A CHARACTERIZATION OF ADDING MACHINE MAPS

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ABSTRACT. Let $f : X \to X$ be a continuous map of a compact metric space to itself. We prove that f is topologically conjugate to an adding machine map if and only if X is an infinite minimal set for f and each point of X is regularly recurrent. Moreover, if X is an infinite minimal set for f and one point of X is regularly recurrent, then f is semiconjugate to an adding machine map.

1. Introduction and Preliminaries

Let $f : X \to X$ be a continuous map with X a compact metric space. Then a *minimal set* for f is a minimal, nonempty, closed invariant subset of X under f. Minimal sets were defined by G. D. Birkhoff and are of considerable interest and importance in dynamical systems. The simplest examples of minimal sets are periodic orbits. For periodic orbits, the dynamics on the orbit is very simple and uninteresting. However, there may be very interesting and complicated dynamics on infinite minimal sets. Infinite minimal sets may have simple geometry. For instance, the circle is a minimal set under any irrational rotation. Somewhat more complicated, but still very geometric are homeomorphisms on the n-torus which make the n-torus a minimal set. For any solenoid, there are homeomorphisms of the solenoid which make the solenoid a minimal set. So, the geometry of an infinite minimal set need not be simple. Other examples of minimal sets are more abstract such as the Sturmian minimal sets [9], the Morse minimal set [8], and substitution minimal sets [7].

The adding machines form a special class of minimal sets. These are all Cantor sets topologically, but have a certain geometric appeal when their structure as compact Abelian topological groups is in view [10]. Adding machines arise frequently and naturally in dynamical systems. In [11] Markus and Meyers show that generically a C^∞ Hamiltonian on a compact symplectic manifold has adding machine minimal sets

1991 *Mathematics Subject Classification.* 37B20, 54H20, 54H11.

Key words and phrases. adding machine, minimal set, regularly recurrent.

for every possible α. Buescu and Stewart have shown that Lyapunov stable Cantor sets are adding machines [5]. Nitecki has used adding machines in describing the dynamics of piecewise monotone maps of the interval [12]. In fact, adding machine maps play a major role in the description of the dynamics of maps of a one-dimensional space. The spectral decomposition theorem of Blokh [4] shows this.

It is obvious that for any minimal set, every forward orbit must be dense in the set. However, for the closure of an orbit of a point x to be a minimal set, x must be *strongly recurrent.* Moreover, in a minimal set each point is strongly recurrent. This characterization is a classic result of Birkhoff [2]. A proof may also be found on page 93 of [3].

In this paper we show that there is a characterization for adding machines which is similar to the characterization by Birkhoff for minimal sets. Our characterization of adding machines (Corollary 2.5) is that they are infinite minimal sets in which each point is *regularly recurrent.* In Section 3 we give an example of a map $f : X \to X$ on a compact metric space such that X is an infinite minimal set for f and there is a point x in X which is regularly recurrent but f is not conjugate to an adding machine map. Thus, it is essential that every point in the space X be regularly recurrent to guarantee that f is conjugate to an adding machine map. However, we show (Theorem 2.4) that if one point in an infinite minimal set is regularly recurrent, then f is semiconjugate to an adding machine map.

A related result has been obtained by Thomas [15] for one-dimensional minimal sets of flows. Translating this result to the setting of maps yields a sufficient condition for a map on a zero-dimensional minimal set to be topologically conjugate to an adding machine map. In Section 4 we show that this translated result for maps follows from Corollary 2.5.

The main results of this paper are valid for a compact Hausdorff space X. Of course, the reader may choose to focus on the case where X is a compact metric space.

Let $f : X \to X$ be a continuous map. Let x be a point in X. Then x is said to be *strongly recurrent* if for every neighborhood V of x, there is a sequence of positive integers $k_1 < k_2 < \cdots$ and a positive integer M such that $f^{k_i}(x) \in V$ for all k_i and $k_{i+1} - k_i \leq M$ for all i. The point $x \in X$ is said to be *regularly recurrent* if for every neighborhood V of x, there is a positive integer n such that for every nonnegative integer k, $f^{kn}(x) \in V$.

