SEMI-CONJUGACIES AND INVERSE LIMIT SPACES

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Abstract. Let f be a continuous map of the interval to itself. We prove that if f has a k-horseshoe, then f is topologically semi-conjugate to a tent map with slope $\pm k$. Using this result, we prove that if f has positive topological entropy, and $n \geqslant 2$, then there is a continuous surjective map from the inverse limit space (I, f) to the Knaster continuum K_n. In particular, for $m, n \geqslant 2$, there is a continuous surjective map from K_m to K_n. On the other hand, we prove that the analogous statement does not hold for solenoids.

1. Introduction

In studying the dynamics of continuous maps, the notion of topological conjugacy plays a fundamental role. Let $f : X \to X$ and $g : Y \to Y$ be continuous functions. We say that f and g are **topologically conjugate** if there exists a homeomorphism $\alpha : X \to Y$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow{g} & Y \end{array}$$

If f and g are topologically conjugate, we think of the dynamical properties of the two maps as being the same.

A related idea is the notion of topological semi-conjugacy. We say that f is topologically semi-conjugate to g if there exists a continuous, surjective function $\alpha : X \to Y$ such that the diagram above commutes. Note, when we say "f is semi-conjugate to g," we mean that f is the top map and g is the bottom map in the diagram. Sometimes, in the literature, the phrase "f is an extension of g" is used with the same meaning.

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If f is topologically semi-conjugate to g we think of the dynamics of f as being at least as complicated as the dynamics of g. Many results in the theory of dynamical systems involve topological semi-conjugacy.

As our first example we mention a result of M. Shub [17, Theorem 2, p. 192]. As noted by Shub in the remark on page 184, a particular case of this result implies that if g is a continuous map of the circle onto itself of degree n with $|n| > 1$, then g is topologically semi-conjugate to the map of the circle to itself given by $z \to z^n$. A simple proof of this result appears in [6]. Similar results may also be found in [7].

A second example is a result of Bowen [5, Theorem 28, p. 741]. The result is that if f is the restriction of a diffeomorphism satisfying S. Smale's Axiom A to a basic set, then there exists a subshift of finite type having the same topological entropy as f which is topologically semi-conjugate to f.

Our final example is a result of Milnor and Thurston [11] (compare with the result of Parry [13]). The result is that if f is a piecewise monotone continuous map of the interval to itself with topological entropy $\log s$, and $s > 1$, then f is topologically semi-conjugate to a piecewise linear map with slope everywhere equal to $\pm s$. Also, the topological semi-conjugacy is monotone.

In this paper, we obtain a different result dealing with topological semi-conjugacies for maps of the interval to itself. We let T_n and T_{n^-} denote the piecewise linear maps of $[0, 1]$ onto itself with slopes alternating between $+n$ and $-n$ (see Definition 2.1). We prove the following result in the first part of the paper.

Theorem 3.31. *Let $f : [0, 1] \to [0, 1]$ be continuous and have an n-horseshoe. Then f is topologically semi-conjugate to T_n or T_{n^-}.*

We remark that the topological semi-conjugacy obtained in Theorem 3.31 is not in general monotone. In fact, we give an example (see Proposition 3.19) where the topological semi-conjugacy is nowhere differentiable and a dense set of points have uncountably many inverse images.

The second part of this paper exploits the connection between topological semi-conjugacies and inverse limit spaces. For each integer $m > 1$, we let K_m denote the corresponding Knaster continuum (see Definition 4.2). Also, given $f : [0, 1] \to [0, 1]$ we let (I, f) denote the corresponding inverse limit space. We prove the following result:

Theorem 5.1. *Let $f : I \to I$ be a continuous function with positive topological entropy. Then for any $m \geqslant 2$, there is a continuous surjective map from (I, f) to K_m.*

We obtain two corollaries to this theorem. The first is a well-known result of Barge and Martin [3]. A stronger result has recently been obtained by Mouron [12].

Corollary 5.2. *Let $f : I \to I$ be a continuous function with positive topological entropy. Then (I, f) has an indecomposable subcontinuum.*

Corollary 5.3. *For any $m, n \geqslant 2$, there is a continuous surjective map from K_m to K_n.*

We also show that the analogous statement to Corollary 5.3 does not hold for solenoids. We let Σ_n denote the n-solenoid and prove the following:

Theorem 5.12. *Suppose m, n are positive integers, $m, n \geqslant 2$, such that there is a prime p with $p|n$ but $p \nmid m$. Then there does not exist a continuous surjective map from Σ_m to Σ_n.*

We conclude the paper with the following result:

Theorem 6.7. *Suppose f and g are maps of the circle to itself with deg(f) $= m > 0$, deg(g) $= n > 1$, and $m \neq n$. Then f is not topologically semi-conjugate to g.*

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2. Definitions and Preliminary Lemmas

Definition 2.1. *The **full tent map**, $T : I \to I$, where I is the unit interval $[0, 1]$, is defined as*

$$T(x) = \begin{cases} 2x & \text{if } x \leqslant 1/2 \\ -2(x-1) & \text{if } x > 1/2 \end{cases}$$

In general, for a positive integer $n \geqslant 2$, let $T_n : I \to I$ be piecewise monotone and linear with slope $\pm n$ defined as follows:

$$T_n(k/n) = \begin{cases} 0 & \text{if } k \text{ is an even integer, } 0 \leqslant k \leqslant n \\ 1 & \text{if } k \text{ is an odd integer, } 0 \leqslant k \leqslant n \end{cases}$$

*and T_n is linear bewteen these points. The maps T_n are sometimes called **sawtooth maps**.*
In particular, the full tent map is T_2, but when we only write T, we will mean the full tent map, T_2.
We also define $T_{n^-} : I \to I$ similar to T_n except with slopes reversed, i.e.

$$T_{n^-}(k/n) = \begin{cases} 1 & \text{if } k \text{ is an even integer, } 0 \leqslant k \leqslant n \\ 0 & \text{if } k \text{ is an odd integer, } 0 \leqslant k \leqslant n \end{cases}$$

Remark 2.2. *For n even, T_n and T_{n^-} are topologically conjugate via $\alpha(x) = 1 - x$. This is not true when n is odd; simply consider the dynamics of the endpoints.*

Definition 2.3. *Let $f : I \to I$ be continuous. The map f is **turbulent** if there exist two closed subintervals of I, A_1 and A_2, with disjoint interiors, such that*

$$(A_1 \cup A_2) \subset (f(A_1) \cap f(A_2)).$$

*In general, we say a continuous map of the interval has an n**-horseshoe** if there exist n closed subintervals of I, $A_1, ..., A_n$, with pairwise disjoint interiors, such that*

$$(A_1 \cup ... \cup A_n) \subset (f(A_1) \cap ... \cap f(A_n)).$$

Turbulence is then simply having a 2-horseshoe.

