ITERATED FUNCTION SYSTEMS AND THE CODE SPACE

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Abstract. The code space plays a significant role in the study of self-similar fractals. It is used to give coordinates to the points of a self-similar set. The code space also plays an important role in dynamical systems as well. The shift map on the code space is a valuable an example of a dynamical system. The shift map restricted to certain invariant subspaces is also important. The subshifts of finite type are used to analyze many common dynamical systems. This is the basis for the theory of symbolic dynamics.

There is a particularly useful metric on the code space. With this metric the dimension of the code space and the subshifts of finite type can be computed using results of K. Falconer on sub-self-similar sets. We show that the code space can be embedded in Euclidean space, R^n, by a map which is bi-Lipschitz. The shift or subshift of finite type on this embedded image can be extended to a C^∞-map $F : R^n \to R^n$ having the embedded set as a maximal compact invariant set which contains the nonwandering set of F in this case. The map F restricted to this set will be equivalent to a power of the shift or the subshift of finite type. The Hausdorff dimension of the invariant set, the Lyapunov exponents of F at various points in the invariant set, and the topological entropy of F can all be computed. This provides a general method of constructing examples in which the relationship between these quantities can be studied.

§1. Introduction

Let $\{f_1, \ldots, f_N\}$ be an *Iterated Function System*, *IFS*, on R^n. This is a collection of *contraction similitudes*, that is, each f_i is a function $f_i : R^n \to R^n$ such that for all $x, y \in R^n$, $||x - y|| = c_i \cdot ||f(x) - f(y)||$ for some $0 < c_i < 1$. Let the contraction factors for this *IFS* be denoted $\{c_1, \ldots, c_N\}$, respectively. Let K be the unique compact invariant set for the *IFS*. The set K is known as the *self-similar fractal defined by the IFS*. Let $\Sigma = \prod_{i=1}^{\infty}\{1, \ldots, N\}$. Let Σ be endowed with the product topology. There is a well-known map $g : \Sigma \to K$ which is used to coordinatize the points of K. The map g is defined by

$$g((i_j)) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(K) = \cap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(K).$$

1991 *Mathematics Subject Classification.* Primary 37B40, 37B10, 37B25, 37C45, 37C05, 37D05 Secondary 37A05.

Key words and phrases. code space, subshift of finite type, Hausdorff dimension, self-similar set, sub-self-similar set, topological entropy, Lyapunov exponent, smooth map.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

In §2 we describe a metric on Σ which has a natural connection with the invariant set K. The metric is well-known and described in the book by G. Edgar [E1]. We give some additional applications of the metric in this paper.

The code space is usually denoted by Ω when associated with self-similar sets. In this paper we use the symbol Σ to denote the space. This notation is more common in dynamical systems.

K. Falconer introduced the concept of sub-self-similar set [F3]. These are closed subsets E of K which have the property that $E \subset \cup_{i=1}^{N} f_i(E)$. He gave a way of computing the Hausdorff dimension of sub-self-similar sets whenever the *IFS* satisfies the *Open Set Condition*, *OSC*, introduced by J. Hutchinson [H]. Although the formula is abstract, in many instances practical calculations can be made. P. Duvall and J. Keesling [DK] determined the dimension of the boundary of the Lévy Dragon. P. Duvall, J. Keesling, and A. Vince [DKV] determined a general formula for the dimension of the boundary of many self-similar tiles by adapting the formula of Falconer. Some similar calculations were also done by R. Strichartz and Y. Wang [SW] and R. Kenyon, J. Li, R. Strichartz, and Y. Wang [KLSW] using different methods.

In this paper we show that the formula of K. Falconer can also be adapted to calculate the dimension of subshifts of finite type in the case of the special metric ρ on Σ. We also show that Σ can be embedded in Euclidean space R^n for some large n in such a way that the embedding and its inverse are both Lipschitz. We also show that this image can be the invariant set for a smooth map $F : R^n \to R^n$ in such a way that the Hausdorff dimension of the invariant set can be calculated. The map F restricted to this invariant set will be conjugate to a power of the full one-sided shift or subshift of finite type, respectively. The Lyapunov exponents of F on the set can be calculated, and the topological entropy of the map on this invariant set is also readily determined. The value of the construction is that all of the before-mentioned values are either known from classical results or are straightforward calculations. The relationships between these invariants for smooth maps has been a widespread object of study. See for instance the book by Y. Pesin [P] dealing with this subject, and other relationships between dynamics and dimension.

