First-Year Analysis Examination
September 2009

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let s_n, t_n be bounded sequences of nonnegative real numbers. Prove that
$$\limsup_{n\to\infty}(s_n t_n) \leq (\limsup_{n\to\infty} s_n)(\limsup_{n\to\infty} t_n).$$
Give an example to show that the inequality can be strict.

2. Let X be a metric space. Prove that if A, B are connected subsets of X and $A \cap B$ is nonempty, then $A \cup B$ is connected.

3. Prove that every sequence in a compact metric space has a convergent subsequence. Deduce (from this fact or otherwise) that every compact metric space is complete.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Prove that the sequence of functions
$$f_n(x) := f(x + \frac{1}{n})$$
is uniformly convergent.

5. Prove that a continuous function on a compact metric space is uniformly continuous.

6. Let f_n be a sequence of functions on $[a, b]$, and suppose that

   1) each f_n is differentiable on $[a, b]$ and f_n' is continuous,
   2) the sequence f_n' converges uniformly on $[a, b]$, and
   2) the sequence $(f_n(x_0))$ converges for some $x_0 \in [a, b]$.

   Prove that f_n converges uniformly to a differntiable function f on $[a, b]$, and
$$f' = \lim_{n\to\infty} f_n'$$

7. a) State the monotone convergence theorem.

   b) State and prove Fatou's theorem.

8. Give an example (with justificiation) of each of the following, if possible. If no such example exists, briefly explain why.

   a) A Lebesgue integrable function on $[0, 1]$ that is not Riemann integrable.

   b) A sequence of integrable functions on $\mathbb{R}$ that converges uniformly but not in $L^1(\mathbb{R})$.

   c) A sequence of functions that converges in $L^1([0, 1])$ but not in $L^2([0, 1])$.

9. Prove that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.