3. Results on semi-conjugacies to tent maps

Our main goal in this section is to prove Theorem 3.31. We will first deal with the case of the tent map $T = T_2$. We begin with a simple, basic fact about the tent map. A proof may be found in [15, page 385].

Proposition 3.1. *Given a sequence $B = b_0, b_1, b_2, ...$ of L's and R's, there exists a unique $y \in [0, 1]$ such that for the tent map T, the following holds: For each nonnegative integer k, $b_k = L$ implies $T^k(y) \in [0, 1/2]$ and $b_k = R$ implies $T^k(y) \in [1/2, 1]$.*

Definition 3.2. *Let B be a sequence of L's and R's. Let y be the unique point given in Proposition 3.1. We will call y the point whose **itinerary** under T is B.*

We wish to use Definition 3.2 to define a map which is a potential semi-conjugacy from a map f to the tent map T. With this in mind, we make the following three definitions:

Definition 3.3. *A **labeled partition of** $[0, 1]$ **with respect to** f is an ordered pair (C, ϕ) such that*

1. *C is a nonempty finite subset of the open interval $(0, 1)$,*
2. *ϕ is a function whose domain is the set of connected components of $[0, 1] - C$ and whose range is the set containing the two symbols L and R, and*
3. *ϕ alternates between L and R on consecutive connected components.*

Definition 3.4. *The* ***itinerary of*** x ***with respect to*** f ***and*** (C, ϕ) *is the sequence* $B = b_0, b_1, b_2, ...$ *of* L*'s and* R*'s such that* $b_k = R$ *if* $f^k(x)$ *lies in some component* J *of* $[0,1] - C$ *with* $\phi(J) = R$*, and* $b_k = L$ *otherwise.*

Note that in the above definition, $b_k = L$ if $f^k(x) \in C$. The choice of L is arbitrary. We will see later that this choice does not matter.

Definition 3.5. *Given a labeled partition* (C, ϕ) *with respect to* f *we define a map* $\alpha : [0,1] \to [0,1]$ *as follows:*
Form the itinerary of x *under* f *to obtain a sequence* B *of* L*'s and* R*'s. Set* $\alpha(x) = y$ *where* y *is the point whose itinerary under* T *is* B*.*

Definition 3.6. *Let* A *and* B *be finite disjoint subsets of* I*. By the* ***distance between*** A ***and*** B*, we mean the minimum value of* $|a - b|$ *where* $a \in A$*,* $b \in B$*.*

Proposition 3.7. *Suppose* $f : [0,1] \to [0,1]$ *is continuous,* (C, ϕ) *is a labeled partition with respect to* f*, and* α *is the associated map. Then for all points* x *such that* $f^n(x) \notin C$ *for all nonnegative integers* n*,* α *is continuous at* x*.*

Proof. Let $x \in I$ be such that $f^n(x) \notin C$ for all nonnegative integers n. Since f is continuous, all of its iterates are continuous as well. Let $\epsilon > 0$. Choose N large enough so that $2^{-(N+1)} < \epsilon$. Let δ' denote the distance between the sets $\{x, f(x), ..., f^N(x)\}$ and C. There is a $\delta > 0$ such that if $y \in I$ with $|x - y| < \delta$, then $|f^k(x) - f^k(y)| < \delta'$ for each $k = 0, ..., N$.

Suppose $y \in I$ with $|x - y| < \delta$. Then the itineraries of x and y under f agree on the first $N + 1$ places. Hence the itineraries of $\alpha(x)$ and $\alpha(y)$ under T agree on the first $N + 1$ places. Hence $|\alpha(x) - \alpha(y)| \leqslant 2^{-(N+1)} < \epsilon$.

□

Proposition 3.8. $\alpha \circ f = T \circ \alpha$*.*

Proof. Put $(\alpha \circ f)(x) = y$. Then y is the point that satisfies $\underline{I}_f(f(x)) = \underline{I}_T(y)$, where $\underline{I}_f$ and $\underline{I}_T$ are the itineraries of the point under f and T, respectively. If $\underline{I}_f(x) = b_0 b_1 b_2 ...$, then y has itinerary $\underline{I}_T(y) = b_1 b_2 ...$. Now let $z = \alpha(x)$. Then z has the itinerary $\underline{I}_T(z) = b_0 b_1 b_2 ...$. Applying T to z, we have $\underline{I}_T(T(z)) = b_1 b_2 ...$. Since y and $T(z)$ have the same itinerary under T, $y = T(z)$.

□

Proposition 3.9. *Let* B *and* B' *be sequences of* L*'s and* R*'s. Let* x *and* x' *be the corresponding points whose itineraries are* B *and* B' *under* T*,* *respectively. Let A be a finite sequence of L's and R's with length n. Let y and y' be the corresponding points whose itineraries are AB and AB' under T, respectively. If $|x - x'| < \epsilon$, then $|y - y'| < \frac{\epsilon}{2^n}$.*

Proof. Let us suppose that $A = L$. Then $x = 2y$ and $x' = 2y'$. So $|y - y'| = |\frac{x-x'}{2}| < \frac{\epsilon}{2}$. It is clear that this holds for $A = R$ as well. Formally by induction, we see by concatenating one by one, we have the desired result. □

Theorem 3.10. *Suppose $f : [0,1] \to [0,1]$ is continuous, (C, ϕ) is a labeled partition with respect to f, and α is the associated map. Then the following are equivalent:*

1. *α is continuous.*
2. *α is continuous at each $z \in C$.*
3. *$\alpha(z) = 1/2$ for all $z \in C$.*
4. *Each $z \in C$ has itinerary $LR\overline{L}$, where $\overline{L}$ denotes the sequence $LLL....$*

Moreover, if α is continuous, then α is also surjective, and hence f is topologically semi-conjugate to T.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1): By Proposition 3.7, it suffices to show that α is continous on the points x such that $x \notin C$ but $f^n(x) \in C$ for some positive integer n. Let x be such that $x \notin C$ but $f^n(x) \in C$, where n is the smallest such positive integer. Call $f^n(x) = y$ and let $\epsilon > 0$. Since α is continuous at y, there is a $\delta' > 0$ such that if $|y - y'| < \delta'$, then $|\alpha(y) - \alpha(y')| < \epsilon$.

Let γ denote the distance between the sets $\{x, f(x), ..., f^{n-1}(x)\}$ and C. There exists $\delta > 0$ such that if $x' \in I$ and $|x - x'| < \delta$, then $|f^n(x) - f^n(x')| < \delta'$ and $|f^k(x) - f^k(x')| < \gamma$ for $k = 0, 1, ..., n-1$. Suppose $x' \in I$ and $|x - x'| < \delta$. Then $|\alpha(f^n(x)) - \alpha(f^n(x'))| < \epsilon$ and by Proposition 3.9, $|\alpha(x) - \alpha(x')| < \frac{\epsilon}{2^n} < \epsilon$

(2) $\Rightarrow$ (3): Suppose, by way of contradiction, that α is continuous at z for some $z \in C$ but $\alpha(z) \neq 1/2$. Now suppose that $\alpha(z) = k < 1/2$ and that the connected component of $[0,1] - C$ to the left of z is labeled L and the connected component of $[0,1] - C$ to the right of z is labeled R. Take a sequence z_n that converges and decreases to z. For some N, the itinerary of all z_n starts with R for all $n > N$. Since $\alpha(z_n) \geqslant 1/2$ for all n, and α is continuous at z, $\alpha(z_n) \to \alpha(z) \geqslant 1/2$, a contradiction

since $\alpha(z)$ was assumed to be strictly less than 1/2. A similar proof works in the other cases.