We now define the α-adic adding machine Δ_α. Let $\alpha = (j_1, j_2, \dots)$ be a sequence of integers where each $j_i \geq 2$. Let Δ_α denote all sequences $(x_1, x_2, \dots)$ where $x_i \in \{0, 1, \dots, j_i - 1\}$ for each i. We put a metric

d_α on Δ_α given by

$$d_\alpha((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{i=1}^{\infty} \frac{\delta(x_i, y_i)}{2^i},$$

where $\delta(x_i, y_i) = 1$ if $x_i \neq y_i$ and $\delta(x_i, y_i) = 0$ if $x_i = y_i$. Addition in Δ_α is defined as follows. We set

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (z_1, z_2, \dots)$$

where $z_1 = (x_1 + y_1) \mod j_1$ and $z_2 = (x_2 + y_2 + t_1) \mod j_2$. Here $t_1 = 0$ if $x_1 + y_1 < j_1$ and $t_1 = 1$ if $x_1 + y_1 \geq j_1$. So, we *carry* a one in the second case. Continue adding and carrying in this way for the whole sequence.

We define $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ by

$$f_\alpha(x_1, x_2, \dots) = (x_1, x_2, \dots) + (1, 0, 0, \dots).$$

We will refer to the map $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ as the *adding machine map.*

Using an ergodic theory approach, Buescu and Stewart in [5] give a classification of adding machine maps up to topological conjugacy. We obtain the same classification as a corollary to our main result (Corollary 2.8). It follows in particular that one could always assume that α is a sequence of primes when studying the adding machine map f_α.

The adding machine map has also been called the *odometer map* and the *solenoid*. This last term is used for a quite different but related object in topology. The dynamical suspension of f_α is what topologists call the α-adic solenoid. What the dynamicists call the *suspension* is called the *mapping torus* by topologists. One can think of the α-adic adding machine map as the Poincaré first return map in the α-adic solenoid for a certain flow and a certain cross section of the solenoid.

We would like to thank the referee for helpful suggestions to improve the paper.

2. Main Theorem and Corollaries

The following lemma is an important tool. It characterizes when a map f on a minimal set is semiconjugate to a cyclic map on a finite discrete set.

Lemma 2.1. *Let $f : X \to X$ be a continuous map of a compact Hausdorff space to itself. Suppose that X is a minimal set for f. Let n be a positive integer. Then for some positive integer $t \leq n$ and some f^n-minimal set M, we have each of the following:*
(1) *X is the disjoint union of M, $f(M)$, $\dots$, $f^{t-1}(M)$.*
(2) *Each of the sets M, $f(M)$, $\dots$, $f^{t-1}(M)$ is clopen.*

(3) *The collection* $\{M, f(M), \ldots, f^{t-1}(M)\}$ *is the collection of all subsets of* X *which are* f^t*-minimal and also the collection of all subsets of* X *which are* f^n*-minimal.*

Also, for each $x \in X$*, the closure of the* f^n*-orbit of* x *is an* f^n*-minimal set.*

Proof. Let M be an f^n-minimal set. Let t be the least positive integer with $f^t(M) \cap M \neq \emptyset$. It is easy to see that $f^i(M)$ is an f^n-minimal set for each positive integer i. Since any two f^n-minimal sets are disjoint or equal, we have $f^t(M) = M$. By choice of t, the collection of sets $\{M, f(M), \ldots, f^{t-1}(M)\}$ is pairwise disjoint. Since $M \cup f(M) \cup \cdots \cup f^{t-1}(M)$ is invariant under f, we see that (1) holds. Statement (2) follows. Since X is f-minimal, we have that M is f^t-minimal. So statement (3) follows.

Finally, let $x \in X$. By (1) and (3), there is an f^n-minimal set M_1 with $x \in M_1$. Then M_1 is the closure of the f^n-orbit of x.

□

Corollary 2.2 follows simply from Lemma 2.1 and gives a useful restatement of that result.

Corollary 2.2. *Let* $f : X \to X$ *be continuous with* X *compact Hausdorff. Suppose that* X *is a minimal set for* f*. Let* $n > 1$ *be an integer. Then the following are equivalent.*
(1) *There is a positive integer* n *and a continuous map* $\pi : X \to Z_n$ *such that* $(\pi \circ f)(x) = \pi(x) + 1 \mod n$.
(2) *There is a proper closed subset* M *in* X *which is a minimal set for* f^n *such that* M *is not minimal for* f^t *for any positive integer* $t < n$.

