq \frac{\log \lambda_n}{2^k}$. Since $\lambda_n > \sqrt{2}$ we have $h(F) > \frac{\log 2}{2^{k+1}}$.

It follows that $H(f_\alpha) \geq \frac{\log 2}{2^{k+1}}$. We show that $H(f_\alpha) \leq \frac{\log 2}{2^{k+1}}$. Let N be a positive integer. Choose a positive integer t, satisfying the condition in the hypothesis of Theorem 3.7, with $t = 2^k \cdot r$ where r is odd and $r \geq N$. By Proposition 3.4, there is a cycle π of length t with $h(\pi) = \frac{\log \lambda_r}{2^k}$. By Theorem 3.7, $H(f_\alpha) \leq h(\pi) \leq \frac{\log \lambda_N}{2^k}$. Since $\lim_{i \to \infty} \lambda_i = \sqrt{2}$, it follows that $H(f_\alpha) \leq \frac{\log 2}{2^{k+1}}$.

□

We conclude with the following remark. Let $\alpha = (j_1, j_2, \dots)$ where each j_i is prime. Suppose that there are infinitely many values of i with $j_i = 2$. By a similar argument to that of Theorem 3.10, using Theorem 3.7, it follows that $H(f_\alpha) = 0$. If there is at least one j_i which is not 2, then for any extension F of any embedding, $h(F) > 0$. However, the example of Delahaye [9] shows that if $\alpha = (2, 2, \dots)$, then there is an embedding $e : \Delta_\alpha \to I$ and an extension F such that $h(F) = 0$.

4. Extension of Theorem 1.2

We begin this section with the following definition.

Definition 4.1. Let $f : I \to I$ and $g : I \to I$ be continuous. Suppose that A, B, C, and D are finite subsets of I with $\#A = \#C$, $\#B = \#D$, $f(A) \subset B$, and $f(C) \subset D$. Let $A = \{x_1, \dots, x_k\}$ with $x_1 < x_2 < \cdots < x_k$, $B = \{y_1, \dots, y_n\}$ with $y_1 < y_2 < \cdots < y_n$, $C = \{v_1, \dots, v_k\}$ with

$v_1 < v_2 < \cdots < v_k$, and $D = \{w_1, \ldots, w_n\}$ with $w_1 < w_2 < \cdots < w_n$. We say that $f|_A : A \to B$ and $g|_C : C \to D$ are equivalent if

(1) $A \cap B = \emptyset$ and $C \cap D = \emptyset$.

(2) $y_1 < x_1 < x_k < y_n$ and $w_1 < v_1 < v_k < w_n$.

(3) For each $i = 1, \ldots, k$ and $j = 1, \ldots, n-1$, $x_i \in (y_j, y_{j+1})$ if and only if $v_i \in (w_j, w_{j+1})$.

(4) For each $i = 1, \ldots, k$ and $j = 1, \ldots, n$, $f(x_i) = y_j$ if and only if $g(v_i) = w_j$.

We will use the following lemma which follows by repeated application of the intermediate value theorem.

Lemma 4.2. *Suppose that g is a continuous, realvalued function on an interval $[a, b]$. Suppose that $y_1, y_2, \ldots, y_n$ are real numbers and either $g(a) < y_1 < y_2 < \cdots < y_n < g(b)$ or $g(a) > y_1 > y_2 > \cdots > y_n > g(b)$. then there exist $x_1, x_2, \ldots, x_n$ with $a < x_1 < x_2 < \cdots < x_n < b$ such that $f(x_i) = y_i$ for $i = 1, \ldots, n$.*

Our next theorem is the main step in obtaining the desired result of this section. We use the following definition. Let θ be a cyclic permutation of $\{1, 2, \ldots, m\}$. We say θ is a double if $m = 2k$ for some positive integer k, $\theta^k(1) = 2$, $\theta^k(2) = 1$, and for each $j = 1, \ldots, k$, $\theta^j(1)$ and $\theta^j(2)$ are adjacent integers.

Theorem 4.3. *Suppose that X is a compact subset of I which contains the endpoints of I. Suppose that f and g are continuous maps of I to itself with $f(X) \subset X$, $g(X) \subset X$, and $f|_X = g|_X$. Suppose that for each component K of $I \setminus X$ the restriction of f to K is monotonic. Let θ be a cyclic permutation of $\{1, 2, \ldots, m\}$, and suppose that θ is not a double. Suppose that f has a periodic orbit P which is equivalent to θ. Then g has a periodic orbit Q which is equivalent to θ.*

Proof. If some point of P is in X, then $P \subset X$ and we may take $Q = P$. So, we may assume that $P \subset I \setminus X$.