(3) $\Rightarrow$ (2): The first thing we observe is that the point that corresponds to the itineraries $B = LRLLL...$ and $B' = RRLLL...$ under T is 1/2. The finite itinerary $LRLLL...L$ with n characters defines points in the interval $[1/2 - 1/2^n, 1/2]$ and the finite itinerary $RRLLL...L$ with n characters defines points in the interval $[1/2, 1/2 + 1/2^n]$. Let $z \in C$ and suppose $\alpha(z) = 1/2$. Let $\epsilon > 0$. Take n to be such that $2^{-n} < \epsilon$. By hypothesis (3), each element of C has itinerary $LR\overline{L}$. Thus, if $z \in C$, we have $f^k(z) \in I - C$ for each positive integer k. In the same manner as Propostion 3.7, we may find a $\delta > 0$ such that if $|z - z'| < \delta$, z and z' share itineraries starting with, at worst, the second character up to the $(n+1)^{st}$ character. Since z is mapped to 1/2 by α, the images of both points must lie in the interval $[1/2 - 1/2^{n+1}, 1/2 + 1/2^{n+1}]$. So $|\alpha(z) - \alpha(z')| < 2^{-n} < \epsilon$.

(3) $\Leftrightarrow$ (4) is clear.

Moreover, if α is continuous, then α is surjective: Let $z \in C$. Then $\alpha(z) = 1/2$ and $\alpha(f(z)) = 1$ and $\alpha(f^2(z)) = 0$. Hence α is surjective. It follows from Proposition 3.8 that f is topologically semi-conjugate to T.

□

We now use Theorem 3.10 to obtain the desired result for turbulent maps. We will use the following lemma from Block and Coppel [4, page 26].

Lemma 3.11. *Let f be a turbulent continuous map of the interval. Then there exist points a, b and c such that $f(a) = a$, $f(b) = a$, and $f(c) = b$ for $a < c < b$ or $b < c < a$.*

Theorem 3.12. *Let f be a turbulent continuous map of the interval. Then f is topologically semi-conjugate to the tent map, T.*

Proof. Since f is turbulent, there exist points a, b, and c such that $f(a) = a$, $f(b) = a$, and $f(c) = b$ and either $a < c < b$ or $b < c < a$. Let us assume the first set of inequalities is true. Define a labeled partition (C, ϕ) by $C = \{c\}$ and $\phi([0, c)) = L$ and $\phi((c, 1]) = R$. We see that the itinerary of c under f is $LR\overline{L}$. Therefore by Theorem 3.10, f is topologically semi-conjugate to T.

□

Theorem 3.13. *If f is topologically semi-conjugate to the tent map, T, then f need not be turbulent.*

Proof. Let f be the function with $f(0) = 1/2$, $f(1/4) = 1$, $f(1/2) = 1/2$, $f(3/4) = 0$, $f(1) = 1/2$, and linear between these points (see Figure 1). It is easy to see this map is not turbulent since there do not exist points a, b, and c with the property $f(a) = a$, $f(b) = a$, $f(c) = b$ with c in between a and b. However, if we define a labeled partition with $C = \{1/4, 3/4\}$ with the labeling in Figure 1, we see the itineraries of $1/4$ and $3/4$ are $LR\overline{L}$, hence by Theorem 3.10, f is topologically semi-conjugate to T.

□

Remark 3.14. *The topological semi-conjugacy can be verified to be $\alpha(x) = T_{2^-}$, as shown in Figure 2.*

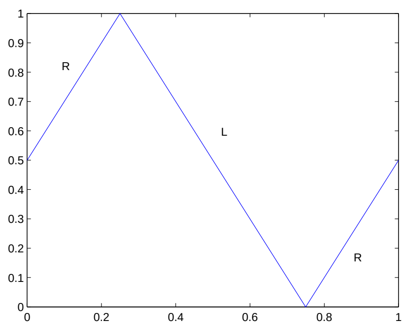

Figure 1

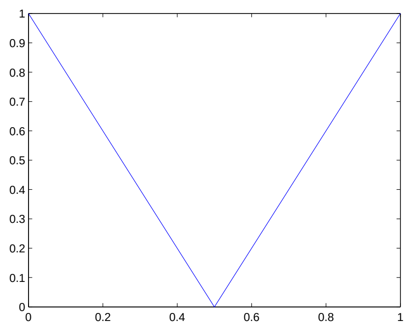

Figure 2

Remark 3.15. *It follows from Theorem 3.12 that for $n \geqslant 2$, T_n and T_{n^-} are semi-conjugate to the tent map, T, since it is easy to see that T_n and T_{n^-} are turbulent.*

Remark 3.16. *If f is semi-conjugate to the tent map, T, a turning point of f need not map to the turning point of T. Moreover, a point that is not a turning point of f may map to the turning point of T.*

To see this, consider the map f with $f(0) = 0$, $f(3/8) = 1$, $f(3/4) = 0$, $f(1) = 1/2$, and f linear between these points, as shown in Figure 3. We see that f is turbulent with $a = 0$, $b = 3/4$, $c = 9/32$. We define α as in the proof of of Theorem 3.12. It is clear that $\alpha(c) = 1/2$ although c is not a turning point of f. By following the itinerary of the turning point of f, $3/8$, we see that $\alpha(3/8) \neq 1/2$. The graph of α is shown in Figure 4.

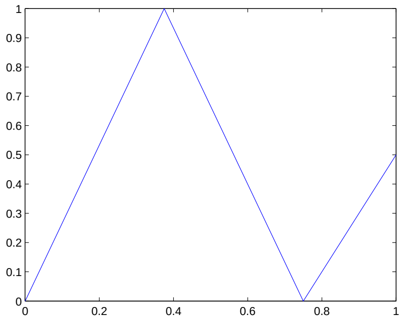

Figure 3

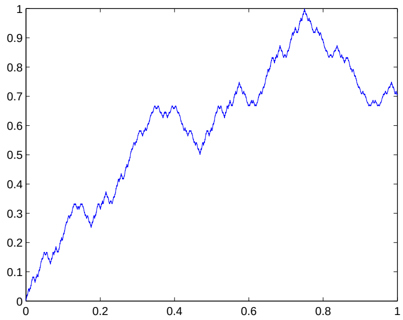

Figure 4

Remark 3.17. *We also note that f may be turbulent but not onto. So we can have a semi-conjugacy to T without f being onto I.*

Remark 3.18. *If f is semi-conjugate to T, then α is not necessarily unique.*

Proof. Let $f = T_4$. Then we may define one labeled partition with $C = \{1/4\}$, L to the left of $1/4$, and R to the right of $1/4$ and another labeled partition with $C = \{3/4\}$, L to the left of $3/4$, and R to the right of $3/4$. These each yield two different semi-conjugacies. We call the first α_1 and the second α_2.