The advantage of the approach we give is that calculations which are easily done in the theory of self-similar sets and sub-self-similar sets are brought to bear in the examples. On the other hand, this feature does limit the invariant sets we construct. The approach can probably be generalized using the theory of self-affine sets [F1].

§2. Preliminaries

The modern basis for self-similar fractal theory was articulated by J. Hutchinson [H]. There are many good books which can be consulted. See for instance the books by G. Edgar [E1], K. Falconer [F2], or P. Mattila [M]. A good reference for dynamical systems theory is the book by C. Robinson [R]. The code space and the shift map on it form the beginning of the theory of symbolic dynamics. See the books by B. Kitchens [K] and by D. Lind and B. Marcus [LM] for more on this subject.

Let $\{1, \dots, N\}$ be the first N integers. The *code space* will be denoted by $\Sigma = \prod_{i=1}^{\infty}\{1, \dots, N\}$. The (full one-sided) *shift map* is the map $\sigma : \Sigma \to \Sigma$ defined by $\sigma((i_1, i_2, i_3, \dots) = (i_2, i_3, \dots)$.

Suppose that $\{c_1, \dots, c_N\}$ is a collection of real numbers with $0 < c_i < 1$ for each $1 \le i \le N$. We define the metric ρ on Σ by the following rule. Let

$$\rho((i_j),(k_j)) = c_{i_1} \cdot \dots \cdot c_{i_n}$$

where the two sequences agree up to index n and disagree at index $n+1$. Let $\rho((i_j),(k_j)) = 0$ if the two sequences are identical and $\rho((i_j),(k_j)) = 1$ if they differ in the first element of the sequence. This puts a non-Archimedean metric or ultrametric on Σ intimately connected with the mapping g. Recall that an *ultrametric* is a metric $d(x,y)$ which obeys the strong triangle inequality $d(x,y) = \max\{d(x,z), d(z,y)\}$. Some properties of the metric ρ are developed in [E1]. The code space is used in the study of ergodic theory, probability theory, stochastic processes, and information theory. The map g gives a connection between the code space and the theories associated with it to the geometry of self-similar sets.

Let $\{f_1, \dots, f_N\}$ be an *IFS* on R^n with contraction constants $\{c_1, \dots, c_N\}$, respectively. Let $g : \Sigma \to K$ be the mapping defined in §1. With the metric ρ on Σ defined in §2 the mapping g is Lipschitz with Lipschitz constant $\operatorname{diam}(K)$, i.e., $d(g(\alpha), g(\beta)) \le \operatorname{diam}(K) \cdot \rho(\alpha, \beta)$. If the map g is one-to-one, then g^{-1} is also Lipschitz with Lipschitz constant $\frac{1}{D}$ where

$$D = \min_{1 \le i \ne j \le N} d(f_i(K), f_j(K)).$$

When g is bi-Lipschitz, then the Hausdorff dimension of Σ and K are the same as are the dimensions of A and $g(A)$ for all $A \subset \Sigma$.

§3. Self-Similar Sets and the Open Set Condition

Let $\{f_1, \dots, f_N\}$ be an *IFS* on R^n with contraction constants $\{c_1, \dots, c_N\}$, respectively. The *IFS* is said to satisfy the Open Set Condition, *OSC*, provided there is a bounded non-empty open set O in R^n which satisfies the following conditions. For each i, $f_i(O) \subset O$ and for all $1 \le i \ne j \le N$, $f_i(O) \cap f_j(O) = \emptyset$. This condition was introduced by J. Hutchinson [H]. He showed that when this condition holds for self-similar sets K in R^n, the Hausdorff dimension of K is given by $\dim_H K = \alpha$ where α is the unique non-negative real number satisfying the equation