The next theorem will be needed for our main theorem.

Theorem 2.3. *Let* $\alpha = (j_1, j_2, \ldots)$ *be a sequence of integers with* $j_i \geq 2$ *for each* i*. Let* $m_i = j_1 \cdot j_2 \cdot \cdots \cdot j_i$ *for each* i*. Let* $f : X \to X$ *be a continuous map of a compact topological space* X*. Then* f *is topologically conjugate to* f_α *if and only if* (1), (2), *and* (3) *hold.*
(1) *For each positive integer* i*, there is a cover* P_i *of* X *consisting of* m_i *pairwise disjoint, nonempty, clopen sets which are cyclically permuted by* f*.*
(2) *For each positive integer* i*,* P_{i+1} *partitions* P_i*.*
(3) *If* $W_1 \supset W_2 \supset W_3 \supset \cdots$ *is a nested sequence with* $W_i \in P_i$ *for each* i*, then* $\cap_{i=1}^{\infty} W_i$ *consists of a single point.*

Moreover, in this case statement (4) *also holds.*
(4) X *is metrizable and if* $\mathrm{mesh}(P_i)$ *denotes the maximum diameter of an element of the cover* P_i*, then* $\mathrm{mesh}(P_i) \to 0$ *as* $i \to \infty$.

Proof. ($\Rightarrow$) First, we show that $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ has the desired properties. For each $j = 0, \dots, j_1 - 1$, let $Y_{1,j}$ denote the set of sequences $(x_1, x_2, \dots)$ in Δ_α such that $x_1 = j$. Let $Q_1 = \{Y_{1,0}, \dots, Y_{1,j_1-1}\}$.

Next for each $j = 0, \dots, j_1 - 1$, let $Y_{2,j}$ denote the set of sequences $(x_1, x_2, \dots)$ in Δ_α such that $x_1 = j$ and $x_2 = 0$. For each $j = j_1, \dots, 2j_1 - 1$, let $Y_{2,j}$ denote the set of sequences $(x_1, x_2, \dots)$ in Δ_α such that $x_1 = j - j_1$ and $x_2 = 1$. For each $j = 2j_1, \dots, 3j_1 - 1$, let $Y_{2,j}$ denote the set of sequences $(x_1, x_2, \dots)$ in Δ_α such that $x_1 = j - 2j_1$ and $x_2 = 2$. Continuing in this way we obtain a cover $Q_2 = \{Y_{2,0}, \dots, Y_{2,m_2-1}\}$ with the desired properites. In the analogous way we obtain inductively for each positive integer i, a cover Q_i of Δ_α such that the appropriate properties corresponding to (1), (2), and (3) hold.

Next, suppose $f : X \to X$ is topologically conjugate to $f_\alpha : \Delta_\alpha \to \Delta_\alpha$. There is a homeomorphism $h : X \to \Delta_\alpha$ such that $h \circ f = f_\alpha \circ h$. Let $Q_i = \{Y_{i,0}, \dots, Y_{i,m_i-1}\}$ denote the covers obtained above. Let $P_i = \{X_{i,0}, \dots, X_{i,m_i-1}\}$ where $X_{i,j} = h^{-1}(Y_{i,j})$ for $i = 1, 2, \dots,$ and $j = 0, \dots, m_i - 1$. Then (1), (2), and (3) hold. This proves the first implication.

Also, it follows from the previous argument that (4) holds.

($\Leftarrow$) Suppose $f : X \to X$ is continuous and for each positive integer i, there is a cover P_i of X such that (1), (2), and (3) hold. We may write $P_i = \{X_{i,0}, \dots, X_{i,m_i-1}\}$ where for each positive integer i we have:
(4) $f(X_{i,j}) = X_{i,j+1 \bmod m_i}$ for $j = 0, 1, \dots, m_i - 1$.
(5) $X_{i+1,0} \subset X_{i,0}$.