Figure 5 below is the graph of α_1, and Figure 6 is the graph of α_2.

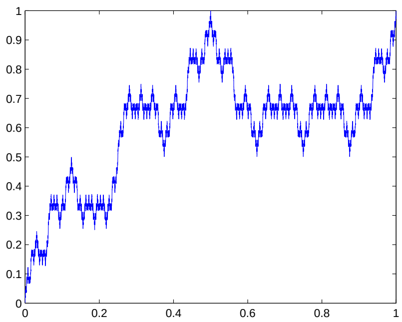

Figure 5

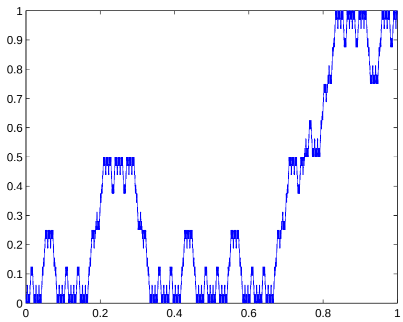

Figure 6

□

Proposition 3.19. *The topological semi-conjugacies α_1 and α_2 from the previous two figures are nowhere differentiable, and for $i = 1, 2$, there is a dense set D such that for each $x \in D$, $\alpha_i^{-1}(x)$ is uncountable.*

Proof. We will prove the proposition only for α_1, as the proof is similar for α_2. Let $A = a_0, a_1, a_2, ...$ be any sequence where each $a_i \in \{1, 2, 3, 4\}$. There is a unique $x \in [0, 1]$ such that if $a_n = k$, then $T_4^n(x) \in [\frac{k-1}{4}, \frac{k}{4}]$. Let $x \in [0, 1]$ correspond to such a sequence A under T_4. We can view α_1 as obtaining a new sequence $A' = b_0, b_1, b_2, ...,$ where $b_i = L$ if $a_i = 1$ and $b_i = R$ if $a_i \in \{2, 3, 4\}$, then mapping x to the point whose itinerary under T_2 is A'.

Since T_2 is transitive, there is a point y with a dense orbit. Let $D = \{y, T_2(y), T_2^2(y), ...\}$. For each point in D, the itinerary under T_2 has infinitely many R's. For each R that appears, there are 3 pre-images in T_4, since $2, 3, 4$ are all converted to R under α_1. Therefore every point in D has uncountably many pre-images under α_1.

Now, we show that α_1 is not differentiable anywhere. If we fix n, we may consider a canonical partition of the interval as $[0, 1/4^n]$, $[1/4^n, 2/4^n]$, and so on. Each interval J in this partition corresponds to a unique finite sequence $\{a_1, a_2, ..., a_n\}$ where each $a_i \in \{1, 2, 3, 4\}$. Consider the sequence $\{b_1, b_2, ..., b_n\}$ where $b_i = L$ if $a_i = 1$ and $b_i = R$ if $a_i \in \{2, 3, 4\}$. There is a unique interval K of length $1/2^n$ corresponding to $\{b_1, b_2, ..., b_n\}$. Then $\alpha_1(J) = K$.

Now, let $x \in [0, 1]$ and fix an integer $n \geqslant 1$. The point x lies in one of these canonically chosen intervals J of length $1/4^n$, with $\alpha_1(J) = K$, as above. Let x_n be a point in J such that $\alpha_1(x_n)$ is an endpoint of K and $|\alpha_1(x) - \alpha_1(x_n)|$ is as large as possible. This distance is at least half the total length of K. Therefore $\frac{|\alpha_1(x)-\alpha_1(x_n)|}{|x-x_n|} \geqslant \frac{2^n}{2} = 2^{n-1}$ (half the expansion rate). As $n \to \infty$, we see that $2^{n-1} \to \infty$ and $x_n \to x$. Hence, α_1 is nowhere differentiable.

□

Theorem 3.20. *Let $f : [0, 1] \to [0, 1]$ be continuous. Suppose f has an n-horseshoe.*

(1) If n is even, then there exist points $c_0, c_1, ..., c_n$ such that $f(c_{2i}) = c_0$ and $f(c_{2i+1}) = c_n$ for $i = 0, ..., n/2$ and either $c_0 < c_1 < ... < c_n$ or $c_n < c_{n-1} < ... < c_0$.

(2) If n is odd, then there exist points $c_0, c_1, ..., c_n$ such that $f(c_{2i}) = c_{k_1}$

and $f(c_{2i+1}) = c_{k_2}$ *for* $i = 0, ..., (n-1)/2$ *with* $c_0 < ... < c_n$ *and either* $k_1 = 0$ *and* $k_2 = n$ *or* $k_1 = n$ *and* $k_2 = 0$.

Proof. The idea for this proof is to show that there are at least n intervals that alternate between increasing and decreasing (with respect to f), on average. Let f have an n-horseshoe. Then there are closed intervals $A_1, ..., A_n$ with disjoint interiors that satisfy the definition. Let us also put them in increasing order by their left endpoints. Let $[v_i, w_i] = A_i$. There are points $x_1, y_1 \in A_1$ such that $f(x_1) = v_1$ and $f(y_1) = w_n$. Let us assume $x_1 < y_1$, so on average $[x_1, y_1]$ increases. Now in A_2 there exists a point x_2 such that $f(x_2) = v_1$. So $x_2 > y_1$ and $[y_1, x_2]$ decreases on average covering $A_1, ..., A_n$.

Continuing in this manner, we find that there are at least n intervals that alternate increasing and decreasing (on average) such that the image under f of each one of these intervals contains the interval $[v_1, w_n]$. Let us call these intervals $B_1, B_2, ..., B_n$, ordering them as before.

Suppose that n is even and that B_1 increases on average. Let c_0 be the smallest fixed point in B_1. Now in B_2, take the largest point, call it c_2, that maps to c_0. We continue finding points c_{2i} in this manner that map to c_0 until we find the last one in B_n. We may now find a point c_1 in B_1 such that c_1 maps to c_n and $c_0 < c_1 < c_2$. Continuing in this manner again, we find the necessary c_{2i+1}. Thus we have the desired result with $c_0 < ... < c_n$.

Similarly, if B_1 is decreasing, we may find the points with $c_n < ... < c_0$. Also, this process holds for n odd.