$$\sum_{i=1}^{N} c_i^\alpha = 1.$$

Now it is always the case that $\dim_H \Sigma = \alpha$ with the metric ρ. So, if g is one-to-one and hence bi-Lipschitz, then it is obvious that $\dim_H K = \alpha$ as well. The *OSC* was introduced to show that $\dim_H K = \alpha$ in a more general setting. It is also the case in R^n that when the *OSC* holds, then $0 < \mathcal{H}^\alpha(K) < \infty$ where $\mathcal{H}^\alpha(X)$ is the *Hausdorff α-measure* of X.

The condition that there is an open set satisfying the *OSC* which in addition satisfies $O \cap K \ne \emptyset$ is called the *Strong Open Set Condition*, *SOSC*. It was an open problem for some time whether *OSC* was equivalent to *SOSC*. A. Schief [S1] was able to show this to be so in Euclidean space. However, in the study of self-similar sets in general complete metric spaces, the conditions are not equivalent. In general complete metric spaces the *OSC* does not imply that $\dim_H K = \alpha$, but the *SOSC* does. However, in general complete metric spaces the *SOSC* may hold with $\mathcal{H}^\alpha(K) = 0$. See [S2] for further developments in this more general setting.

§4. Sub-Self-Similar Sets

The study of the Hausdorff dimension of sub-self-similar sets was introduced by K. Falconer in [F3]. Let $\{f_1, \ldots, f_N\}$ is an *IFS* on R^n, with constants $\{c_1, \ldots, c_N\}$, respectively, and let E be a sub-self-similar set in K. Suppose that the *IFS* satisfies the *OSC*. Then Falconer showed that $\dim_H E = \beta$ where β is determined by the following procedure.

Let $A \subset \Sigma$ be closed with $\sigma(A) \subset A$ and with $g(A) = E$. That such an A exists is a characterization of sub-self-similar sets. See [F3, Proposition 2.1]. This was also proved by C. Bandt [B]. Let A_k denote the sequences in A restricted to the first k coordinates. Then let

$$\tau(s) = \lim_{k\to\infty} \Big(\sum_{I_k \in A_k} c_{I_k}^s \Big)^{\frac{1}{k}}$$

where $c_{I_k} = c_{i_1} c_{i_2} \cdots c_{i_k}$ for $I_k = (i_1 i_2 \cdots i_k)$.

Now this limit exists with $0 \leq \tau(s) < \infty$ and there is a unique s such that $\tau(s) = 1$. Then $\dim_H E = \beta$ where $\tau(\beta) = 1$. It is also the case that the box dimension of E exists and is equal to β as well.

With this procedure, the Hausdorff dimension of E is not always easily computed. However, in certain cases, the computation can be made by determining the largest eigenvalue of a certain matrix with non-negative entries. For instance, the boundary of a self-similar set is sub-self-similar. If K is a self-similar digit tile, then the method of Falconer can be used to compute the Hausdorff dimension of the boundary of K. In this case the calculation is quite nice and is determined by finding the largest eigenvalue of the contact matrix. See [DKV] where the details are carried out. See [SW] and [KLSW] for related results.

The approach of Falconer starts with a sub-self-similar set E and then uses a closed shift invariant set $A \subset \Sigma$ to calculate the Hausdorff dimension of E. On the other hand, one could start with the code space Σ, constants $\{c_1, \ldots, c_N\}$ and a closed shift invariant subset A. One does not need the *IFS* on R^n to define the metric ρ on Σ. We will show that Falconer's formula applies to determine the dimension of any shift invariant set A with the metric ρ. Suppose that the invariant set A is a subshift of finite type with transition matrix B. In this special case Falconer's procedure has an easy formulation to determine the Hausdorff dimension of A. In §5 we will show that we can embed Σ into some Euclidean space R^n in a bi-Lipschitz manner. In §6 we will show how to calculate the dimension of subshifts of finite type using the embedding given in §5 to satisfy the conditions required to use Falconer's results. The Hausdorff dimension of Σ_B can then be calculated by simple linear algebra. We mention here that there is an alternative method for making this calculations using results of D. Mauldin and S. Williams [MW].