As we saw in the first direction of the proof, for the map $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ we have for each positive integer i, a cover $Q_i = \{Y_{i,0}, \dots, Y_{i,m_i-1}\}$ of Δ_α such that the analogous statements to (1), (2), (3), (4), and (5) hold.

Define $h : X \to \Delta_\alpha$ as follows. Let $x \in X$. For each positive integer i, there is a unique $k_i = 0, 1, 2, \dots, m_i - 1$ such that $x \in X_{i,k_i}$. Now $\cap_{i=1}^{\infty} Y_{i,k_i}$ consists of a single point y. Let $h(x) = y$.

We show that h is continuous. Let $V = Y_{i,j}$ for some positive integer i and some $j = 0, 1, \dots, m_i - 1$. Then $h^{-1}(V) = X_{i,j}$, and so $h^{-1}(V)$ is open. Since the collection of all sets $Y_{i,j}$ is a basis for Δ_α, it follows that h is continuous. It is easy to see that h is a bijection; hence h is a homeomorphism.

Finally, let $x \in X$. For each positive integer i, there is a unique $k_i = 0, 1, 2, \dots, m_i - 1$ such that $x \in X_{i,k_i}$. There is a unique $t_i = 0, 1, 2, \dots, m_i - 1$ defined by

$$t_i = (k_i + 1) \mod m_i.$$

Then each of the points $h(f(x))$ and $f_\alpha(h(x))$ is an element of $\cap_{i=1}^{\infty} Y_{i,t_i}$. Since this intersection consists of a single point, we have $h(f(x)) = f_\alpha(h(x))$.

□

We are now in a position to state and prove the main theorem of the paper. We will use the following definition.

Let $f : X \to X$ be a continuous map of a compact space to itself. Suppose that X is a minimal set for f. Let $S(f)$ denote the set of positive integers i such that for some subset M of X, M is f^i-minimal, but M is not f^j-minimal for $j = 1, \ldots, i-1$.

The set $S(f)$ contains the same information as the D-functions of the minimal set X as defined by Ye in [16] and also used by Alsedá and Ye in [1].

Theorem 2.4. *Let $f : X \to X$ be a continuous map of a compact Hausdorff space to itself. Suppose that X is an infinite minimal set for f and some point of X is regularly recurrent. Then there is a sequence α of prime numbers and a continuous onto map $\pi : X \to \Delta_\alpha$ such that $f_\alpha \circ \pi = \pi \circ f$. Moreover, if $x \in X$ is regularly recurrent, then $\pi^{-1}(\pi(x)) = \{x\}$.*

Proof. Let S denote the set $S(f)$ defined above. By Lemma 2.1, for each $n \in S$, there is a unique cover Q_n of X which consists of the n pairwise disjoint f^n-minimal sets. Moreover, these f^n-minimal sets are clopen and are cyclically permuted by f. We proceed to prove several claims about the set S and the covers Q_n.

Claim 1. S is infinite.

Proof of Claim 1. Let k be a positive integer. It suffices to show that there is an $n \in S$ with $n > k$. By hypothesis there is a regularly recurrent point $x \in X$. Let V be an open set with $x \in V$ such that $f^i(x) \notin \overline{V}$ for $i = 1, 2, \ldots, k$.

Since x is regularly recurrent, there is a positive integer t such that $f^{it}(x) \in V$ for each nonnegative integer i. By Lemma 2.1, the closure of the f^t-orbit of x is an f^t-minimal set M. Let n be the least positive integer such that M is f^n-minimal. Then $n \in S$. Since $M \subset \overline{V}$, we have $n > k$. This proves Claim 1.

Claim 2. Suppose $k \in S$, $n \in S$, and k is a multiple of n. Then Q_k refines Q_n.

Proof of Claim 2. Let $M \in Q_k$. Let $z \in M$. Then M is the closure of the f^k-orbit of z. Since the f^k-orbit of z is contained in the f^n-orbit of z, M is contained in the closure of the f^n-orbit of z. By Lemma 2.1, the latter set is an element of Q_n. This proves Claim 2.

Claim 3. Suppose that $k \in S$ and n is a positive integer. If k is a multiple of n, then $n \in S$.