□

Definition 3.21. *An* ***n-labeled partition of*** $[0,1]$ ***with respect to*** f *is an ordered n-tuple* $(C_1, C_2, ..., C_{n-1}, \phi)$ *such that:*

1. *Each* C_i *is a nonempty finite subset of the open interval* $(0,1)$,
2. ϕ *is a function whose domain is the set of connected components of* $[0,1] - C$ *where* $C = \bigcup_{i=1}^{n-1} C_i$ *and whose range is the set containing the numbers* $\{1, 2, ..., n\}$,
3. *Given any two adjacent components* J_1 *and* J_2 *of* $[0,1] - C$ *which are separated by a point* $z \in C_i$, $\phi(\{J_1, J_2\}) = \{i, i+1\}$.

Definition 3.22. *The* ***itinerary of*** x ***with respect to*** f ***and*** $(C_1, ..., C_{n-1}, \phi)$ *is a sequence* B *of numbers in* $\{1, 2, ..., n\}$ *such that* $b_k = i$ *if* $f^k(x)$ *lies in some component* J *of* $[0,1] - C$ *with* $\phi(J) = i$ *or if* $f^k(x) \in C_i$.

Proposition 3.23. *Given any sequence B of $\{1, ..., n\}$, there exists a unique point $y \in [0,1]$ such that for the tent map T_n, the following holds: For each nonnegative integer k, $b_k = i$ implies $T^k(x) \in [\frac{i-1}{n}, \frac{i}{n}]$.*

Proof. The verification is essentially the same as Proposition 3.1. □

Definition 3.24. *We call the point y described in the previous proposition the* ***point whose itinerary under*** *T_n* ***is*** *B.*

Definition 3.25. *Given a labeled partition $(C_1, ..., C_{n-1}, \phi)$ with respect to f, we define a map α as follows:*
Form the itinerary of x under f. We get a sequence B of numbers in the set $\{1, ..., n\}$. Set $\alpha(x) = y$ where y is the point whose itinerary under T_n is B.

Proposition 3.26. *Suppose $f : [0,1] \to [0,1]$ is continuous, $(C_1, ..., C_{n-1}, \phi)$ is an n-labeled partition with respect to f, and α the associated map. Then for all points x such that $f^k(x) \notin C_i$ for each i and k, α is continuous at x.*

Proof. Let $\epsilon > 0$. Choose N such that $n^{-N} < \epsilon$. Let δ denote the distance between the sets C and $\{x, f(x), ..., f^N(x)\}$. If $|x - y| < \delta$, then y is in the same connected component of x on at least the first N iterates of f. Therefore, $|\alpha(x) - \alpha(y)| < n^{-N} < \epsilon$. □

Proposition 3.27. *$\alpha \circ f = T_n \circ \alpha$.*

Proof. The verification of this proposition is essentially identical to Proposition 3.8. □

Remark 3.28. *We may also replace each T_n in Propositions and Definitions 3.23 through 3.27 by T_{n^-} and obtain the same results with T_{n^-}.*

Proposition 3.29. *Let B and B' be sequences of $\{1, ..., n\}$. Let x and x' be the points whose itineraries under T_n are B and B', respectively. Let A be a finite sequence of $\{1, ..., n\}$ of length j. Let y and y' be the points whose itineraries under T_n be AB and AB', respectively. If $|x - x'| < \epsilon$, then $|y - y'| < \frac{\epsilon}{n^j}$. Moreover, the same result holds for T_{n^-}.*

Proof. The proof is essentially the same as the proof of Proposition 3.9. □

Theorem 3.30. *Suppose $f : [0,1] \to [0,1]$ is continuous and $(C_1, ...C_{n-1}, \phi)$ is an n-labeled partition with respect to f, and α is the associated map. Then the following are equivalent:*

(1) *α is continuous.*
(2) *α is continuous at each $z \in C$, where C is the union of C_i's.*
(3) *$\alpha(z) = \frac{i}{n}$ for all $z \in C_i$ and all i.*

Moreover, if α is continuous, then α is also surjective, and hence f is topologically semi-conjugate to either T_n or T_{n^-}.

Proof. The proof is nearly identical to the proof of Theorem 3.10 and will be omitted.

□

Theorem 3.31. *Let $f : [0,1] \to [0,1]$ be continuous and have an n-horseshoe. Then f is topologically semi-conjugate to T_n or T_{n^-}.*

Proof. There are actually 4 cases to prove; when n is even or odd and whether it is to T_n or T_{n^-}. The following is the proof for n even and T_n. The other 3 cases are virtually identical.

Let $c_0, c_1, ..., c_n$ be the points that satisfy Theorem 3.20 and suppose $c_0 < ... < c_n$. Then we set an n-labeled partition $(C_1, ..., C_{n-1}, \phi)$ as $C_i = \{c_i\}$ and $\phi[0, c_1) = 1$, $\phi(c_1, c_2) = 2$, and so on, increasing by one on each consecutive connected component. The itinerary of c_i when i is odd is the sequence $i, n, 1, 1, 1...$, so $\alpha(c_i) = i/n$. The itinerary of c_i when i is even is $i, 1, 1, 1, ...$, so again $\alpha(c_i) = i/n$. So by Theorem 3.30, α is a topological semi-conjugacy.

□

Remark 3.32. *We mention a few of the above properties as major differences between maps of the interval and maps of the circle. In* [6], *Boyland proves the following about degree two maps of the circle:*
Let g be a continuous degree two map of the circle with α a light (which means every pre-image is totally disconnected) semi-conjugacy to the angle doubling map, d. Then the following are equivalent:

(1) *α is not injective.*
(2) *There exists a full measure, dense G_δ-set, $\Lambda \in S^1$ so that $\theta \in \Lambda$ implies $\alpha^{-1}(\theta)$ is uncountable.*
(3) *$h(g) > h(d) = \log(2)$.*

Moreover, if g is transitive, then g is locally eventually onto. Also, if α is degree 1, then α is unique.

We see that the tent map on the interval is an analogy to the angle-doubling map on the circle since both map half of the space to the whole space. Moreover, the angle-doubling map d is semi-conjugate to T. However, by looking at the example shown in Figures 1 and 2, the entropy of the map is $\log 2$ and the semi-conjugacy has disconnected point inverses. We have also seen that the map in Figure 1 is transitive. None of the above results that hold on the circle hold for its equivalent, the tent map, on the interval.

