First-Year Analysis Examination
May 2009

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let s_n, t_n be bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(s_n + t_n) \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

   Give an example to show that the inequality can be strict.

2. Recall that a collection of sets $\{F_\alpha\}$ has the *finite intersection property* if the intersection of any finite subcollection is nonempty. Let X be a metric space with the following property:

   Whenever a collection of closed *sets* $\{F_\alpha\}$ *has the finite intersection property, the intersection* $\cap F_\alpha$ *is nonempty.*

   Prove that X is compact.

3. Let X be a metric space with the following property:

   Whenever E_n *is a sequence of nonempty closed subsets of* X *with* $E_{n+1} \subseteq E_n$ *for all* n *and* $diam(E_n) \to 0$*, then* $\cap E_n$ *is nonempty.*

   Prove that X is complete.

4. Let X be a metric space. A function $f : X \to \mathbb{R}$ is called *idempotent* if f only takes on the values 0 or 1 (in other words, $f(x) = f(x)^2$ for all $x \in X$). Prove that X is connected if and only if every *continuous* idempotent $f : X \to \mathbb{R}$ is constant.

5. Suppose K is a compact metric space and (f_n) is an equicontinuous sequence of functions on K. Prove that if $f_n \to f$ pointwise, then $f_n \to f$ uniformly.

6. Let f be Riemann integrable on $[a, b]$ and suppose f is continuous at $x_0 \in (a, b)$. Prove that the function

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable at x_0 and $F'(x_0) = f(x_0)$.

7. a) State Fatou's theorem.

   b) State and prove the dominated convergence theorem.

8. Give an example of each of the following, if possible. If no such example exists, briefly explain why.

   a) A sequence of nonnegative, integrable functions on $[0, 1]$ such that $f_n \to f$ pointwise, f is integrable, but $\lim \int f_n \neq \int f$.

   b) A sequence of uniformly bounded, integrable functions on $\mathbb{R}$ such that $f_n \to f$ pointwise, f is integrable, but $\lim \int f_n \neq \int f$.

   c) A sequence of uniformly bounded, integrable functions on $[0, 1]$ such that $f_n \to f$ pointwise, f is integrable, but $\lim \int f_n \neq \int f$.

9. Let f be a nonnegative, measurable function on $\mathbb{R}$, and suppose that

$$F(x) := \int_{-\infty}^x f(t)\,dt$$

is finite for all x. Prove that F is continuous.