Proof of Claim 3. We have $k = tn$ for some positive integer t. Let M be f^k-minimal. Let $M_1 = M \cup f^n(M) \cup \cdots \cup f^{(t-1)n}(M)$. Then $f^n(M_1) \subset M_1$. So, some subset M_2 of M_1 is f^n-minimal. Since $f^i(M_1) \cap M_1 = \emptyset$ for $i = 1, \dots, n-1$, we have $n \in S$. This proves Claim 3.

Claim 4. Suppose $k \in S$, $n \in S$, where k and n are relatively prime. Then $kn \in S$.

Proof of Claim 4. Let M_1 be f^k-minimal and choose $x \in M_1$. Then x is in some f^n-minimal set M_2. Set $Y = M_1 \cap M_2$.

Now $f^t(Y) \cap Y \neq \emptyset$ if and only if t is a multiple of k while $f^t(M_2) \cap M_2 \neq \emptyset$ if and only if t is a multiple of n. Thus $f^t(Y) \cap Y = \emptyset$ for $t = 1, 2, \dots, kn-1$. Thus, $kn \in S$. This proves Claim 4.

Now, for each integer $j > 1$ and each prime divisor p of j, let $M_j(p)$ denote the multiplicity of p in the prime factorization of j. Let $M_j(p) = 0$ if $j = 1$ or if $j > 1$ and p does not divide j.

Let p be a prime number. If $M_j(p)$ is arbitrarily large for $j \in S$, set $M(p) = \infty$. Otherwise, set $M(p) = \max_{j \in S} M_j(p)$. So defined $M(p)$ is either 0, ∞, or a positive integer.

Let $\alpha = (p_1, p_2, \dots)$ be a sequence of primes such that each prime number p appears in this sequence exactly $M(p)$ times. Let $m_i = p_1 p_2 \cdots p_i$ for each positive integer i. It follows from Claims 3 and 4 that $m_i \in S$ for each positive integer i.

For each positive integer i, let $P_i = \{X_{i,1}, \dots, X_{i,m_i}\}$ denote the cover Q_{m_i} of X.

By Lemma 2.1, statement (1) of Theorem 2.3 holds and by Claim 2, statement (2) of Theorem 2.3 holds. With these two properties we obtain, as in the proof of Theorem 2.3, a continuous surjective map $\pi : X \to \Delta_\alpha$ such that $f_\alpha \circ \pi = \pi \circ f$.

Finally, suppose that $x \in X$ is regularly recurrent. Suppose $y \neq x$ and $y \in \pi^{-1}(\pi(x))$. Then there is a nested sequence of sets W_i with $W_i \in P_i$ for each i such that $\cap_{i=1}^{\infty} W_i$ contains x and y.

There is an open subset V of X with $x \in V$ but $y \notin \overline{V}$. Since x is regularly recurrent there is a positive integer n such that $f^{kn}(x) \in V$ for each positive integer k.

Now, the closure of the f^n-orbit of x is an f^n-minimal set $M \subset \overline{V}$. There is a positive integer $t \leq n$ such that M is f^t-minimal and $t \in S$. By the choice of α, there is a positive integer j such that m_j is a multiple of t. Now $x \in W_j$ and $x \in M$. So by Claim 2 we must have that $W_j \subset M$. This is a contradiction as $y \in W_j$ but $y \notin M$. Thus, $\pi^{-1}(\pi(x)) = \{x\}$.

□

Corollary 2.5. *Let $f : X \to X$ be a continuous map of a compact Hausdorff space to itself. There is a sequence α of prime numbers such that f is topologically conjugate to the adding machine map f_α if and only if X is an infinite minimal set for f and each point of X is regularly recurrent.*

Proof. It follows from Theorem 2.3 that if there is a sequence α such that f is topologically conjugate to f_α, then each point is regularly recurrent. The converse follows from Theorem 2.4.

□

Corollary 2.6. *Let $\beta = (j_1, j_2, \dots)$ and $\gamma = (k_1, k_2, \dots)$ be sequences of integers with $j_i \geq 2$ and $k_i \geq 2$ for each i. Then f_β and f_γ are topologically conjugate if and only if $S(f_\beta) = S(f_\gamma)$.*

Proof. It is easy to see that in general if f and g are topologically conjugate, then $S(f) = S(g)$. Suppose $S(f_\beta) = S(f_\alpha)$. Let α be a sequence of primes formed as in the proof of Theorem 2.4. It follows from the proof of Theorem 2.4 that each of the maps f_β and f_γ is topologically conjugate to f_α. Hence, f_β and f_γ are topologically conjugate.