4. Inverse limits

Definition 4.1. *Let $\{X_i, d_i\}_{i=0}^{\infty}$ be a collection of compact metric spaces each with a metric d_i bounded by 1, and such that for each i, $f_i : X_{i+1} \to X_i$ is a continuous map. The* ***inverse limit space*** *of the inverse limit system $\{X_i, d_i\}_{i=0}^{\infty}$ is the set*

$$\lim_{\leftarrow}\{X_i, f_i\}_{i=0}^{\infty} = \{\bar{x} = (x_0, x_1, \dots) \mid \bar{x} \in \prod_{i=0}^{\infty} X_i, f_i(x_{i+1}) = x_i, i \in \mathbb{N}\},$$

with a metric d given by

$$d(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}.$$

If $X_i = X$ and $f_i = f$ for all i, the inverse limit space is denoted (X, f). The map $\sigma : (X, f) \to (X, f)$ defined by

$$\sigma(x_0, x_1, \dots) = (f(x_0), x_0, x_1, \dots)$$

is called the ***shift homeomorphism****, or the* ***induced homeomorphism****.*

Definition 4.2. *Of particular interest, we will consider later the* ***α-adic solenoid****, denoted Σ_α, and the* ***α-adic Knaster continuum****, denoted K_α. Let $\alpha = (a_0, a_1, a_2, ...)$ be a sequence of integers with $a_k > 1$ for all $k \geqslant 0$. Define Σ_α as the inverse limit space of mappings $z \mapsto z^{a_k}$, $k = 1, 2, ...$ on the circle in the complex plane $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Then*

$$\Sigma_\alpha = \lim_{\leftarrow}\{S^1, z^{a_k}\}_{k=0}^{\infty}.$$

Define K_α as the quotient Σ_α by the relation $\sim$ where $x \sim y$ if and only if $x = y$ or $x = y^{-1}$. For notation, if $\alpha = (n, n, n, ...)$, we will denote the spaces Σ_n and K_n.

Both theorems in this section are well-known, and some interesting results relating to them may be found in Ye [18]. The following theorem establishes a connection between topological semi-conjugacies and inverse limit spaces.

Theorem 4.3. *Suppose $f : X \to X$ and $g : Y \to Y$, are continuous maps on compact spaces X and Y. Suppose also that f is topologically semi-conjugate to g, with semi-conjugacy α. Then $\hat{\alpha}$ is continuous and surjective and the following diagram commutes:*

$$\begin{array}{ccc} (X,f) & \xrightarrow{\hat{f}} & (X,f) \\ \hat{\alpha}\big\downarrow & & \big\downarrow\hat{\alpha} \\ (Y,g) & \xrightarrow{\hat{g}} & (Y,g) \end{array}$$

where $\hat{\alpha}((x_0, x_1, \ldots)) = (\alpha(x_0), \alpha(x_1), \ldots)$ and $\hat{f}$ and $\hat{g}$ are the shift homeomorphisms on their respective spaces.

Proof. We start by showing $\hat{\alpha}$ indeed sends points to (Y, g). Let $(x_0, x_1, \ldots) \in (X, f)$. Since $f(x_n) = x_{n-1}$, $\alpha(f(x_n)) = \alpha(x_{n-1})$ and $\alpha(f(x_n)) = g(\alpha(x_n))$ by the semi-conjugacy, $g(\alpha(x_n)) = \alpha(x_{n-1})$, so $\hat{\alpha}$ sends points to (Y, g). Clearly, $\hat{\alpha}$ is continuous.

Now we show that $\hat{\alpha}$ is surjective.

Let $(y_0, y_1, y_2, ...)$ be a point of (Y, g). Fix a positive integer n. Let x_n be such that $\alpha(x_n) = y_n$. We have that $\alpha f(x_n) = g\alpha(x_n) = g(y_n) = y_{n-1}$, so $f(x_n) \in \alpha^{-1}(y_{n-1})$. We may choose $x_{n-1} \in \alpha^{-1}(y_{n-1})$ such that $f(x_n) = x_{n-1}$.

Let $L = \bigcap_{j=0}^{\infty} f^j(X)$. For a point $(x_0, x_1, x_2, ...) = \bar{x} \in (X, f)$, let $\pi_n(\bar{x})$ be the projection map onto the coordinate x_n. Notice that $f(L) = L$, so for all $x \in L$, there exists an $\bar{x} \in (X, f)$ with $\pi_0(\bar{x}) = x$.

Now, fix $\bar{y} = (y_0, y_1, y_2, ...) \in (Y, g)$ and fix an integer $k \geqslant 0$.

Claim 4.4. *There is a $\bar{w} \in (X, f)$ with $\pi_k(\hat{\alpha}(\bar{w})) = y_k$.*

Proof. Let $E_0 = \alpha^{-1}(\{y_k\})$, $E_1 = \alpha^{-1}(\{y_{k+1}\})$, and so on. We see that each E_j is non-empty, compact, and $E_0 \supset E_1 \supset E_2 \supset \ldots$. So there is an $x \in \bigcap_{k=0}^{\infty} E_k$. Then $x \in L$, so there is an $\bar{x}$ with $\pi_0(\bar{x}) = x$. Now set $\bar{w} = \hat{f}^k(\bar{x})$. Then $\pi_k(\hat{\alpha}(\bar{w})) = y_k$ because $\pi_k(\bar{w}) \in \alpha^{-1}(\{y_k\})$. This proves the claim.

Set $A_k = \{\bar{x} \mid \pi_k(\hat{\alpha}(\bar{x})) = y_k\}$. We see that $A_0 \supset A_1 \supset \ldots$ with each compact and nonempty, so there is an $\bar{x}$ in the intersection as well. Then $\hat{\alpha}(\bar{x}) = \bar{y}$. Hence, $\hat{\alpha}$ is surjective.

Finally, we just check that the diagram commutes. $\hat{\alpha}(\hat{f}((x_0, x_1, x_2...))) = \hat{\alpha}((f(x_0), x_0, x_1, ...)) = (\alpha(f(x_0)), \alpha(x_0), \alpha(x_1), ...) = (g(\alpha(x_0)), \alpha(x_0), \alpha(x_1), ...) = \hat{g}(\hat{\alpha}((x_0, x_1, ...)))$. Thus $\hat{f}$ is topologically semi-conjugate to $\hat{g}$.

□

Definition 4.5. *We say that X is a **continuum** if X is a nonempty, compact, connected metric space. If X is not the union of any two proper subcontinua, we say X is **indecomposable**.*

Theorem 4.6. *Let X and Y be continua. Suppose that Y is indecomposable and $f : X \to Y$ is continuous and surjective. Then X has an indecomposable subcontinuum.*

Proof. Let $\mathcal{S}$ denote the set of all continua $K \subset X$ such that $f(K) = Y$. We have that $\mathcal{S}$ is nonempty since $X \in \mathcal{S}$. We may order $\mathcal{S}$ by reverse inclusion. $\mathcal{S}$ is partially ordered. If we take a chain $X \supset K_1 \supset K_2 \supset ...$, this has an upper bound, namely $\bigcap_\beta K_\beta$. So by Zorn's lemma, there is a maximal element in $\mathcal{S}$. (Notice that since we are ordering by reverse inclusion, maximal means smallest.) Call this element $K_0 \in \mathcal{S}$.