§5. Embedding the Code Space in Euclidean Space

As stated in the previous section, one need not have an *IFS* in order to define the code space Σ and metric ρ. All that one needs is the set of integers $\{1, \ldots, N\}$ and constants

$\{c_1, \ldots, c_N\}$ with each $c_i \in (0,1)$. In this section of the paper we will show how to embed this space into some Euclidean space in a bi-Lipschitz manner provided each $c_i < \frac{1}{2}$. In §8 we handle the general case. Of course, the space Σ is a Cantor set, so topologically it is easy to embed. To preserve the Hausdorff dimension of Σ and its subsets we need the embedding to be bi-Lipschitz.

Theorem 5.1. *Suppose that $\Sigma = \prod_{i=1}^{\infty}\{1, \ldots, N\}$ with the metric ρ. Suppose that the constants $\{c_1, \ldots, c_N\}$ each satisfy $0 < c_i < \frac{1}{2}$. Then there is a bi-Lipschitz embedding $g : \Sigma \to g(\Sigma) \subset R^n$ where n is any integer satisfying $2^n \geq N$. The embedding g is such that $g(\Sigma)$ is a self-similar fractal defined by an IFS satisfying the open set condition.*

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Proof. Let n satisfy $2^n \geq N$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for R^n. Let

$$a_i = \sum_{j=1}^{n} d_{ij} e_j$$

for $1 \leq i \leq N$ where each $d_{ij} \in \{0,1\}$ such that the collection $\{a_1, \ldots, a_N\}$ are distinct points. Such a collection $\{a_i | i = 1, \ldots, N\}$ exists since $2^n \geq N$.

Now for each $1 \leq i \leq N$ let $f_i : R^n \to R^n$ be defined by

$$f_i(x) = c_i \cdot (x - a_i) + a_i.$$

Then $\{f_1, \ldots, f_N\}$ is an *IFS* on R^n. Furthermore, the code space for this *IFS* corresponds precisely to the Σ we started with. So, there is the standard mapping $g : \Sigma \to K \subset R^n$ where K is the invariant set of the *IFS*.

Let $I^n = \prod_{i=1}^{n}[0,1]$ be the unit cube in R^n. Then it is easy to see that $K \subset I^n$. Furthermore, $f_i(I^n) \cap f_j(I^n) = \emptyset$ for all $i \neq j$. Thus, $f_i(K) \cap f_j(K) = \emptyset$. This condition implies that $g : \Sigma \to K = g(\Sigma)$ is one-to-one and the open set condition holds. Thus, g is bi-Lipschitz and is the embedding required in Theorem 5.1. □

§6. Hausdorff Dimension of Subshifts of Finite Type

Let B be an $N \times N$ matrix with $b_{ij} \in \{0,1\}$ for all $1 \leq i, j \leq N$. Let $\Sigma_B = \{(i_j) | b_{i_j i_{j+1}} = 1$ for all $j = 1, 2, \ldots\}$. Let $\sigma : \Sigma \to \Sigma$ be the shift map $\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots)$. Then $\sigma(\Sigma_B) \subset \Sigma_B$. Let σ_B be the restriction of σ to Σ_B. Then σ_B is called the *subshift of finite type* associated to B. Sometimes the space Σ_B is also given this designation. The matrix B is called the *transition matrix* or the *adjacency matrix.* Let λ be the largest eigenvalue of B. Then the *topological entropy* of σ_B is given by $\text{ent}(\sigma_B) = \max\{\ln \lambda, 0\}$. See [R, Theorem IX.1.9].