Let V denote the union of the components of $I \setminus X$ which contain points of P. Then V is a finite union of open intervals. Let A_0 denote the set of endpoints of these intervals. Then A_0 is a finite subset of X.

Now, suppose that $c \in A_0$ and W is a component of V. Suppose that neither endpoint of W is mapped by f to c, but some point of W is mapped by f to c. Then, as $f|_W$ is monotonic, $\{x \in W : f(x) = c\}$ is a closed interval. Denote this closed interval by $K(W, c)$. We may form a finite set B_1 consisting of one point from each closed interval $K(W, c)$ as above. Then B_1 contains one point from each component of $\{x \in V : f(x) \in A_0\}$ except for the components whose closure contains a point of A_0. Set $A_1 = A_0 \cup B_1$.

Next, we form a finite set $B_2 \subset V$ which consists of one point from each component of $\{x \in V : f(x) \in B_1\}$. (observe that no component of $\{x \in V : f(x) \in B_1\}$ has an element of A_0 in its closure.) Let $A_2 = A_1 \cup B_2$. Inductively we may form $B_3, B_4, \ldots$ and $A_3, A_4, \ldots$ in the analogous way.

Claim 1. For some positive integer s, each component of $V \setminus A_s$ contains at most one point of P.

Proof of Claim 1. We prove Claim 1 by contradiction. Suppose the claim is false. Then there are adjacent points x and y of P with $x < y$ such that for each $i \geq 0$, the two points $f^i(x)$ and $f^i(y)$ lie in one component of V. In particular, for each positive integer i, f^i maps the interval $[x, y]$ monotonically onto its image.

There is a unique integer j with $1 \leq j < m$ such that $f^j(x) = y$. We will show that $f^j(y) = x$. First, suppose that $f^j(y) > x$. Then $f^{2j}(x) = f^j(y) > x$ and $f^{2j}(x)$ is in the same component of V as x and y. It follows that $f^{2j}|_{[x,f^{2j}(x)]}$ is monotonically increasing. So, $x < f^{2j}(x) < f^{4j}(x)$, and $f^{2j}|_{[x,f^{4j}(x)]}$ is monotonically increasing. It follows inductively that $f^{nj}(x) > x$ for each positive integer n. This contradicts the fact that x is a periodic point of f^j. If $f^j(y) < x$ we obtain a contradiction by the same argument. Thus, $f^j(y) = x$.

Now, recall that x and y were chosen to be adjacent points of P. Since f maps the closed interval joining $f^{j-1}(x)$ and $f^{j-1}(y)$ monotonically onto the interval $[x, y]$, it follows that $f^{j-1}(x)$ and $f^{j-1}(y)$ are adjacent points of P. It follows inductively that $f^i(x)$ and $f^i(y)$ are adjacent points of P for each $i = 0, 1, \ldots, j-1$. Hence θ is a double. This contradicts our hypothesis. So, Claim 1 holds.

So far we have constructed sets $A_0, A_1, \ldots, A_s$ and $B_1, B_2, \ldots, B_s$ with respect to the map f. We next construct corresponding sets with respect to the map g.

Let $C_0 = A_0$. By the previous lemma, there is a subset D_1 of V such that $g|_{D_1} : D_1 \to C_0$ and $f|_{B_1} : B_1 \to A_0$ are equivalent. Let $C_1 = C_0 \cup D_1$.

We define $D_2, \ldots, D_s$ and $C_2, \ldots, C_s$ inductively. Suppose $1 \leq k < s$ and D_k and C_k have been defined. By the previous lemma, there is a subset D_{k+1} of V such that $g|_{D_{k+1}} : D_{k+1} \to C_k$ and $f|_{B_{k+1}} : B_{k+1} \to A_k$ are equivalent. Set $C_{k+1} = C_k \cup D_{k+1}$.

By construction, $\#A_s = \#C_s$. Let $A_s = \{x_1, \ldots, x_t\}$ with $x_1 < x_2 < \cdots < x_t$ and $C_s = \{y_1, \ldots, y_t\}$ with $y_1 < y_2 < \cdots < y_t$.

Let i be in $\{1, \ldots, t\}$. If $x_i \in A_0$, then $y_i \in A_0$, $x_i = y_i$ and $f(x_i) = g(y_i)$. If $x_i \notin A_0$, x_i and y_i are in the same component of V, $f(x_i) = x_j$ for some $x_j \in A_s$, and $g(y_i) = y_j$.