Claim 4.7. *K_0 is indecomposable.*

By way of contradiction, suppose that K_0 is not indecomposable. Then $K_0 = V \cup W$ where V and W are proper subcontinua of K_0. Now, $f(V) \cup f(W) = Y$, but Y is indecomposable, so it must be the case that one is contained the other, say $f(V) \subset f(W)$. Therefore, $f(W) = Y$, but then W is a proper subset of K_0 and $W \in \mathcal{S}$, a contradiction. So K_0 is indecomposable.

□

The following remarks are well-known and will be stated without proof. Some of these results, and others, may be found in [10], [2].

Remark 4.8. *For $n \geqslant 1$, the inverse limit spaces (I, f) and (I, f^n) are homeomorphic.*

Remark 4.9. *For $n \geqslant 2$, the inverse limit space (I, T_n) is homeomorphic to the Knaster continuum K_n. Note that since $T^2_{n^-}$ and T^2_n are topologically conjugate (for n odd, they are actually equal), we have that (I, T_{n^-}) is homeomorphic to (I, T_n) by the previous remark.*

Remark 4.10. *If X is an indecomposable continuum, then X is not path connected.*

5. Semi-conjugacies and inverse limit spaces

We now use our previous results to prove Theorem 5.1 and two corollaries.

Theorem 5.1. *Let $f : I \to I$ be a continuous function with positive topological entropy. Then for any $m \geqslant 2$, there is a continuous surjective map from (I, f) to K_m.*

Proof. Let k be such that f^k has an m-horseshoe [4, Theorem VIII.29, p. 215] [1, Theorem 4.3.5, p. 207]. Then f^k is semi-conjugate to T_m or T_{m^-}. So by Theorem 4.3, there is a continuous surjective map from (I, f^k) to (I, T_m) or (I, T_{m^-}). Since $(I, f^k) \simeq (I, f)$, $(I, T_m) \simeq K_m$, and $(I, T_{m^-}) \simeq K_m$, we have the desired result.

□

Corollary 5.2. *Let $f : I \to I$ be a continuous function with positive topological entropy. Then (I, f) has an indecomposable subcontinuum.*

Proof. By Theorem 5.1, there is a continuous surjective map from (I, f) to K_2. Since K_2 is indecomposable, the conclusion follows from Theorem 4.6.

□

Corollary 5.3. *For any $m, n \geqslant 2$, there is a continuous surjective map from K_m to K_n.*

Proof. Set $f = T_m$. Then (I, f) is homeomorphic to K_m. The topological entropy of f is $\log m$. By Theorem 5.1, there is a continuous surjective map from (I, f) to K_n.

□

Given the connection between solenoids and Knaster continua, one might expect that the analogous result to Corollary 5.3 holds for Σ_m and Σ_n. It turns out that this is not the case. To see this, we will use some results about topological groups.

Definition 5.4. *Let G be a locally compact abelian group. A* ***character*** *of G is a continuous group homomorphism from G to the circle group, S^1.*

Definition 5.5. *The* ***character group****, also called the* ***Pontryagin dual****, of a locally compact abelian group G is the set of all characters on G.*

Remark 5.6. *The Pontryagin dual, G', of a locally compact abelian group G is also a locally compact abelian group with the compact-open topology. G'', the Pontryagin dual of G', is naturally isomorphic to G, hence the reason for using the term dual. For further reading, we may consult* [14].

Theorem 5.7. *The Pontryagin dual of the solenoid Σ_n is the additive group $\mathbb{Q}_n = \{\frac{j}{{a_1}^{n_1} \cdot {a_2}^{n_2} \cdot \ldots \cdot {a_k}^{n_k}}\}$ where $a_1 \cdot \ldots \cdot a_k$ are the distinct primes in the prime factorization of n, $n_1, \ldots n_k$ are positive integers, and $j \in \mathbb{Z}$. If there is a one-to-one continuous homomorphism from $\mathbb{Q}_m$ to $\mathbb{Q}_n$, then this induces a continuous surjective homomorphism from Σ_n to Σ_m. Dually, if there is a continuous surjective homomorphism from Σ_n to Σ_m, then this induces a one-to-one continuous homomorphism from $\mathbb{Q}_m$ to $\mathbb{Q}_n$.*

These theorems are detailed in [9, p. 392, p. 402]. In a more general setting, these theorems are stated as having "dense" in place of "surjective." However, since Σ_n and Σ_m are compact and connected, we have the stated stronger result.

Theorem 5.8. *Let m, n be integers, $m, n \geqslant 2$. Suppose there is a continuous homomorphism $f : \mathbb{Q}_n \to \mathbb{Q}_m$ that does not map every element of $\mathbb{Q}_n$ to the identity. Then f is one-to-one.*

Proof. Suppose, by way of contradiction, that $f(a) = 0$ for some $a \neq 0$. We may assume a is an integer since if $f(c) = 0$ and $c = a/b$ where a and b are integers, then $0 = bf(a/b) = f(a)$. Then, $a/n \in \mathbb{Q}_n$. So, $0 = f(a) = nf(a/n) = naf(1/n)$, so $f(1/n) = 0$. Similarly, $f(1/n^2) = 0$. By induction, $f(1/n^k) = 0$ for $k = 1, 2, 3, \ldots$. Since the set $\{1/n, 1/n^2, \ldots\}$ generates $\mathbb{Q}_n$, f must send everything to 0, a contradiction. Hence, f must be one-to-one.

□

Corollary 5.9. *If $f : \Sigma_m \to \Sigma_n$ is a continuous homomorphism that does not take every element to the identity, then f is surjective.*

Proof. This follows by duality since f induces a map $f^* : \mathbb{Q}_n \to \mathbb{Q}_m$ which does not take every element to the identity.

□

Definition 5.10. *Let g be an element of a group $(G, *)$. Given a positive integer p, we say g has **infinite** p^{th} **roots** if for any positive integer k, there is an element $h \in G$ such that $\underbrace{h * h * \ldots * h}_{p^k \text{ times}} = g$.*

Lemma 5.11. *Suppose m and n are positive integers such that there is a prime p with $p | n$ but $p \nmid m$. Then there does not exist a continuous one-to-one homomorphism from $\mathbb{Q}_n$ to $\mathbb{Q}_m$.*

Proof. We first note that the only element in $\mathbb{Q}_m$ with infinite p^{th} roots is 0. Now suppose there is a continuous one-to-one homomorphism $f : \mathbb{Q}_n \to \mathbb{Q}_m$. Then $f(1) = b$ for some $b \neq 0$. So $b = f(1) = p^k f(1/p^k)$

for any positive integer k. But this means b has infinite p^{th} roots, so $b = 0$, a contradiction.

□

We are now ready to prove that the analogous result to Corollary 5.3 does not hold for solenoids.