□

Lemma 2.7. *Let $\beta = (j_1, j_2, \dots)$ be a sequence of integers with $j_i \geq 2$ for each i. Let $m_i = j_1 \cdot j_2 \cdot \dots \cdot j_i$ for each i. Then $S(f_\beta)$ is precisely the set of positive integers k such that k divides m_i for some i.*

Proof. Using the fact that statement (1) of Theorem 2.3 holds for f_β, we see that $m_i \in S(f_\beta)$ for each positive integer i.

So, if k divides m_i for some i, it follows from Claim 3 of the proof of Theorem 2.4, that $k \in S(f_\beta)$.

On the other hand, suppose that $k \in S(f_\beta)$. By Lemma 2.1, there is a cover P of Δ_β consisting of k pairwise disjoint clopen sets which are cyclically permuted by f_β. As in the proof of the first direction of Theorem 2.3, we obtain for each positive integer i a cover Q_i of Δ_β consisting of m_i pairwise disjoint clopen sets which are cyclically permuted by f_β. Moreover, the diameter of the cover Q_i approaches zero as i increases.

It follows that Q_i refines P for i sufficiently large. This implies that k divides m_i for i sufficiently large.

□

Corollary 2.8. *Let $\beta = (j_1, j_2, \dots)$ and $\gamma = (k_1, k_2, \dots)$ be sequences of integers with $j_i \geq 2$ and $k_i \geq 2$ for each i. We let M_β denote a function whose domain is the set of all prime numbers and which maps* *to the extended natural numbers* $\{0, 1, 2, \ldots, \infty\}$*. The function* M_β *is defined by*

$$M_\beta(p) = \sum_{i=1}^{\infty} n_i$$

where n_i *is the power of the prime* p *in the prime factorization of* j_i*.*

Then f_β *and* f_γ *are topologically conjugate if and only if the functions* M_β *and* M_γ *are equal.*

We will use the following definition in our final corollary. Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact Hausdorff spaces. We say that g is a quotient of f if there is a continuous surjective map $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$.

Corollary 2.9. *Suppose* $g : Y \to Y$ *is a quotient of an adding machine map. Then either* g *is topologically conjugate to an adding machine map or* Y *is a periodic orbit.*

Proof. Suppose that g is a quotient of an adding machine map f_α. Since each point of Δ_α is regularly recurrent under f_α, it follows easily that each point of Y is regularly recurrent under g. Also, it is easy to see that Y is a minimal set for g. Also, it is easy to see that Y is a minimal set for g. If Y is infinite, it follows from Corollary 2.5 that g is topologically conjugate to an adding machine map. Of course, if Y is finite, then Y is a finite minimal set and hence a periodic orbit.

□

3. An Example

In this section we give an example of a mapping $f : X \to X$ on a compact metric space such that there is a regularly recurrent point in X, but f is not topologically conjugate to an adding machine map. We state the example as a theorem. We first need a lemma.

Lemma 3.1. *Let* $f : X \to X$ *be a continuous map of a compact metric space to itself. Suppose that* x *and* y *are distinct points of* X *with* $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$*. Then at least one of the two points* x *and* y *is not regularly recurrent.*

Proof. Suppose both x and y are regularly recurrent. Let $r = d(x, y)$. Let V denote the open ball of radius $\frac{r}{4}$ about x and let W denote the open ball of radius $\frac{r}{4}$ about y. There is a positive integer n_0 such that if $n \geq n_0$, then $d(f^n(x), f^n(y)) < \frac{r}{4}$. Also, there are positive integers n_1 and n_2 such that $f^{kn_1}(x) \in V$ and $f^{kn_2}(y) \in W$ for each positive integer k. Let $n = n_0 \cdot n_1 \cdot n_2$. Then $n \geq n_0$, $f^n(x) \in V$, and $f^n(y) \in W$. By the triangle inequality $d(x, y) < r$, a contradiction.