Now, choose $p \in P$. For each $j = 0, \ldots, m-1$, $f^j(p) \in (x_{i_j}, x_{i_{j+1}})$ for some $i_j = 1, \ldots, t-1$. Moreover, the intervals $(x_{i_0}, x_{i_0+1}), \ldots (x_{i_{m-1}}, x_{i_{m-1}+1})$ are distinct by Claim 1.

Hence the intervals (y_{i_0}, y_{i_0+1}) are also distinct. By construction, $g(y_{i_0}, y_{i_0+1}) \supset (y_{i_1}, y_{i_1+1})$, $g(y_{i_1}, y_{i_1+1}) \supset (y_{i_2}, y_{i_2+1})$, $\ldots$, $g(y_{i_{m-2}}, y_{i_{m-2}+1}) \supset (y_{i_{m-1}}, y_{i_{m-1}+1})$, and $g(y_{i_{m-1}}, y_{i_{m-1}+1}) \supset (y_{i_0}, y_{i_0+1})$. By Lemma 3.12, there is a fixed point q of g^m with $g^j(q) \in [y_{i_j}, y_{i_j+1}]$ for $j = 0, \ldots, m-1$. Let Q denote the orbit of q under g. Then no point of Q is in $\{y_1, \ldots, y_t\}$, unless $Q \subset A_0$. In either case, it follows from the construction that Q is a periodic orbit of g of period m which is equivalent to θ.

□

Theorem 4.4. *Suppose that X is a compact subset of I which contains the endpoints of I. Suppose that f and g are continuous maps of I to itself with $f(X) \subset X$, $g(X) \subset X$, and $f|_X = g|_X$. Suppose that for each component K of $I \setminus X$, the restriction of f to K is monotonic. Then $h(f) \leq h(g)$.*

Proof. It is known (see Proposition VIII.33, page 217 of [6] or Theorem 4.4.10, page 225 of [2]) that

$$h(g) = \sup\{h(\pi)\}$$

where the supremum is taken over all cycles π such that g has a periodic orbit equivalent to π.

Now, suppose that g has a periodic orbit equivalent to π, where π is a cycle which is a double of a cycle θ. Then it follows from Lemma 4.4.16, page 229 of [2] that $h(\pi) = h(\theta)$, and it follows from Proposition VII.8, page 175 of [6] that g has a periodic orbit equivalent to θ. It follows by repeating this argument inductively that there is a cycle γ which is not a double such that $h(\pi) = h(\gamma)$ and g has a periodic orbit equivalent to γ. Hence, we see from (1) that $h(g) = \sup\{h(\pi)\}$ where the supremum is taken over all cycles π such that π is not a double and g has a periodic orbit equivalent to π. Since this statement holds for f as well as g, it follows from the previous theorem that $h(f) \leq h(g)$. □

References

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309-319.

[2] Lluís Alsedà, Jaume Llibre, and Michal Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, World Scientific, 1993.

[3] Harold Bell and Kenneth R. Meyer, *Limit period functions, adding machines, and solenoids*, Jour. of Dyn. and Diff. Eq. **7** (1995), 409-422.

[4] L. S. Block, *Simple periodic orbits of mappings of the interval*, Trans. Amer. Math. Soc., **254** (1979), 391-398.

[5] A. Blokh, *The spectral decomposition for one-dimensional maps*, Dynamics Reported **4** (1995), 1-59.

[6] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Mathematics **1513**, Springer Verlag, 1992.

[7] L. S. Block and J. E. Keesling, *A characterization of adding machine maps*, Topology and its Applications, to appear.

[8] Jorge Buescu and Ian Stewart, *Liapunov stability and adding machines*, Ergod. Th. and Dynam. Sys. **15** (1995), 271-290.

[9] J.-P. Delahaye, *Fonctions admettant des cycles d'ordre n'importe queele puissance de 2 et aucun autre cycle* C. R. Acad. Sci. Paris Sér. A-B, **291** (1980), A323-325.

[10] Edwin Hewitt and Kenneth Ross, *Abstract Harmonic Analysis*, Springer-Verlag, 1963.

[11] L. Markus and K. R. Meyer, *Periodic orbits and solenoids in generic Hamiltonian dynamical systems*, Amer. J. Math. **102** (1980), 25-92.

[12] Welington de Melo and Sebastian van Strien, *One-Dimensional Dynamics*, Springer-Verlag, 1993.

[13] M. Misiurewicz and Z. Nitecki, *Combinatorial patterns for maps of the interval*, Mem. Amer. Math. Soc. **94** (1991), no. 456.

[14] Xiang Dong Ye, *D-function of a minimal set and an extension of Sharkovskiĭ's Theorem to minimal sets*, Ergodic Theory and Dynam. Systems **12** (1992), 365-376.

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