Theorem 5.12. *Suppose m, n are positive integers, $m, n \geqslant 2$, such that there is a prime p with $p|n$ but $p \nmid m$. Then there does not exist a continuous surjective map from Σ_m to Σ_n.*

Proof. Suppose, by way of contradiction, that there exists a continuous surjective map $g : \Sigma_m \to \Sigma_n$. By Scheffer's theorem [16], there is a unique continuous homomorphism $\hat{g} : \Sigma_m \to \Sigma_n$ such that g is homotopic to $\hat{g}$. Since Σ_n is not path connected, $\hat{g}$ does not map every element of Σ_m to the identity of Σ_n. By Corollary 5.9, $\hat{g}$ is surjective. By Theorem 5.7, $\hat{g}$ induces a continuous, one-to-one homomorphism from $\mathbb{Q}_m$ to $\mathbb{Q}_n$. This contradicts Lemma 5.11.

□

Corollary 5.13. *Suppose m and n are positive integers such that there is a prime p with $p|n$ but $p \nmid m$. Let f and g be the maps of the circle to itself defined by $f(z) = z^m$ and $g(z) = z^n$. Then f is not topologically semi-conjugate to g.*

Proof. Suppose, by way of contradiction, that f is topologically semi-conjugate to g with semi-conjugacy α. By Theorem 4.3, there exists a continuous surjective map $\hat{\alpha} : \Sigma_m \to \Sigma_n$. This contradicts Theorem 5.12.

□

While this corollary follows immediately, we will prove a more general result in the next section.

6. A Result on Topological Semi-Conjugacies For Maps of the Circle

Definition 6.1. *Let $p : \tilde{X} \to X$ be a map of topological spaces. A **lift** of a map $f : Y \to X$, Y a topological space, is a map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.*

Definition 6.2. *A **covering space** of a topological space X is a topological space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying: There exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α.*

Theorem 6.3 (The Homotopy Lifting Property)**.** *Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.*

Remark 6.4. *The circle and the map $p : S^1 \to S^1$, $p(z) = z^n$, is a covering space when n is any positive integer.*

Remark 6.5. *Let f and g be maps of the circle to itself. Let deg(f) denote the degree of f. We have the following results: The maps f and g are homotopic if and only if deg(f) = deg(g). Also, deg($f \circ g$) = deg(f)deg(g). Finally, deg(f) = 0 if and only if f is homotopic to a constant map.*

The previous results are well-known. The formulations of the definitions as well as the proofs to the remarks can be found in Hatcher's text [8].

Lemma 6.6. *Suppose f and g are maps of the circle to itself, with deg(f) $= m > 0$, deg(g) $= n > 1$, and $m \neq n$. If f is topologically semi-conjugate to g, then f is topologically semi-conjugate to the map $p(z) = z^n$. Moreover, this topological semi-conjugacy has degree 0.*

Proof. By hypothesis, f is topologically semi-conjugate to g via a semi-conjugacy α_1. Also, by the Theorem of Shub [17] mentioned in the introduction, g is topologically semi-conjugate to p via a semi-conjugacy α_2. We have the following diagram:

$$\begin{array}{ccc} S^1 & \xleftarrow{f} & S^1 \\ \alpha_1 \big\downarrow & & \big\downarrow \alpha_1 \\ S^1 & \xleftarrow{g} & S^1 \\ \alpha_2 \big\downarrow & & \big\downarrow \alpha_2 \\ S^1 & \xleftarrow{p} & S^1 \end{array}$$

Then f is semi-conjugate to p via $\alpha_2 \circ \alpha_1$. Also, by Remark 6.5, α_1 has degree 0, so $\alpha_2 \circ \alpha_1$ also has degree 0.

□

We are now in position to prove our final result.

Theorem 6.7. *Suppose f and g are maps of the circle to itself with deg(f) $= m > 0$, deg(g) $= n > 1$, and $m \neq n$. Then f is not topologically semi-conjugate to g.*

Proof. Suppose, by way of contradiction, that with the above hypotheses, f is topologically semi-conjugate to g.

Step 1. By Lemma 6.6, there is a topological semi-conjugacy h from f to the map $p(z) = z^n$. There are homotopies $(h_0)_t$ and $(h_1)_t$ such that $(h_0)_t \circ f = p \circ (h_1)_t$ for all t, where $(h_k)_0 = h$ and $(h_k)_1$ is a constant map, $k = 0, 1$.

Proof of Step 1. By Lemma 6.6, we have $\deg(h) = 0$. This means that h is homotopic to a constant map c. Let $(h_0)_t$ be a homotopy with $(h_0)_0 = h$ and $(h_0)_1 = c$. Since p is a covering map, we may apply Theorem 6.3. The reader might want to take a look at the first box in the diagram below at this point.

Consider the homotopy $(h_0)_t \circ f$. We have $(h_0)_0 \circ f = h \circ f$ and $(h_0)_1 \circ f = c$. There is a map that lifts $(h_0)_0 \circ f$, namely h. Thus, by Theorem 6.3, there exists a homotopy $(h_1)_t$ with $(h_1)_0 = h$ such that $(h_0)_t \circ f = p \circ (h_1)_t$ for all t. So at $t = 1$, we have $c \circ f = c = p \circ (h_1)_1$. Since the pre-image of a single point under p is a discrete set of n points and $(h_1)_1$ is a continuous function of a connected space, $(h_1)_1$ is a constant map as well.

Step 2. There exists a homotopy $(h_\infty)_t : (S^1, f) \to \Sigma_n$, where $(h_\infty)_0 = \hat{h}$, the induced map from h, and $(h_\infty)_1$ is a constant map.

Proof of Step 2. By applying Theorem 6.3 inductively we obtain a homotopy $(h_n)_t$ for each positive integer n, such that $(h_n)_0 = h$, $(h_n)_1$ is a constant map, and the diagram below is commutative.

$$\begin{array}{ccccccc} S^1 & \xleftarrow{f} & S^1 & \xleftarrow{f} & S^1 & \xleftarrow{f} & \dots \\ {\scriptstyle (h_0)_t}\downarrow & & {\scriptstyle (h_1)_t}\downarrow & & {\scriptstyle (h_2)_t}\downarrow & & \\ S^1 & \xleftarrow{p} & S^1 & \xleftarrow{p} & S^1 & \xleftarrow{p} & \dots \end{array}$$

Define $(h_\infty)_t : (S^1, f) \to \Sigma_n$ by

$$(h_\infty)_t(x_0, x_1, x_2, ...) = ((h_0)_t(x_0), (h_1)_t(x_1), (h_2)_t(x_2), ...).$$

Then $(h_\infty)_0 = \hat{h}$ and $(h_\infty)_1$ is a constant map.

Step 3. There cannot be a homotopy h_t from a topological space to Σ_n from a surjective map to a constant. Hence, we come to a contradiction, so f cannot be topologically semi-conjugate to g.

Proof of Step 3. If there is a homotopy from a surjective map to a constant map, it would imply that Σ_n is path-connected, but Σ_n is not path-connected. We have reached the desired contradiction. □

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