Theorem 6.1. *Let B be an $N \times N$ transition matrix. Let Σ_B be the subshift of finite type associated with B in Σ. Suppose that ρ is the metric on Σ associated with the constants $\{c_1, \ldots, c_N\}$ with $0 < c_i < 1$ for all i. Then*

$$\dim_H \Sigma_B = \beta$$

where β is the unique value of s such that the matrix $M(s)$ has largest real eigenvalue $\lambda = 1$.

$$M(s) = \begin{bmatrix} b_{11}c_1^s & b_{12}c_2^s & \cdots & b_{1N}c_N^s \\ b_{12}c_1^s & b_{22}c_2^s & \cdots & b_{2N}c_N^s \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1}c_1^s & b_{N2}c_2^s & \cdots & b_{NN}c_N^s \end{bmatrix}$$

Proof. First let us consider the case that the constants $\{c_1, \ldots, c_N\}$ defining the metric ρ on Σ are all less than $\frac{1}{2}$.

Case 1. *The constants $\{c_1, \ldots, c_N\}$ are all less than $\frac{1}{2}$.*

Proof for Case 1. Let n be an integer satisfying $2^n \geq N$. By Theorem 5.1, there is a bi-Lipschitz embedding $g : \Sigma \to g(\Sigma) \subset R^n$ such that $g(\Sigma)$ is a self-similar fractal defined by an *IFS* satisfying the *OPS*. Since Σ_B is shift invariant, $g(\Sigma)$ is a sub-self-similar set. So, the formula of Falconer given in §4 may be applied to compute the Hausdorff dimension of $g(\Sigma_B)$. Since g is bi-Lipschitz, this formula also gives the Hausdorff dimension of Σ_B. So the Hausdorff dimension of Σ_B is β where β is the unique value of s such that $\tau(s) = 1$ where $\tau(s)$ is defined by

$$\tau(s) = \lim_{k\to\infty} \Big(\sum_{I_k \in A_k} c_{I_k} \Big)^{\frac{1}{k}}.$$

Here A_k is the collection of all sequences I_k of length k such that there is a sequence I in Σ_B whose first k coordinates are I_k.

We define a matrix $D(s)$ using $M(s)$ by the following algorithm. If row i in $M(s)$ is all zeros, then change all the entries of $M(s)$ in the i column to zeros. Repeat this process until it produces no new zeros in the matrix. The resulting matrix is $D(s)$. With $D(s)$ so defined, we can show that

$$[\, c_1^s \quad c_2^s \quad \cdots c_N^s \,] \times D(s)^{k-1} \times \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{I_k \in A_k} c_{I_k}.$$

Thus,

$$\tau(s) = \lim_{k\to\infty} \left([\, c_1^s \quad c_2^s \quad \cdots c_N^s \,] \times D(s)^{k-1} \times \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{\frac{1}{k}}.$$

From this formula, one can show that $\tau(s) = 1$ if and only if the largest eigenvalue of $D(s)$ is $\lambda = 1$. Also, one can show that the eigenvalues of $D(s)$ are the same as those of $M(s)$. This proves Theorem 6.1 for Case 1.

Case 2. *There is no restriction on the constants* $\{c_1, \dots, c_N\}$ *other than the standing one that* $0 < c_i < 1$.

Proof for Case 2. We want to prove the result for Case 2 using Case 1. Let $t > 0$ be large enough that $c_i^t < \frac{1}{2}$ for all i. Then for this t we have a new set of constants $\{c_1^t, \dots, c_N^t\}$ which we could then use to define a new metric ρ' on Σ. Let us use Σ' to denote the space having ρ' as metric.

Let $f : \Sigma \to \Sigma'$ be the identity map. Then f satisfies the *Hölder condition of exponent* t [F2, page 28] and f^{-1} satisfies the Hölder condition of exponent $\frac{1}{t}$. Hence

$$\dim_H \Sigma'_B \leq \frac{1}{t} \cdot \dim_H \Sigma_B$$

and

$$\dim_H \Sigma_B \leq t \cdot \dim_H \Sigma'_B.$$

The inequalities hold as a result of [F2, Proposition 2.3]. These two inequalities imply equality. In the case being dealt with here, $\rho' = \rho^t$, so the Hausdorff measure in dimension α of any subset of Σ is the same as the measure of its image in dimension $t \cdot \alpha$. See [F2, Proposition 2.2].