□

Theorem 3.2. *There is a continuous map $f : X \to X$ with X a compact metric space and an infinite minimal set for f with the following property. There is a point x in X which is regularly recurrent, but f is not topologically conjugate to any adding machine map.*

Proof. We begin with a construction of Delahaye [6]. In [6] a function $g : [0,1] \to [0,1]$ is given. If $C \subset [0,1]$ is the standard Cantor middle-third set, then $g|_C$ is conjugate to the dyadic adding machine. The map g is defined as follows. Let $g(0) = \frac{2}{3}$ and $g(1) = 0$. Then for each integer $i > 0$ let $g(1 - \frac{2}{3^i}) = \frac{1}{3^{i-1}}$ and $g(1 - \frac{1}{3^i}) = \frac{2}{3^{i+1}}$. The function is then extended linearly between successive points. Note that since $g|_C$ is conjugate to the dyadic adding machine, it is a homeomorphism of C to itself.

Note also that g is strictly increasing on the intervals $[0, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, $[\frac{8}{9}, \frac{25}{27}], \ldots, [\frac{3^i-1}{3^i}, \frac{3^{i+1}-2}{3^{i+1}}], \ldots$. In fact, g is linear with slope one on these intervals.

We now modify the map g just given. We describe the modification of g in words. Let x_0 be a point in the Cantor set C such that its orbit $\{(g|_C)^n(x_0)\}_{n\in Z}$ does not contain any endpoints of the Cantor set. Now we want to replace the points on this orbit by intervals $\{L_i\}_{i\in Z}$ whose diameters converge to zero reminiscent of the Denjoy construction. See Robinson [14], the proof of Theorem 8.8 on page 56. The Denjoy construction is for a homeomorphism on S^1. Our map g is strictly increasing in an open interval containing each point where we add an L_i. That is all that is needed to guarantee continuity of the map $\hat{g}$ described next.

Observe first that the addition of the intervals $\{L_i\}_{i\in Z}$ to I creates a new interval. We call this extended interval $\hat{I}$. The new function $\hat{g}$ will be continuous $\hat{g} : \hat{I} \to \hat{I}$.

Now the original middle-third Cantor set has been modified to contain the sequence of intervals $\{L_i\}_{i\in Z}$ whose diameters converge to zero. Let $\hat{C}$ be the original C with the point $g^i(x_0)$ expanded to an interval L_i. Of course, $\hat{C}$ will *not* be a Cantor set. However, there is a continuous, monotonically increasing, surjective map $p : \hat{C} \to C$ such that for each $x \in C$ not in $\{(g|_C)^n(x_0)\}_{n\in Z}$, $p^{-1}(x)$ is a single point, while if $x = (g|_C)^n(x_0)$, then $p^{-1}(x)$ is the interval L_n.

Define $\hat{g}$ on $\hat{C}$ as follows. Suppose $x \in \hat{C}$ is not in any of the intervals $\{L_i\}_{i\in Z}$. Then set $\hat{g}(x) = p^{-1}(g|_C)(p(x))$. We also let $\hat{g}$ be linear increasing from one added interval to the next so that $\hat{g}(L_i) = L_{i+1}$. So we obtain a continuous map $\hat{g} : \hat{C} \to \hat{C}$. Define $\hat{g}$ on the interval

$\hat{I}$ by extending linearly on the intervals in the complement of $\hat{C}$ in $\hat{I}$. The function so defined will be continuous $\hat{g} : \hat{I} \to \hat{I}$.

Now define X to be $\hat{C} \setminus \cup_{i \in Z} \mathring{L}_i$ where $\mathring{L}_i$ is the interior of L_i. Then $\hat{g}$ will map X to X and X will topologically be a Cantor set. Set $f = \hat{g}|_X$. Note that the effect of the construction is to replace each point on the (forward and backward) orbit of x_0 by two points. So, we now have two orbits in place of the original orbit of x_0.

Let x be any point of X which is not in any of the intervals $\{L_i\}_{i \in Z}$. If V is an open neighborhood of x, then $p(V)$ contains an open neighborhood of $p(x)$. Since $p(x)$ is regularly recurrent under $g|_C$, it follows easily that x is regularly recurrent under f.

