First Year Analysis Exam, January 2009

Answer each question on a separate sheet of paper. Write all solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose (a_n) and (b_n) are bounded sequences of real numbers and let $(c_n = a_n + b_n)$. Prove,

$$\limsup c_n \leq \limsup a_n + \limsup b_n.$$

Give an example where strict inequality holds.

2. Let X and Y be metric spaces. Prove, if $f : X \to Y$ is uniformly continuous, Y is complete, and (x_n) is a Cauchy sequence in X, then the sequence $(f(x_n))$ converges.

Give an example which shows that the uniform continuity of f can not be replaced with continuity.

3. Prove, if $f : X \to Y$ is continuous, one-one and onto, and if X is compact, then $f^{-1} : Y \to X$ is continuous.

Give an example which shows that the hypothesis that X is compact is essential.

4. Prove, if $f : \mathbb{R} \to \mathbb{R}$ has bounded derivative, then f is uniformly continuous.

5. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{n}$ if $x \neq 0$ is rational and $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$, $n > 0$, and $\gcd(m, n) = 1$. Define $f(0) = 0$. Prove that f is Riemann integrable.

Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable.

6. Suppose $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. Suppose $f_n : [0,1] \to \mathbb{R}$ is a sequence of Lebesgue integrable functions. Define,

$$F_n(x) = \int_0^1 K(x,y) f_n(y) dy.$$

If there is a C such that

$$\int_0^1 |f_n(y)| dy \leq C$$

for all n, must (f_n) have a uniformly convergent subsequence?

7. State the Monotone Convergence Theorem. State and prove Fatou's Lemma. Give an example where strict inequality holds in Fatou.

8. Suppose (X, Σ, μ) is a measure space and $f : X \to \mathbb{R}$ is a measurable function. Prove, if $f \geq 0$ and $\int_X f \, d\mu = 0$, then $f = 0$ almost everywhere (μ).

9. Let $C \subset [0,1]$ be a nonempty closed set. Let K_C denote the characteristic function of C and define $\rho_C : [0,1] \to \mathbb{R}$ by, $\rho_C(x) = \inf\{|x-t| : t \in C\}$.

Define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \frac{1}{1 + n\rho_C(x)}.$$

Prove

$$\lim_n \int_0^1 |g_n - K_C|^2 dx = 0,$$

giving reasons for the existence of the Lebesgue integrals implicit in the statement.