Since Σ' has constants all less than $\frac{1}{2}$, by Case 1, $\dim_H \Sigma'_B$ is the value γ of s so that the matrix $M'(s)$ has largest eigenvalue $\lambda = 1$, where

$$M'(s) = \begin{bmatrix} b_{11}(c_1^t)^s & b_{12}(c_2^t)^s & \cdots & b_{1N}(c_N^t)^s \\ b_{12}(c_1^t)^s & b_{22}(c_2^t)^s & \cdots & b_{2N}(c_N^t)^s \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1}(c_1^t)^s & b_{N2}(c_2^t)^s & \cdots & b_{NN}(c_N^t)^s \end{bmatrix}.$$

Then the Hausdorff dimension of Σ_B will be $\beta = t \cdot \gamma$. But β is precisely the value of s such that $M(s)$ has largest eigenvalue $\lambda = 1$. This proves Case 2 and completes the proof of Theorem 6.1. □

The calculations in this section could also be done by the directed graph constuction methods developed by D. Mauldin and S. Williams [MW]. Using Falconer's paper one can claim that $\mathcal{H}^\beta(\Sigma_B) > 0$. One would need the results in [MW] to claim that if B is transitive, then $\mathcal{H}^\beta(\Sigma_B) < \infty$.

§7. Consruction of a Smooth Map in R^n

Let B be an $N \times N$ transition matrix. Let Σ_B be as in §6. Let b_{ij} denote the ij element in the matrix B. If B is the matrix with with $b_{ij} = 1$ for all $0 \leq i, j \leq N$, then Σ_B is just Σ.

Let $\{c_1, \dots, c_N\}$ be constants with $0 < c_i < \frac{1}{2}$ for $1 \leq i \leq N$. Let Σ and ρ be as defined in §5 and §6 above. Let $2^n \geq N$.

We construct a smooth map $F : R^n \times (0, \infty) \to R^n \times (0, \infty)$ having an invariant set Y_B with the following properties:

(1) Y_B is a maximal compact invariant subset of $R^n \times (0, \infty)$, and Y_B contains the nonwandering set of F.

(2) $Y_B \subset R^n \times \{1\}$.

(3) $F|_{Y_B}$ is topologically conjugate to $\sigma|_{\Sigma_B}$ and the conjugacy is bi-Lipschitz.

Since $(0, \infty)$ is homeomorphic to R, we easily obtain from F a map of R^{n+1} to itself with the analogous properties.

We now proceed with the construction. Let $\{e_1, \ldots, e_n\}$ be the standard basis for R^n. Let $\{a_1, \ldots, a_N\}$ be distinct elements of R^n such that for each $i = 1, \ldots, n$, there are scalars $\{\gamma_{i1}, \ldots, \gamma_{iN}\}$ with $\gamma_{ij} \in \{0, 1\}$ such that

$$a_i = \sum_{j=1}^{n} \gamma_{ij} \cdot e_j.$$

Now for each $1 \leq i \leq N$, let $f_i : R^n \to R^n$ be the map defined by $f_i(x) = \frac{1}{c_i} \cdot (x - a_i) + a_i$.

Note that $f_i(I^n) \supset I^n$ where $I^n = \prod_{i=1}^{n} [0, 1] \subset R^n$. We also note that the collection $\{D_1 = f_1^{-1}(I^n), \ldots, D_N = f_N^{-1}(I^n)\}$ is pairwise disjoint using the fact that $c_i < \frac{1}{2}$ for all i. We note that each of the maps f_i is smooth being an expansion similitude on R^n. Let $\{U_1, \ldots, U_N\}$ be a collection of bounded open sets with disjoint closures such that $U_i \supset D_i$ for $i = 1, \ldots, N$.

Now for each $1 \leq i \leq N$ let $w_i : R^n \to [0, 1]$ be a smooth map such that $\{w_1, \ldots, w_N\}$ is a partition of unity on R^n satisfying $w_i(x) = 1$ precisely on the closure of the set U_i.