Any such point x as in the previous paragraph will also have a dense orbit in X. Since regular recurrence implies strong recurrence, we have that X will be a minimal set.

Finally, let y and z be the two endpoints of the interval L_0. By construction,

$$\lim_{n \to \infty} d(f^n(y), f^n(z)) = 0.$$

By Lemma 3.1, at least one of the two points y and z is not regularly recurrent. Hence f is not topologically conjugate to an adding machine map.

□

We remark that the example given in the proof of Theorem 3.2 has an obvious map $\pi : X \to \Delta_\alpha$ where $\alpha = (2, 2, \ldots)$ which commutes with the dyadic adding machine. So, the example illustrates Theorem 2.4.

4. The Result of Thomas

In this section, we use Corollary 2.5 to obtain a sufficient condition for a map $f : X \to X$ to be topologically conjugate to an adding machine map in the case that X has dimension zero. The result that we obtain is the result of Thomas [15], except that the result of Thomas was given for flows and the result here is given for maps.

Definition 4.1. Let $f : X \to X$ and let $x \in X$. We say that x is almost periodic in the sense of Nemytskii and Stepanov if for every $\epsilon > 0$ there is a relatively dense subset D of the positive integers such that for each $k \in D$ and each integer $j \geq 0$,

$$d(f^j(x), f^{j+k}(x)) < \epsilon.$$

The set D is *relatively dense* provided that for some positive integer L, each set of L consecutive positive integers contains an element of D.

Proposition 4.2. *Let $f : X \to X$ be continuous, with X a compact zero-dimensional metric space. Suppose that $x \in X$ is almost periodic in the sense of Nemytskii and Stepanov. Then x is regularly recurrent.*

Proof. Let V be a neighborhood of x. There is a clopen neighborhood W of x with $W \subset V$. Let $\epsilon = d(W, X \setminus W)$. Then $\epsilon > 0$. Since x is almost periodic in the sense of Nemytskii and Stepanov, there is a relatively dense set D as in the previous definition. There is a positive integer $k \in D$.

Now, $d(x, f^k(x)) < \epsilon$. It follows that $f^k(x) \in W$.

Similarly, $d(f^k(x), f^{k+k}(x)) < \epsilon$ and $f^{2k}(x) \in W$.

By induction, $f^{nk}(x) \in W$ for each positive integer n. Since $W \subset V$, $f^{nk}(x) \in V$ for each positive integer n.

So x is regularly recurrent.

□

Proposition 4.3. *Let $f : X \to X$ be continuous, with X a compact metric space. Suppose that X is a minimal set for f. Suppose that there exists $x \in X$ which is almost periodic in the sense of Nemytskii and Stepanov. Then each $y \in X$ is almost periodic in the sense of Nemytskii and Stepanov.*

Proof. Let $y \in X$. Let $\epsilon > 0$. There is a set D which satisfies Definition 4.1 for x and $\frac{\epsilon}{2}$. We show this same set D satisfies that definition for y and ϵ.

Let $j \geq 0$ and let $k \in D$. We show that $d(f^j(y), f^{j+k}(y)) < \epsilon$. Proceeding by contradiction suppose $d(f^j(y), f^{j+k}(y)) \geq \epsilon$. Then there is a neighborhood W of $f^j(y)$ such that $d(z, f^k(z)) > \frac{\epsilon}{2}$ for all $z \in W$.

For some integer $n \geq 0$, $f^n(x) \in W$. Since $k \in D$, $d(f^n(x), f^{n+k}(x)) < \frac{\epsilon}{2}$. This is a contradiction.

□

We now obtain the result of Thomas [15], translated from the setting of flows to the setting of maps.

Corollary 4.4. *Let $f : X \to X$ be continuous, with X a compact zero-dimensional metric space. Suppose that X is an infinite minimal set for f. Suppose also that some point $x \in X$ is almost periodic in the sense of Nemytskii and Stepanov. Then f is topologically conjugate to an adding machine map.*

Proof. By Propositions 4.2 and 4.3 each point of X is regularly recurrent. So the conclusion follows from Corollary 2.5.

□

Finally, we remark that Corollary 4.4 does not hold without the requirement that X has dimension zero. Any irrational rotation on the circle serves as an example.

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