Now define $G : R^n \to R^n$ by

$$G(x) = \sum_{i=1}^{N} w_i(x) \cdot f_i(x).$$

By this definition G is a smooth map. The maps $\{f_1^{-1}, \ldots, f_N^{-1}\}$ form an *IFS* with contraction constants $\{c_1, \ldots, c_N\}$, respectively. Let E denote the invariant set for this *IFS*.

Let $g : \Sigma \to E$ be the embedding given in §1. Thus, by Theorem 5.1 g is bi-Lipschitz. Let E_B denote the image $g(\Sigma_B)$ in E. If B is the matrix with $b_{ij} = 1$ for all ij, then $E_B = E$.

It is easy to see that E is precisely the set of points $x \in R^n$ such that $G^k(x) \in D_1 \cup \cdots \cup D_N$ for each $k = 0, 1, 2, \ldots$, and that $g : \Sigma \to E$ is a conjugacy, i.e., the following diagram commutes.

(7-1)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ E & \xrightarrow{G|_E} & E \end{array}$$

Let $D_{ij} = f_i^{-1}(D_j)$. Define $f : R^n \to [0, 2]$ to be a smooth map so that it has the following properties

i. $f(x) = 0$ if $x \in D_{ij}$ and $b_{ij} = 1$;

ii. $f(x) = 2$ if $x \in D_{ij}$ and $b_{ij} = 0$;

iii. $f(x) = 2$ if $x \notin \cup_{i=1}^{N} U_i$.

Define $F : R^n \times (0, \infty) \to R^n \times (0, \infty)$ by $F(x, t) = (G(x), t^2 + f(x))$ where the map f is defined above.

We observe that each of the following holds.

(a) Let $x \in R^n \setminus I^n$. For each $i = 1, \dots, N$, $d(f_i(x), I^n) \geq \frac{1}{c_i} \cdot d(x, I^n)$. Hence, there exists a $w > 1$ independent of x such that $d(G(x), I^n) \geq w \cdot d(x, I^n)$. In particular, $G^k(x) \to \infty$ as $k \to \infty$.

(b) If $x \in U_i \setminus D_i$ for some i, then $G(x) \notin I^n$. So, by (a), $G^k \to \infty$ as $k \to \infty$.

(c) If $x \in R^n \setminus \cup_{i=1}^{N} U_i$, then $f(x) = 2$. Thus, the second coordinate of $F^k(x, t)$ goes to ∞ as $k \to \infty$.

(d) If $x \in D_{ij}$ with $b_{ij} = 0$, then the second coordinate of $F^k(x, t)$ goes to ∞ as $k \to \infty$.

(e) Suppose that for each nonnegative integer k, $G^k(x)$ is in some D_{ij} with $b_{ij} = 1$. Then the second doordinate of $F^k(x, t)$ goes to ∞ if $t > 1$ and goes to 0 if $t < 1$.

It follows that if we let $Y_B = \{(x, t) \in R^n \times (0, \infty) | x \in E_B \text{ and } t = 1\}$, then (1), (2), and (3) hold.

The relationship between dimension and dynamics has been a recent topic of research. For example, there are relationships between Hausdorff dimension, topological entropy, and Lyapunov exponents. The book by Y. Pesin [P] is a good reference.

Consider the invariant set Y_B for the map F constructed above. It follows from (3) that the Hausdorff dimension of Y_B may be computed by Theorem 6.1. It also follows from (3) that the topological entropy of $F|_{Y_B}$ equals the topological entropy of σ_B. This is known to be $\max\{\ln \lambda, 0\}$ where λ is the largest eigenvalue of the matrix B.

The Lyapunov exponents for $F|_{Y_B}$ can also be calculated. By definition

$$\lambda(x, v) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \ln(|DF(F^i(x))(v)|)$$

when this limit exists. If $(x, 1) \in Y_B$ and x corresponds to $(i_j) \in \Sigma_B$, then

$$\lambda((x, 1), (v, 0)) = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} - \ln(c_{i_j})$$

when this limit exists.

§8. Arbitrary Contraction Constants

Let $\{c_1, \dots, c_N\}$ be such that $0 < c_i < 1$. In this section we adapt the construction given in §7 so that it applies without the restriction that the c_i's are all less than $\frac{1}{2}$. Let $c_{\max}$ be the maximum of the c_i's. Let m be such that $c_{\max}^m < \frac{1}{2}$. Let

$$\Sigma_{(m)} = \prod_{i=1}^{\infty} (\prod_{j=1}^{m} \{1, \dots, N\}).$$

Now $\Sigma_{(m)}$ is a code space with the set $\prod_{j=1}^{m}\{1,\ldots,N\}$ as the factor in the product. We now associate with each point in $\prod_{j=1}^{m}\{1,\ldots,N\}$ a constant between 0 and 1. Let $J=(i_j)_{j=1}^{m}$ be an element of $\prod_{j=1}^{m}\{1,\ldots,N\}$. Let $c_J=\prod_{j=1}^{m}c_{i_j}$. Let $\rho_{(m)}$ be the metric on $\Sigma_{(m)}$ using the constants $\{c_J|J\in\prod_{j=1}^{m}\{1,\ldots,N\}\}$.

We now show that $\Sigma_{(m)}$ with the metric $\rho_{(m)}$ is bi-Lipschitz equivalent to Σ with the metric ρ. Let $(i_j)_{j=1}^{\infty}\in\Sigma$. Then define

$$h((i_j)_{j=1}^{\infty})=(J_k)_{k=1}^{\infty}$$

where J_1 is the first m elements in the sequence $(i_j)_{j=1}^{\infty}$, J_2 is the second m elements in the sequence, etc.

Clearly, h so defined is one-to-one and onto. It is also easy to see that h is bi-Lipschitz with respect to the metrics ρ and $\rho_{(m)}$. The Lipschitz constant C for h is just $C=c_{\min}^{1-m}$, where $c_{\min}$ is the minimum of $\{c_1,\ldots,c_N\}$. The Lipschitz constant C' for h^{-1} is just $C'=1$. We summarize what we have shown for reference.

Proposition 8.1. *Let $h:\Sigma\to\Sigma_{(m)}$ be defined as above. Then h is bi-Lipschitz.*

It should be pointed out that the bi-Lipschitz map h is not a conjugacy for the maps σ on Σ and σ' on $\Sigma_{(m)}$. It is a conjugacy between σ^m and σ' so that the following diagram commutes.

(8-1)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma^m} & \Sigma \\ h\downarrow & & \downarrow h \\ \Sigma_{(m)} & \xrightarrow{\sigma'} & \Sigma_{(m)} \end{array}$$

Let B be an $N\times N$ transition matrix and Σ_B the subshift of finite type associated with it. Now we want to show that $h(\Sigma_B)$ is a subshift of finite type in $\Sigma_{(m)}$ corresponding to some $N^m\times N^m$ transition matrix B'. We define the matrix B' as follows. Let J and J' be two elements in $\prod_{i=1}^{m}\{1,2,\ldots,N\}$. Let $J=(i_1,\ldots,i_m)$ and $J'=(i'_1,\ldots,i'_m)$. Define $b'_{JJ'}=0$ if either $b_{i_p i_{p+1}}=0$ for some $1\le p\le m-1$ or $b_{i'_p i'_{p+1}}=0$ for some $1\le p\le m-1$. Also, let $b'_{JJ'}=0$ if $b_{i_m i'_1}=0$. Otherwise let $b_{JJ'}=1$. Then it is easy to check that $h(\Sigma_B)=\Sigma_{(m)B'}$.

Now $\dim_H\Sigma_B=\dim_H\Sigma_{(m)B'}$ by Proposition 8.1. In §6 we showed how to compute the Hausdorff dimension of Σ_B

The construction in §7 can now be repeated on the set $\Sigma_{(m)B'}$ to obtain an embedding $g:\Sigma_{(m)B'}\to R^{n+1}$ where $2^n\ge N^m$. Then we obtain the embedding $g\circ h:\Sigma_B\to R^{n+1}$ with properties as in §7. The only difference is that $F|_{Y_B}$ in this case is conjugate to σ^m on Σ_B rather than σ.

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